Optimum Design - Sheet 1 - Solution
Problem Formulation

1. Enterprising chemical engineering students have set up a still in a bathtub. They can produce 225 bottles of pure alcohol each week. They bottle two products from alcohol: (i) wine, 20 proof, and (ii) whiskey, at 80 proof. Recall that pure alcohol is 200 proof. They have an unlimited supply of water but can only obtain 800 empty bottles per week because of stiff competition. The weekly supply of sugar is enough for either 600 bottles of wine or 1200 bottles of whiskey. They make $1.00 profit on each bottle of wine and $2.00 profit on each bottle of whiskey. They can sell whatever they produce. How many bottles of wine and whiskey should they produce each week to maximize profit. Formulate the design optimization problem.

Solution:

Given: The amount of bottles of pure alcohol which can be produced each week, the two types of alcohol which are produced, the amount of empty bottles available per week, the amount of each alcohol which can be produced based on the weekly sugar supply, and the profits for each alcohol type.

Required: It is desired to find the amount of bottles of wine and whisky which should be produced, each week, to maximize profit.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement

Shown above

Step 2: Data and Information Collection

Shown above

Step 3: Definition of Design Variables

x_1 = bottles of wine produced/week

x_2 = bottles of whiskey produced/week

Step 4: Optimization Criterion

Optimization criterion is to maximize profit, and the cost function is defined as

$Profit = x_1 + 2x_2$

Step 5: Formulation of Constraints

Supply of Bottles Constraint: $x_1 + x_2 \le 800$

Supply of Alcohol Constraint: $0.1x_1 + 0.4x_2 \le 225$

Sugar Limitation Constraint: $x_1/600 + x_2/1200 \le 1$

Explicit Design Variable Constraints:

$x_1 \ge 0, \; x_2 \ge 0$

2. Design a can closed at one end using the smallest area of sheet metal for a specified interior volume of 600 cm^3. The can is a right circular cylinder with interior height *h* and radius *r*. The ratio of height to diameter must not be less than 1.0 nor greater than 1.5. The height cannot be more than 20 cm. Formulate the design optimization problem.

Solution:

Given: The desired interior can volume, the minimum and maximum ratio of height to diameter, and the maximum height.

Required: It is desired to find the design which minimizes the area of sheet metal for the can.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement

Shown above

Step 2: Data and Information Collection

Shown above

Step 3: Definition of Design Variables

h = interior height of the can in cm

r = interior radius of the can in cm

Step 4: Optimization Criterion

Optimization criterion is to minimize area of sheet metal, and the cost function is defined as

$Area = \pi r^2 + 2\pi rh$, , cm^2

Step 5: Formulation of Constraints

Volume Constraint: $\pi r^2 h = 600$, , cm^3

Height/Diameter Constraints:

$h/2r \geq 1$

$h/2r \leq 1.5$

Explicit Design Variable Constraints:

$h \leq 20$, cm; $h \geq 0$, cm; $r \geq 0$, cm

3. A company has m manufacturing facilities. The facility at the ith location has capacity to produce b_i units of an item. The product should be shipped to n distribution centers. The distribution center at the jth location requires at least a_j units of the item to satisfy demand. The cost of shipping an item from the ith plant to the jth distribution center is c_{ij}. Formulate a minimum cost transportation system to meet each distribution center's demand without exceeding the capacity of any manufacturing facility.

<u>Solution:</u>

Given: The number of manufacturing facilities the company owns, the capacity of the ith facility to produce b_i units of an item, the number of distribution centers the product should be shipped too, the minimum number of items, a_j, required by the jth distribution center, and the cost to ship an item from the ith plant to the jth distribution center.

Required: It is desired to design a transportation system which minimizes costs and meets the constraints set by the two types of facilities.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement

Shown above

Step 2: Data and Information Collection

Shown above

Step 3: Definition of Design Variables

x_{ij} : number of items produced at the ith facility shipped to jth distribution center where $i = 1$ to m; $j = 1$ to n

Step 4: Optimization Criterion

Optimization criterion is to minimize the cost, and the cost function is defined as

$$Cost = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Step 5: Formulation of Constraints

Capacity of Manufacturing Facility Constraint: $\sum_{j=1}^{n} x_{ij} \le b_i$ for $i = 1$ to m

Demand Constraint: $\sum_{i=1}^{m} x_{ij} \ge a_j$ for $j = 1$ to n; $x_{ij} \ge 0$ for all i and j

4. Design of a two-bar truss. Design a symmetric two-bar truss (both members have the same cross section) shown in Fig. 1 to support a load W. The truss consists of two steel tubes pinned together at one end and supported on the ground at the other. The span of the truss is fixed at s. Formulate the minimum mass truss design problem using height and the cross-sectional dimensions as design variables. The design should satisfy the following constraints:
    a. Because of space limitations, the height of the truss must not exceed b_1, and must not be less than b_2.
    b. The ratio of the mean diameter to thickness of the tube must not exceed b_3.
    c. The compressive stress in the tubes must not exceed the allowable stress σ_a for steel.
    d. The height, diameter, and thickness must be chosen to safeguard against member buckling.

    Use the following data: W = 10 kN; span s = 2 m; b_1 = 5 m; b_2 = 2 m; b_3 = 90 m; allowable stress, σ_a = 250 MPa; modulus of elasticity, E = 210 GPa; mass density, ρ = 7850 kg/m^3; factor of safety against buckling, FS = 2; $0.1 \le D \le 2$ (m); and $0.01 \le t \le 0.1$ (m).

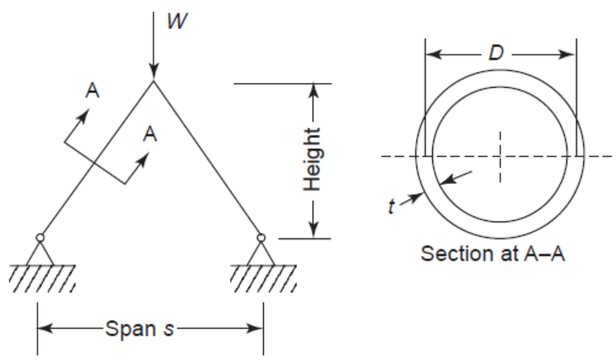


Fig. 1

Solution:

Given: Constraints 1-4 listed above and the factor of safety against buckling in the data section above.
Required: It is desired to design a truss which minimizes mass using height and the cross sectional dimensions as design variables.
Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Depending on the units used for various parameters, the final expressions for various function will look different. The following table give values of various parameters depending on the units used:

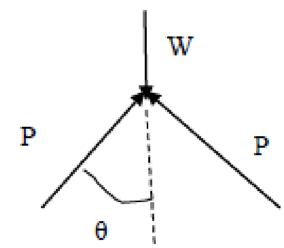
Variable	N & m	N & mm	N & cm	KN & m	MN & m
Load, W	10,000	10,000	10,000	10	1×10^{-2}
σ_a	250×10^6	250	250×10^2	250×10^3	250
Modulus, E	210×10^9	210×10^3	210×10^5	210×10^6	210×10^3
Density, ρ	7850	7.85×10^{-6}	7.85×10^{-3}	7850	7850
Span, s	2	2000	200	2	2
b_1	5	5000	500	5	5
b_2	2	2000	200	2	2
D_{min}	0.1	100	10	0.1	0.10
D_{max}	2	2000	200	2	2
t_{min}	0.01	10	1	0.01	0.01
t_{max}	0.1	100	10	0.1	0.1

Other data/expressions that need to be collected are:

Member length, $l = \sqrt{H^2 + (0.5s)^2}$

Member force: Draw the free-body diagram of the loaded node and sum up the forces in the vertical direction:

$-W + 2P\cos\theta = 0; \quad or \quad P = \frac{W}{2\cos\theta}; \quad \cos\theta = \frac{H}{l}$



Member stress: $\sigma = \frac{P}{A}$

Cross-sectional area: The expression will depend on what variables are used:

$$A = \frac{\pi}{4}(D_o^2 - D_i^2) = \pi Dt$$

Moment of inertia: $I = \frac{\pi}{64}(D_o^4 - D_i^4) = \frac{\pi}{8}(D^3 t + Dt^3)$

Buckling load (critical load) for pin-pin column: $P_{cr} = \frac{\pi^2 EI}{l^2}$

FORMULATION 1: In terms of intermediate variables

Step 3: Definition of Design Variables

H = height of the truss, m
D = mean diameter of the tube, m
t = thickness of the tube, m

Step 4: Optimization Criterion

Optimization criterion is to minimize mass, and the cost function is defined as

$Mass = 2\rho Al$

where ρ is the mass density of the material.

Step 5: Formulation of Constraints

Stress Constraint: $\sigma \le \sigma_a$

Buckling Constraint: $P \le \frac{P_{cr}}{FS}$

Explicit Design Variable Constraints:

$H \le b_1;\ H \ge b_2;\ D/t \le b_3;$

$0.1 \le D \le 2$ m; $0.1 \le t \le 0.1$ m

FORMULATION 2: Explicitly in terms of the design variables.

Use N and m as the units, and the corresponding values for various parameters.

Member Force: $P = W(s^2/4 + H^2)^{\frac{1}{2}}/2H$

Step 3: Definition of Design Variables

H = height of the truss, m
D = mean diameter of the tube, m
t = thickness of the tube, m

Step 4: Optimization Criterion

Optimization criterion is to minimize mass, and the cost function is defined as

$Mass = 2\rho Al = 2\rho(\pi Dt)\left(s^2/4 + H^2\right)^{\frac{1}{2}};$

where ρ is the mass density of the material.

Substituting the given values, we get

$Mass = 2(7850)(\pi Dt)(1 + H^2)^{\frac{1}{2}} = 49323\ Dt(1 + H^2)^{\frac{1}{2}}$, kg

Step 5: Formulation of Constraints

Stress Constraint: $P/A \le \sigma_a$; $W(s^2/4 + H^2)^{\frac{1}{2}}/2H(\pi Dt) \le \sigma_a$

Buckling Constraint: $P \le P_{cr}/(\mathrm{FS})$; $P_{cr} = \pi^2 EI/l^2 = \dfrac{\pi^2 E\left[\pi\left(D^3t + Dt^3\right)/8\right]}{\left(s^2/4 + H^2\right)}$

Or, $$\frac{W\left(s^2/4 + H^2\right)^{\frac{1}{2}}}{2H} \le \frac{\pi^2 E\left[\pi\left(D^3t + Dt^3\right)/8\right]}{(\mathrm{FS})\left(s^2/4 + H^2\right)}$$

Explicit Design Variable Constraints: $H \le b_1;\ H \ge b_2;\ D/t \le b_3;$

$0.1 \le D \le 2$ m; $0.1 \le t \le 0.1$ m

Substituting the given data, we obtain the final form of the constraints as

$10000(1 + H^2)^{\frac{1}{2}}/2\pi HDt \le 250 \times 10^6$

$10000(1 + H^2)^{\frac{1}{2}}/2H \le \left(210 \times 10^9\right)\pi^3\left(D^3t + Dt^3\right)/16\left(1 + H^2\right)$

$H \le 5$, m;

$H \ge 2$, m;

$D/t \le 90$;

$0.1 \le D \le 2$, m;

$0.01 \le t \le 0.1$, m

$H \le 5$, m;

5. A beam of rectangular cross section (Fig. 2) is subjected to a maximum bending moment of M and a maximum shear of V. The allowable bending and shearing stresses are σ_a and τ_a, respectively. The bending stress in the beam is calculated as

$$\sigma = \frac{6M}{bd^2}$$

and average shear stress in the beam is calculated as

$$\tau = \frac{3V}{2bd}$$

where d is the depth and b is the width of the beam. It is also desired that the depth of the beam shall not exceed twice its width. Formulate the design problem for minimum cross-sectional area using the following data: M = 140 kN· m, V = 24 kN, σ_a = 165 MPa, τ_a = 50 MPa.

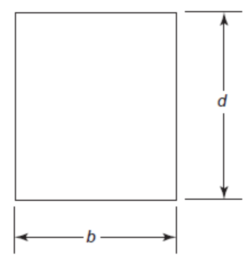


Fig.2 Cross section of a rectangular beam.

Solution:

Given: The equations to calculate bending and average shear stress in a beam, the constraint that the depth of the beam will not exceed twice its width, the applied moment, the applied shear force, and the maximum allowable bending and shear stresses in the beam.

Required: It is desired to design a beam which minimizes cross-sectional area without yielding due to shear or bending stresses.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement

Shown above

Step 2: Data and Information Collection

$M = 140$ kN.m $= 1.4\times10^7$ N.cm;

$V = 24$ kN $= 2.4\times10^4$ N;

$\sigma_a = 165$ MPa $= 1.65\times10^4$ N/cm^2;

$t_a = 50$ MPa $= 5000$ N/cm^2

Step 3: Definition of Design Variables

b = width of the beam, cm

d = depth of the beam, cm

Step 4: Optimization Criterion

Optimization criterion is to minimize the cross-sectional area, and the cost function is defined as

$Area = bd$, cm^2

Step 5: Formulation of Constraints

Bending Stress Constraint: $6M/bd^2 \le \sigma_a$ or $6(1.4 \times 10^7)/bd^2 \le 1.65 \times 10^4$

Shear Stress Constraint: $3V/2bd \le t_a$ or $3(2.4 \times 10^4)/2bd \le 5000$

Constraint: d ≤ 2b or $d - 2b \le 0$

Explicit Design Variable Constraints: b, d ≥ 0

From the graph for the problem, we get the optimum solution as
$b^* \doteq 10.8$ cm, $d^* \doteq 21.6$ cm, $Area \doteq 233$ cm² where constraint numbers 1 and 3 are active.

6. A vegetable oil processor wishes to determine how much shortening, salad oil, and margarine to produce to optimize the use of his current oil stock supply. At the present time, he has 250,000 kg of soybean oil, 110,000 kg of cottonseed oil, and 2000 kg of milk base substances. The milk base substances are required only in the production of margarine. There are certain processing losses associated with each product; 10% for shortening, 5% for salad oil, and no loss for margarine. The producer's back orders require him to produce at least 100,000 kg of shortening, 50,000 kg of salad oil, and 10,000 kg of margarine. In addition, sales forecasts indicate a strong demand for all products in the near future. The profit per kilogram and the base stock required per kilogram of each product are given in Table 1. Formulate the problem to maximize profit over the next production scheduling Period.

Table 1.

		Parts per kg of base stock requirements		
Product	*Profit per kg*	*Soybean*	*Cottonseed*	*Milk base*
Shortening	0.10	2	1	0
Salad oil	0.08	0	1	0
Margarine	0.05	3	1	1

Solution:

Given: The current supply of soybean oil, cottonseed oil, and milk-base substances, milk-base substances are required in the production of margarine only, the amount of processing loss which occurs in shortening, salad oil, and margarine, the minimum production requirement of each product, and the data shown in Table E2.18.

Required: It is desired to create a production schedule which will maximize profit.

Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement

Shown above

Step 2: Data and Information Collection

Shown above

Step 3: Definition of Design Variables

x_1 = shortening produced after losses, kg

x_2 = salad oil produced after losses, kg

x_3 = margarine produced, kg

Step 4: Optimization Criterion

Optimization criterion is to maximize the profit, and the cost function is defined as

$Profit = x_1 + 0.8\,x_2 + 0.5\,x_3$

Step 5: Formulation of Constraints

The ingredients used cannot exceed current stocks

Soybean Constraint: $(2x_1/3)(1/0.9)+(3x_3/5)\le 250{,}000$

Milk Base Constraint: $(x_3/5)\le 2000$

Cottonseed Constraint: $(x_1/3)(1/0.9)+(x_2)(1/0.95)+(x_3/5)\le 110{,}000$

The demand for the needs of the products to be satisfied

Explicit Design Variable Constraints: $x_1\ge 100{,}000$; $x_2\ge 50{,}000$; $x_3\ge 10{,}000$

7.

Answer True or False.

1. Design of a system implies specification for the design variable values. *True*
2. All design problems have only linear inequality constraints. *False*
3. All design variables should be independent of each other as far as possible. *True*
4. If there is an equality constraint in the design problem, the optimum solution must satisfy it. *True*
5. Each optimization problem must have certain parameters called the design variables. *True*
6. A feasible design may violate equality constraints. *False*
7. A feasible design may violate "≥type' constraints. *False*
8. A "≤ type" constraint expressed in the standard form is active at a design point if it has zero value there. *True*
9. The constraint set for a design problem consists of all the feasible points. *True*
10. The number of independent equality constraints can be larger than the number of design variables for the problem. *True*
11. The number of "≤ type" constraints must be less than the number of design variables for a valid problem formulation. *False*
12. The feasible region for an equality constraint is a subset of that for the same constraint expressed as an inequality. *True*
13. Maximization of $f(x)$ is equivalent to minimization of $1/f(x)$. *False*
14. A lower minimum value for the cost function is obtained if more constraints are added to the problem formulation. *False*
15. Let f_n be the minimum value for the cost function with n design variables for a problem. If the number of design variables for the same problem is increased to, say $m = 2n$, then $f_m > f_n$ where f_m is the minimum value for the cost function with m design variables. *False*

8. A cantilever beam is subjected to the point load P (kN), as shown in Fig 3. The maximum bending moment in the beam is Pl (kN·m) and the maximum shear is P (kN). Formulate the minimum mass design problem using a hollow circular cross section. The material should not fail under bending stress or shear stress. The maximum bending stress is calculated as

$$\sigma = \frac{Pl}{I} R_o$$

where I = moment of inertia of the cross section. The maximum shearing stress is calculated as

$$\tau = \frac{P}{3I}(R_o^2 + R_o R_i + R_i^2)$$

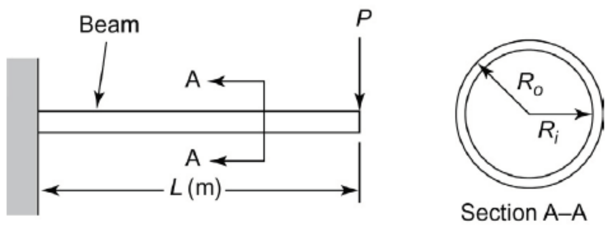


Fig.3 Cantilever beam.

Transcribe the problem into the standard design optimization model (also use $R_o \leq$ 40 cm, $R_i \leq$ 40 cm). Use the following data: P = 14 kN; L = 10 m; mass density, ρ = 7850 kg/m^3; allowable bending stress, σ_b= 165 MPa; allowable shear stress, τ_a = 50 MPa.

Solution:

Given: The equations to calculate maximum bending and shearing stress in the beam, the force applied to the beam, the length of the beam, the density of the beam, the maximum values of R_o and R_i, and the allowable bending and shear stress for the beam.
Required: It is desired to create a beam design, as shown in Figure E2.23, which will minimize the mass of the beam. The beam should not fail due to bending or shear at any point.
Procedure: We follow the five step process to formulate the problem as an optimization problem.

Step 1: Problem Statement
Shown above

Step 2: Data and Information Collection
Using kg, N and cm as units
Given Data: (this data will change if different units are used)
$P = 14 \text{ kN} = 1.4\times10^4$ N
$L = 10 \text{ m} = 1000$ cm
$\sigma_b = 165 \text{ MPa} = 1.65\times10^4$ N/cm^2;

$\tau_a = 50$ MPa = 5000 N/cm

$\rho = 7850$ kg/m^3 = 7.85×10^{-3} kg/cm^3;

Cross-sectional area of hollow tubes: $A = \pi(R_o^2 - R_i^2)$

Moment of inertia of a hollow tube is $I = \pi\left(R_o^4 - R_i^4\right)/4$

Maximum bending stress:

$$\sigma = \frac{PL}{I} R_o$$

Maximum shearing stress:

$$\tau = \frac{P}{3I}(R_o^2 + R_o R_i + R_i^2)$$

In addition, it must be ensured that $R_o > R_i$ which can be imposed as a constraint on the wall thickness as $t \geq t_{min}$ with t_{min} as, say 0.5 cm.

Thickness: $t = R_o - R_i$

Step 3: Definition of Design Variables

R_o = outer radius of hollow tube, cm

R_i = inner radius of hollow tube, cm

FORMULATION 1: Using Intermediate Variables

Step 4: Optimization Criterion

Optimization criterion is to minimize mass of hollow tube, and the cost function is defined as $f = \rho\pi AL$

Step 5: Formulation of Constraints

g_1 : bending stress should be smaller than the allowable bending stress; $\sigma \leq \sigma_a$

$$g_1 = \sigma - \sigma_a \leq 0$$

g_2 : shear stress smaller than allowable shear stress: $\tau \leq \tau_a$

$$g_2 = \tau - \tau_a \leq 0$$

$$g_3 = R_o - 40 \leq 0$$

$$g_4 = R_i - 40 \leq 0$$

$$g_5 = -R_o \leq 0$$

$$g_6 = -R_i \leq 0$$

$$g_7 = t_{min} - t \leq 0$$

FORMULATION 2: Using only Design Variables

Step 4: Optimization Criterion

Optimization criterion is to minimize mass of hollow tube, and the cost function is defined as

$f = \rho\pi\left(R_o^2 - R_i^2\right)L$ *or*

$f = \rho\pi L\left(R_o^2 - R_i^2\right) = \left(7.85\times10^3\right)\pi(1000)\left(R_o^2 - R_i^2\right) = 24.66\left(R_o^2 - R_i^2\right)$, kg

Step 5: Formulation of Constraints

g_1 : bending stress should be smaller than the allowable bending stress
g_2 : shear stress smaller than allowable shear stress

Using the standard form, we get

g_1 : $4PlR_o / \pi\left(R_o^4 - R_i^4\right) \le \sigma_b$; or $4\left(1.4\times10^4\right)\left(10^3\right)R_o / \pi\left(R_o^4 - R_i^4\right) - 1.65\times10^4 \le 0$; or

$g_1 = 1.7825\times10^7\, R_o / \left(R_o^4 - R_i^4\right) - 1.65\times10^4 \le 0$

g_2 : $4P\left(R_o^2 + R_oR_i + R_i^2\right) / 3\pi\left(R_o^4 - R_i^4\right) \le \tau_a$; or

$4\left(1.4\times10^4\right)\left(R_o^2 + R_oR_i + R_i^2\right) / 3\pi\left(R_o^4 - R_i^4\right) - 5000 \le 0$; or

$g_2 = 5941.78\left(R_o^2 + R_oR_i + R_i^2\right) / \left(R_o^4 - R_i^4\right) - 5000 \le 0$

$g_3 = R_o - 40 \le 0$;

$g_4 = R_i - 40 \le 0$;

$g_5 = -R_o \le 0$;

$g_6 = -R_i \le 0$

$g_7 = t_{min} - (R_o - R_i) \le 0$