Optimum Design - Sheet 4 - Solution
Optimality Conditions

1.

Write the Taylor series expansion for the following function up to quadratic terms.

cos$x$ about the point $x^* = \pi/4$

Solution

$f(x) = \cos x;\ f(\pi/4) = \cos(\pi/4) = 1/\sqrt{2};\ \ f'(\pi/4) = -\sin(\pi/4) = -1/\sqrt{2};$

$f''(\pi/4) = -\cos(\pi/4) = -1/\sqrt{2};\ \ \bar{f}(x) = f(x^*) + f'(x^*)(x - x^*) + 0.5 f''(x^*)(x - x^*)^2 \cos x$

$= (1/\sqrt{2}) - (1/\sqrt{2})(x - \pi/4) + 0.5(-1/\sqrt{2})(x - \pi/4)^2 = 1.0444 - 0.15175x - 0.35355x^2$

Write the Taylor series expansion for the following function up to quadratic terms.

cos$x$ about the point $x^* = \pi/3$

Solution

$f(x) = \cos x;\ f(\pi/3) = \cos(\pi/3) = 1/2;\ f'(\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2;$

$f''(\pi/3) = -\cos(\pi/3) = -1/2;\ \bar{f}(x) = f(x^*) + f'(x^*)(x - x^*) + 0.5 f''(x^*)(x - x^*)^2$

$\cos x = (1/\sqrt{2}) - (\sqrt{3}/2)(x - \pi/3) + 0.5(-1/2)(x - \pi/3)^2 = 1.1327 - 0.34243x - 0.25x^2$

Write the Taylor series expansion for the following function up to quadratic terms.

sin$x$ about the point $x^* = \pi/6$

Solution

$f(x) = \sin x;\ f(\pi/6) = \sin(\pi/6) = 1/2;\ f'(\pi/6) = \cos(\pi/6) = -\sqrt{3}/2;$

$f''(\pi/3) = -\cos(\pi/3) = -1/2;\ \ \bar{f}(x) = f(x^*) + f'(x^*)(x - x^*) + 0.5 f''(x^*)(x - x^*)^2$

$\sin x = 1/2 - (\sqrt{3}/2)(x - \pi/6) + 0.5(-1/2)(x - \pi/6)^2 = -0.02199 + 1.12783x - 0.25x^2$

Write the Taylor series expansion for the following function up to quadratic terms.

sin x about the point $x^* = \pi/4$

Solution

$f(x) = \sin x;\ f\ (\pi/4) = \sin\ (\pi/4) = 1/\sqrt{2};\ f'(\pi/4) = \cos\ (\pi/4) = -1/\sqrt{2};$

$f''(\pi/4) = -\sin\ (\pi/4) = -1/\sqrt{2};\ \bar{f}(x) = f(x^*) + f'(x^*)(x - x^*) + 0.5f''(x^*)(x - x^*)^2 \sin x$

$= 1/\sqrt{2} + (1/\sqrt{2})(x - \pi/4) + 0.5(-1/\sqrt{2})(x - \pi/4)^2 = (1 + x - \pi/4 - x^2/2 + \pi x/4 - \pi^2/32)/\sqrt{2}$

$= \left[(1 - \pi/4 - \pi^2/32) + (1 + \pi/4)x - x^2/2\right]/\sqrt{2} = 0.06634 + 1.2625x - 0.35355x^2$

Write the Taylor series expansion for the following function up to quadratic terms.

e^x about the point $x^*=0$

Solution

$f(x) = e^x; f(x) = f'(x) = f''(x) = e^x; f(0) = f'(0) = f''(0) = 1$

$\overline{f}(x) = f(x^*) = +f'(x^*) = (x - x^*) + 0.5f''(x)(x - x^*)^2$

$e^x = e^0 + e^0(x - 0) + 0.5e^0(x - 0)^2 = 1 + x + 0.5x^2$

Write the Taylor series expansion for the following function up to quadratic terms.

e^xabout the point $x^*=2$

Solution

$f(x) = e^x, x^* = 2;\ f'(x) = e^x;\ f''(x) = e^x;\ f(x^*) = f'(x^*) = f''(x^*) = e^2 = 7.389$

$\overline{f}(x) = f(x^*) + f'(x^*)(x - x^*) + 0.5f''(x^*)(x - x^*)^2$

$e^x = 7.389 + 7.389(x - 2) + 0.5(7.389)(x - 2)^2 = 7.389 - 7.389x + 3.6945x^2$

Write the Taylor series expansion for the following function up to quadratic terms.

$f(x_1, x_2) = 10x_1^4 - 20x_1^2x_2 + 10x_2^2 + x_1^2 - 2x_1 + 5$ about the point (1,1). Compare approximate and exact values of the function at the point (1.2,0.8).

Solution

$$f(x_1, x_2) = 10x_1^4 - 20x_1^2x_2 + 10x_2^2 + x_1^2 - 2x_1 + 5;\ x^* = (1, 1)$$

$$\nabla f(x_1, x_2) = \begin{bmatrix} 40x_1^3 - 40x_1x_2 + 2x_1 - 2 \\ -20x_1^2 + 20x_2 \end{bmatrix};\ \mathbf{H}(x_1, x_2) = \begin{bmatrix} 120x_1^2 - 40x_2 + 2 & -40x_1 \\ -40x_1 & 20 \end{bmatrix}$$

$$\overline{f}(x_1, x_2) = f(x^*) + \nabla f^T(x^*)(x - x^*) + 0.5(x - x^*)^T \mathbf{H}(x^*)(x - x^*)$$

$$f(x^*) = 4;\ \nabla f(x^*) = \begin{bmatrix} 0 \\ 0 \end{bmatrix};\ \mathbf{H}(x^*) = \begin{bmatrix} 82 & -40 \\ -40 & 20 \end{bmatrix}$$

$$\overline{f}(x_1, x_2) = 4 + \tfrac{1}{2}\begin{bmatrix} (x_1 - 1) \\ (x_2 - 1) \end{bmatrix}^T \begin{bmatrix} 82 & -40 \\ -40 & 20 \end{bmatrix} \begin{bmatrix} (x_1 - 1) \\ (x_2 - 1) \end{bmatrix} = 41x_1^2 - 42x_1 - 40x_1x_2 + 20x_2 + 10x_2^2 + 15$$

$$f(1.2, 0.8) = 8.136;\ \ \overline{f}(1.2, 0.8) = 7.64;\ \text{Error} = f - \overline{f} = 0.496$$

2.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = x_1^2 + 4x_1x_2 + 2x_1x_3 - 7x_2^2 - 6x_2x_3 + 5x_3^2$

Solution

$$F(\mathbf{x}) = x_1^2 + 4x_1x_2 + 2x_1x_3 - 7x_2^2 - 6x_2x_3 + 5x_3^2$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & -7 & -3 \\ 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -7 & -3 \\ 1 & -3 & 5 \end{bmatrix}$$

Principal Minors: $M_1 = 1 > 0$; $M_2 = (-7) - (2)(2) = -11 < 0$; $M_3 = -69 < 0$. Since $M_1 > 0$, $M_2 < 0$ and $M_3 < 0$, **A** is **indefinite**, so is the quadratic form.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = 2x_1^2 + 2x_2^2 - 5x_1x_2$

Solution

$$F(\mathbf{x}) = 2x_1^2 + 2x_2^2 - 5x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -2.5 \\ -2.5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\mathbf{A} = \begin{bmatrix} 2 & -2.5 \\ -2.5 & 2 \end{bmatrix}$; Principal Minors: $M_1 = 2 > 0$; $M_2 = -2.25 < 0$

Since $M_1 > 0$ and $M_2 < 0$, **A** is indefinite, so the quadratic form is **indefinite**.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = x_1^2 + x_2^2 + 3x_1x_2$

Solution

$F(\mathbf{x}) = x_1^2 + x_2^2 + 3x_1x_2$; (note that the factor of 0.5 does not affect the form of the matrix)

$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1.5 \\ 1.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; $\mathbf{A} = \begin{bmatrix} 1 & 1.5 \\ 1.5 & 1 \end{bmatrix}$; Eigenvalue problem:

$\begin{vmatrix} 1-\lambda & 1.5 \\ 1.5 & 1-\lambda \end{vmatrix} = 0$; $(1-\lambda)(1-\lambda) - 1.5^2 = 0$; $\lambda^2 - 2\lambda - 1.25 = 0$; $\lambda_1 = -0.5$, $\lambda_2 = 2.5$

The matrix and the quadratic form are **indefinite** since one eigenvalue is positive and the other negative.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = x_1^2 - x_2^2 + 4x_1x_2$

Solution

$$F(\mathbf{x}) = x_1^2 - x_2^2 + 4x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$; Principal Minors: $M_1 = 1 > 0$; $M_2 = -5 < 0$

Since $M_1 > 0$ and $M_2 < 0$, the matrix is **indefinite** and so is the quadratic form.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = x_1^2 - x_2^2 + x_3^2 - 2x_2x_3$

Solution

$F(\mathbf{x}) = x_1^2 - x_2^2 + x_3^2 - 2x_2x_3$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Principal Minors: $M_1 = 1 > 0$, $M_2 = -1 < 0$, $M_3 = |\mathbf{A}| = 1(-1-1) = -2 < 0$

Since $M_1 > 0$, $M_2 < 0$ and $M_3 < 0$, so the quadratic form is **indefinite**.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = x_1^2 - 2x_1x_2 + 2x_2^2$

Solution

$$F(\mathbf{x}) = x_1^2 - 2x_1x_2 + 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$; Principal Minors: $M_1 = 1 > 0$, $M_2 = 1 > 0$

Since $M_1 > 0$ and $M_2 > 0$, the quadratic form is **positive definite**.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = x_1^2 - x_1x_2 - x_2^2$

Solution

$$F(\mathbf{x}) = x_1^2 - x_1x_2 - x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -0.5 \\ -0.5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -0.5 \\ -0.5 & -1 \end{bmatrix}; \text{ Principal Minors: } M_1 = 1 > 0, M_2 = -1.25 < 0$$

Since $M_1 > 0$ and $M_2 < 0$, the quadratic form is **indefinite**.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = x_1^2 + 2x_1x_3 - 2x_2^2 + 4x_3^2 - 2x_2x_3$

Solution

$$F(\mathbf{x}) = x_1^2 + 2x_1x_3 - 2x_2^2 + 4x_3^2 - 2x_2x_3$$

$$= [\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}] \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 4 \end{bmatrix}; \text{Principal Minors: } M_1 = 1 > 0, M_2 = -2 < 0, M_3 = -7 < 0$$

Since $M_1 > 0$, $M_2 < 0$ and $M_3 < 0$, the quadratic form is **indefinite**.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = 2x_1^2 + x_1x_2 + 2x_2^2 + 4x_3^2 - 2x_1x_3$

Solution

$$F(\mathbf{x}) = 2x_1^2 + x_1x_2 + 2x_2^2 + 4x_3^2 - 2x_1x_3$$

$$= [\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}] \begin{bmatrix} 2 & 0.5 & -1 \\ 0.5 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0.5 & -1 \\ 0.5 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}; \text{Principal Minors: } M_1 = 2 > 0, M_2 = 3.75 > 0, M_3 = 9.25 > 0$$

Since $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$, the quadratic form is **positive definite**.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = x_1^2 + 2x_2x_3 + x_2^2 + 4x_3^2$

Solution

$$F(\mathbf{x}) = x_1^2 + 2x_2x_3 + x_2^2 + 4x_3^2 = [[x_1 \quad x_2 \quad x_3]]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix}$; Principal Minors: $M_1 = 1 > 0, M_2 = 1 > 0, M_3 = 3 > 0$

Since $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$, the quadratic form is **positive definite**.

Determine the nature of the following quadratic form.
$F(\boldsymbol{x}) = 4x_1^2 + 2x_1x_3 - x_2^2 + 4x_3^2$

Solution

$$F(\mathbf{x}) = 4x_1^2 + 2x_1x_3 - x_2^2 + 4x_3^2 = [[x_1 \quad x_2 \quad x_3]]\begin{bmatrix} 4 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$; Principal Minors: $M_1 = 4 > 0, M_2 = -4 < 0, M_3 = -15 < 0$

Since $M_1 > 0$, $M_2 < 0$ and $M_3 < 0$, the quadratic form is **indefinite**.

3. Write optimality conditions and find stationary points for the following functions (use MATLAB, or Mathematica, if needed to solve the optimality conditions). Also determine the local minimum, local maximum, and inflection points for the functions (inflection points are those stationary points that are neither minimum nor maximum).

$f(x_1, x_2) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + 7$

Solution

$f(x_1, x_2) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + 7$

The gradient and Hessian of $f(\mathbf{x})$ are

$$\tilde{N}f = \begin{bmatrix} 6x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}.$$

Setting gradient to zero gives $\mathbf{x} = (0, 0)$ as the only candidate minimum point.
Principal Minors of the Hessian: $M_1 = 6 > 0$, $M_2 = 20 > 0$. Since $M_1 > 0$ and $M_2 > 0$, the Hessian is positive definite. Therefore, the point (0, 0) is a local minimum point $(f = 7)$.

$f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + 3$

Solution

$f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + 3;$

The gradient and Hessian of $f(\mathbf{x})$ are

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} 2x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

Setting gradient to zero gives $\mathbf{x} = (0, 0)$ as the only candidate minimum point.
Eigenvalue test: $|\mathbf{H} - \lambda\mathbf{I}| = (2-\lambda)(2-\lambda) - 16 = 0;\ \lambda_1 = 6,\ \lambda_2 = -2$
Therefore, the Hessian is indefinite and second order necessary condition is violated. The stationary point (0, 0) is an inflection point.

$f(x_1, x_2) = x_1^3 + 12x_1x_2^2 + 2x_2^2 + 5x_1^2 + 3x_2$

Solution

$f(x_1, x_2) = x_1^3 + 12x_1x_2^2 + 2x_2^2 + 5x_1^2 + 3x_2;$

The gradient and Hessian of $f(\mathbf{x})$ are

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} 3x_1^2 + 12x_2^2 + 10x_1 \\ 24x_1x_2 + 4x_2 + 3 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 6x_1 + 10 & 24x_2 \\ 24x_2 & 24x_1 + 4 \end{bmatrix}$$

Setting the gradient to zero gives a nonlinear system of equations. Using Newton-Raphson method or any nonlinear equation solver, we find two solutions, as

$\mathbf{x}^{*1} = (-3.332,\ 0.0395); \quad \mathbf{x}^{*2} = (-0.398,\ 0.5404)$

$$\mathbf{H}(\mathbf{x}^{*1}) = \begin{bmatrix} -9.992 & 0.948 \\ 0.948 & -75.968 \end{bmatrix}; \ \mathrm{M}_1 = -9.992 < 0, \ \mathrm{M}_2 = 758.17 > 0$$

$\mathbf{H}(\mathbf{x}^{*1})$ is negative definite. Therefore $\mathbf{x}^{*1} = (-3.332,\ 0.0395)$ is a local maximum point.

$$\mathbf{H}(\mathbf{x}^{*2}) = \begin{bmatrix} 7.612 & 12.970 \\ 12.970 & -5.552 \end{bmatrix}; \ \mathrm{M}_1 = 7.612, \ \mathrm{M}_2 = -210.483$$

$\mathbf{H}(\mathbf{x}^{*2})$ is indefinite. Therefore $\mathbf{x}^{*2} = (-0.398,\ 0.5404)$ is an inflection point.

$f(x_1, x_2) = 5x_1 - x_1^2 x_2/16 + x_2^2/4x_1$

Solution

$f(x_1, x_2) = 5x_1 - x_1^2 x_2/16 + x_2^2/4x_1$

The gradient and Hessian of $f(\mathbf{x})$ are

$$\nabla f = \begin{bmatrix} 5 - x_1 x_2/8 - x_2^2/4x_1^2 \\ -x_1^2/16 + x_2/2x_1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} -x_2/8 + x_2^2/2x_1^3 & -x_1/8 - x_2/2x_1^2 \\ -x_1/8 - x_2/2x_1^2 & 1/2x_1 \end{bmatrix}$$

When $\tilde{N}f$ is set to zero the second equation gives $x_2 = x_1^3/8$.

Substituting into the first equation, we get

$5 - x_1^4/64 - x_1^4/256 = 0,\ (5/256)x_1^4 = 5;\ x_1 = \pm 4$.

For $x_1 = 4, x_2 = 8$, and $x_1 = -4, x_2 = -8$.

For the first point $(4, 8)$

$$\mathbf{H} = \begin{bmatrix} -8/8 + 64/2(64) & -4/8 - 8/2(16) \\ -4/8 - 8/2(16) & 1/2(4) \end{bmatrix} = \begin{bmatrix} -1/2 & -3/4 \\ -3/4 & -1/8 \end{bmatrix};$$

$M_1 = -1/2 < 0,\ M_2 = -5/8 < 0$

Since **H** is indefinite, the second order necessary condition is violated. Thus, point $(4, 8)$ is an **inflection** point.

For the second point $(-4, -8)$,

$$\mathbf{H} = \begin{bmatrix} -(-8)/8 + 64/2(-64) & -(-4)/8 - (-8)/2(16) \\ -(-4)/8 - (-8)/2(16) & 1/2(-4) \end{bmatrix} = \begin{bmatrix} 1/2 & 3/4 \\ 3/4 & -1/8 \end{bmatrix};.$$

$M_1 = 1/2 > 0, M_2 = -5/8 < 0$

Since **H** is indefinite, the second order necessary condition is violated. Thus, point $(-4, -8)$ is an **inflection** point.

$f(x) = \cos x$

Solution

$f(x) = \cos x$

The necessary condition gives $f'(x) = -\sin x = 0$

The solution of necessary condition gives: $x = n\pi,\ n = 0,\ \pm 1,\ \pm 2, \ldots$

$f''(x) = -\cos x$; For $x = (2n+1)\pi, n = 0, \pm 1, \pm 2, \ldots,$

$f''(x) = -\cos[(2n+1)\pi] = 1 > 0.$

Thus, $x = (2n+1)\pi, n = 0, \pm 1, \pm 2, \ldots,$ are local minimum points $(f = -1)$.

For $x = 2n\pi, n = 0,\ \pm 1,\ \pm 2, \ldots f''(x) = -\cos\ (2n\pi) = -1 < 0.$

Thus, $x = 2n\pi, n = 0,\ \pm 1,\ \pm 2, \ldots$. are local maximum points $(f = 1)$.

$f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$

Solution

$f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$

The gradient and Hessian of $f(\mathbf{x})$ are

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Solution of necessary conditions of $\tilde{\mathbf{N}}f = \mathbf{0}$ gives $\mathbf{x}^* = (0, 0)$. The Hessian at $\mathbf{x}^*$ is positive definite since $M_1 = 2 > 0$ and $M_2 = 3 > 0$. Thus $(0, 0)$ is a local minimum point $(f = 0)$.

$f(x) = x^2e^{-x}$

Solution

$f(x) = x^2e^{-x}$

The necessary conditions gives $f'(x) = 2xe^{-x} - x^2e^{-x} = 0$; or $2x - x^2 = 0$.

Therefore, $x = 0, 2$ are the stationary points.

$f''(x) = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x} = (x^2 - 4x + 2)e^{-x}$

$f''(0) = 2 > 0$. Therefore, $x = 0$ is a local minimum point. $f^* = 0$.

$f''(2) = -0.27067 < 0$. Therefore, $x = 2$ is a local maximum point. $f^* = 0.541$.

$f(x_1, x_2) = x_1 + 10/(x_1 x_2) + 5x_2$

Solution

$f(x_1, x_2) = x_1 + 10/(x_1 x_2) + 5x_2$

The necessary condition gives:

$\partial f/\partial x_1 = 1 - 10/(x_1^2 x_2) = 0;\ \ \partial f/\partial x_2 = -10/(x_1 x_2^2) + 5 = 0;$ or $x_1^2 x_2 = 10,\ \ 5x_1 x_2^2 = 10$

These equations give $x_1 = 5x_2$. Substituting the equation, we obtain $x_2 = 0.7368$. Therefore, $\mathbf{x}^*$ $= (3.684,\ 0.7368)$ is a stationary point. Hessian is given as

$$\mathbf{H}(\mathbf{x}^*) = \begin{bmatrix} 20/(x_1^3 x_2) & 10/(x_1^2 x_2^2) \\ 10/(x_1^2 x_2^2) & 20/(x_1 x_2^3) \end{bmatrix};\quad \mathbf{H} = \begin{bmatrix} 0.5429 & 1.3572 \\ 1.3572 & 13.572 \end{bmatrix}\quad \begin{matrix} M_1 = 0.5429 > 0 \\ M_2 = 5.526 > 0 \end{matrix}$$

Since Hessian is positive definite, $\mathbf{x}^* = (3.684,\ 0.7368)$ is a local minimum point $(f = 11.0521)$.

$f(x_1, x_2) = x_1^2 - 2x_1 + 4x_2^2 - 8x_2 + 6$

Solution

$f(x_1, x_2) = x_1^2 - 2x_1 + 4x_2^2 - 8x_2 + 6$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 2x_1 - 2 \\ 8x_2 - 8 \end{bmatrix};\quad \mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = (1,1)$. For the Hessian **H**, $M_1 = 2 > 0$, $M_2 = 16 > 0$; so it is positive definite, and $(1,1)$ is a local minimum point $(f = 1)$.

$f(x_1, x_2) = 3x_1^2 - 2x_1 x_2 + 5x_2^2 + 8x_2$

Solution

$f(x_1, x_2) = 3x_1^2 - 2x_1 x_2 + 5x_2^2 + 8x_2$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 6x_1 - 2x_2 \\ -2x_1 + 10x_2 + 8 \end{bmatrix};\quad \mathbf{H} = \begin{bmatrix} 6 & -2 \\ -2 & 10 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = (-2/7,\ -6/7)$. For the Hessian, $M_1 = 6 > 0$, $M_2 = 56 > 0$, so it is positive definite, and the point $(-2/7, -6/7)$ is a local minimum point $(f = -\frac{24}{7})$.

$f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_1 - 2x_1x_2$

Solution

$f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_1 - 2x_1x_2$

The gradient and Hessian are given as

$$\nabla f = \begin{bmatrix} 2x_1 - 4 - 2x_2 \\ 4x_2 - 2x_1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}.$$

Solution of $\nabla f = 0$ gives $\mathbf{x}^* = (8, 4)$. For the Hessian $\mathbf{H}$, $M_1 = 2 > 0$, $M_2 = 4 > 0$; so it is positive definite, and $(8, 4)$ is a local minimum point $(f = 0)$.

$f(x_1, x_2) = 12x_1^2 + 22x_2^2 - 1.5x_1 - x_2$

Solution

$f(x_1, x_2) = 12x_1^2 + 22x_2^2 - 1.5x_1 - x_2$

The gradient and Hessian are given as

$$\nabla f = \begin{bmatrix} 2x_1 - 1.5 \\ 44x_2 - 1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 44 \end{bmatrix}.$$

Solution of $\nabla f = 0$ gives $\mathbf{x}^* = \left(0.75, \dfrac{1}{44}\right)$. For the Hessian $\mathbf{H}$, $M_1 = 2 > 0$, $M_2 = 88 > 0$; so it is positive definite, and $\left(0.75, \dfrac{1}{44}\right)$ is a local minimum point $(f = 5.6136)$.

$f(x_1, x_2) = 7x_1^2 + 12x_2^2 - x_1$

Solution

$f(x_1, x_2) = 7x_1^2 + 12x_2^2 - x_1$

The gradient and Hessian are given as

$$\nabla f = \begin{bmatrix} 14x_1 - 1 \\ 24x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 14 & 0 \\ 0 & 24 \end{bmatrix}.$$

Solution of $\nabla f = 0$ gives $\mathbf{x}^* = \left(\dfrac{1}{14}, 0\right)$. For the Hessian $\mathbf{H}$, $M_1 = 14 > 0$, $M_2 = 336 > 0$; so it is positive definite, and $\left(\dfrac{1}{14}, 0\right)$ is a local minimum point $(f = -0.035714)$.

$f(x_1, x_2) = 12x_1^2 + 21x_2^2 - x_2$

Solution

$f\left(x_1, x_2\right) = 12x_1^2 + 21x_2^2 - x_2$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 12x_1 \\ 42x_2 - 1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 12 & 0 \\ 0 & 42 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = \left(0, \dfrac{1}{42}\right)$. For the Hessian **H**, $M_1 = 12 > 0$, $M_2 = 504 > 0$; so it is positive definite, and $\left(0, \dfrac{1}{42}\right)$ is a local minimum point $\left(f = -0.0119\right)$.

$f(x_1, x_2) = 25x_1^2 + 20x_2^2 - 2x_1 - x_2$

Solution

$f\left(x_1, x_2\right) = 25x_1^2 + 20x_2^2 - 2x_1 - x_2$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 50x_1 - 2 \\ 40x_2 - 1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 50 & 0 \\ 0 & 40 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = \left(\dfrac{1}{25}, \dfrac{1}{40}\right)$. For the Hessian **H**, $M_1 = 50 > 0$, $M_2 = 2000 > 0$; so it is positive definite, and $\left(\dfrac{1}{25}, \dfrac{1}{40}\right)$ is a local minimum point $\left(f = -0.0525\right)$.

$f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$

Solution

$f\left(x_1, x_2, x_3\right) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 2x_1 + 2x_2 \\ 4x_2 + 2x_1 + 2x_3 \\ 4x_3 + 2x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = (0, 0, 0)$. For the Hessian **H**, $M_1 = 2 > 0$, $M_2 = 24 > 0$; so it is positive definite, and $(0, 0, 0)$ is a local minimum point $\left(f = 0\right)$.

$$f(x_1, x_2) = 8x_1^2 + 8x_2^2 - 80\sqrt{x_1^2 + x_2^2 - 20x_2 + 100} - 80\sqrt{x_1^2 + x_2^2 + 20x_2 + 100} - 5x_1 - 5x_2$$

Solution

$f(x_1, x_2) = 8x_1^2 + 8x_2^2 - 80\sqrt{x_1^2 + x_2^2 - 20x_2 + 100} - 80\sqrt{x_1^2 + x_2^2 + 20x_2 + 100} - 5x_1 - 5x_2$;
The gradient is given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 16x_1 - 80x_1\left(x_1^2 + x_2^2 - 20x_2 + 10\right)^{-\frac{1}{2}} - 80x_1\left(x_1^2 + x_2^2 + 20x_2 + 10\right)^{-\frac{1}{2}} - 5 \\ 16x_2 - 40(2x_2 - 20)\left(x_1^2 + x_2^2 - 20x_2 + 10\right)^{-\frac{1}{2}} - 40(2x_2 + 20)\left(x_1^2 + x_2^2 + 20x_2 + 10\right)^{-\frac{1}{2}} - 5 \end{bmatrix};$$

Solution of $\tilde{\mathbf{N}}f = 0$ and the hessian would be solved numerically using a program such as Mathematica or MATLAB.

$$f(x_1, x_2) = 9x_1^2 + 9x_2^2 - 100\sqrt{x_1^2 + x_2^2 - 20x_2 + 100} - 64\sqrt{x_1^2 + x_2^2 + 16x_2 + 64} - 5x_1 - 41x_2$$

Solution

$f(x_1, x_2) = 9x_1^2 + 9x_2^2 - 100\sqrt{x_1^2 + x_2^2 - 20x_2 + 100} - 64\sqrt{x_1^2 + x_2^2 + 20x_2 + 100} - 5x_1 - 41x_2$; The gradient is given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 18x_1 - 100x_1\left(x_1^2 + x_2^2 - 20x_2 + 10\right)^{-\frac{1}{2}} - 64x_1\left(x_1^2 + x_2^2 + 20x_2 + 10\right)^{-\frac{1}{2}} - 5 \\ 18x_2 - 50(2x_2 - 20)\left(x_1^2 + x_2^2 - 20x_2 + 10\right)^{-\frac{1}{2}} - 32(2x_2 + 20)\left(x_1^2 + x_2^2 + 20x_2 + 10\right)^{-\frac{1}{2}} - 41 \end{bmatrix};$$

Solution of $\tilde{\mathbf{N}}f = 0$ and the hessian would be solved numerically using a program such as Mathematica or MATLAB.

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

Solution

$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1^2)$; The gradient is given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} -400x_1(x_2 - x_1^2) - 2 + 2x_1 \\ 200(x_2 - x_1^2) \end{bmatrix};$$

Solution of $\tilde{\mathbf{N}}f = 0$ and the hessian would be solved numerically using a program such as Mathematica or MATLAB.