

## Optimum Design - Sheet 5 Optimality Conditions - Constrained

Find points satisfying KKT necessary conditions for the following problems; check if they are optimum points using the graphical method for two variable problems.

1.

$$\begin{aligned} \text{Maximize } F(x_1, x_2) &= 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8 \\ \text{subject to } x_1 + x_2 &\leq 4 \end{aligned}$$

### Solution

Minimize  $f = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8$  subject to  $g = x_1 + x_2 - 4 \leq 0$

$L = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8 + u(x_1 + x_2 - 4 + s^2)$ ; the KKT necessary conditions are

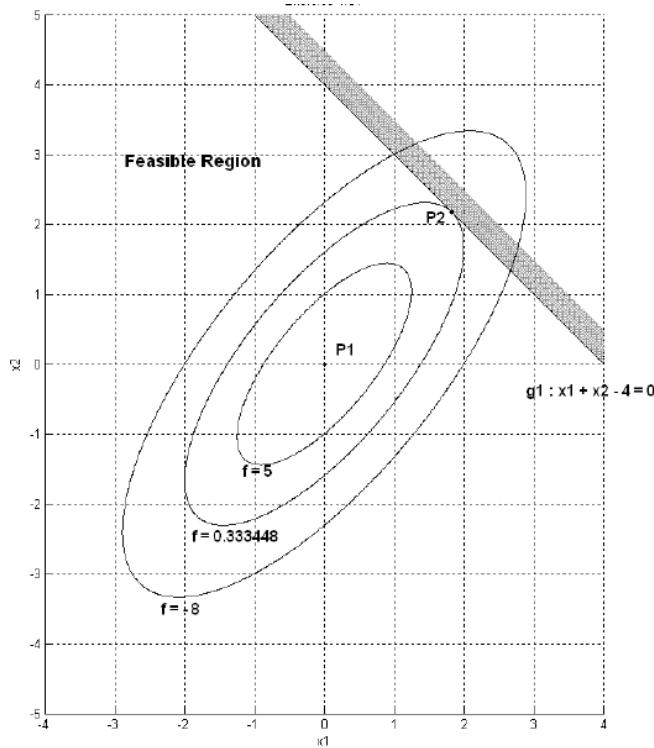
$$\partial L / \partial x_1 = -8x_1 + 5x_2 + u = 0; \quad \partial L / \partial x_2 = -6x_2 + 5x_1 + u = 0;$$

$$\partial L / \partial u = x_1 + x_2 - 4 + s^2 = 0; \quad \partial L / \partial s = 2us = 0$$

**Case 1.**  $u = 0$ ; gives a KKT point as  $(0, 0)$ ;  $F^* = -8$ .

**Case 2.**  $s = 0$  (or  $g = 0$ ); gives a KKT point as  $(11/6, 13/6)$ ;  $u^* = 23/6$ ,  $F^* = -1/3$ .

### Optimum Point (1.83, 2.1667)



## **MATLAB Code**

```
clear all
axis equal
[x1,x2]=meshgrid(-4:0.01:4, -5:0.01:5);
f=(-1)*(4*x1.^2+3*x2.^2-5*x1.*x2-8);
g1=x1+x2-4;
cla reset
axis equal
axis ([-4 4 -5 5])
xlabel('x1'),ylabel('x2')
title('Exercise 4.54')
hold on
cv1=[0:0.03:0.5];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');
fv=[-8 0.333448 5];
fs=contour(x1,x2,f,fv,'b');
a=[0 1.83333];
b=[0 2.16667];
plot(a,b,'.k');
grid
hold off
```

2.

$$\text{Minimize } f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8 \\ \text{subject to } x_1 + x_2 \leq 4$$

### Solution

$$\text{Minimize } f = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8 \text{ subject to } g = x_1 + x_2 - 4 \leq 0$$

$L = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8 + u(x_1 + x_2 - 4 + s^2)$ ; the KKT necessary conditions are

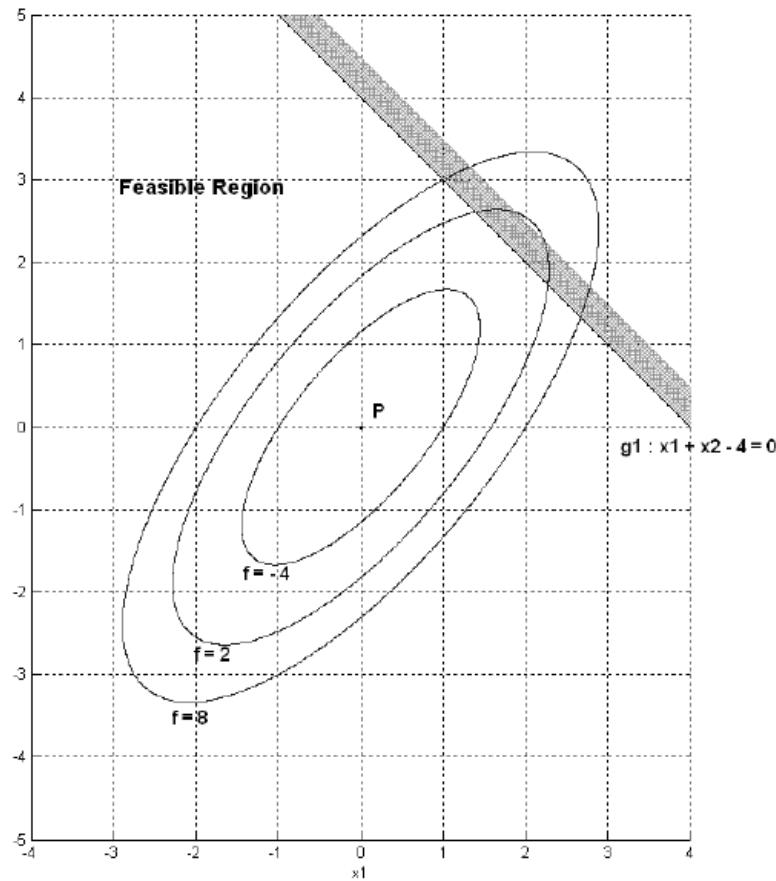
$$\partial L / \partial x_1 = 8x_1 - 5x_2 + u = 0; \quad \partial L / \partial x_2 = 6x_2 - 5x_1 + u = 0;$$

$$\partial L / \partial u = x_1 + x_2 - 4 + s^2 = 0; \quad \partial L / \partial s = 2us = 0$$

**Case 1.**  $u = 0$ ; gives a KKT point as  $(0, 0)$ ;  $f(\mathbf{x}^*) = -8$ .

**Case 2.**  $s = 0$ ; gives no candidate point. ( $u < 0$ )

Optimum point P (0, 0)



## **MATLAB Code**

```
clear all
axis equal
[x1,x2]=meshgrid(-4:0.01:4, -5:0.01:5);
f=(4*x1.^2+3*x2.^2-5*x1.*x2-8);
g1=x1+x2-4;
cla reset
axis equal
axis ([-4 4 -5 5])
xlabel('x1'),ylabel('x2')
title('Exercise 4.55')
hold on
cv1=[0:0.03:0.5];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');
fv=[-8 -4 2 8];
fs=contour(x1,x2,f,fv,'b');
a=[0];
b=[0];
plot(a,b,'.k');
grid
hold off
```

3.

$$\text{Maximize } F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 \\ \text{subject to } x_1 + x_2 \leq 4$$

### Solution

Minimize  $f = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8x_1$  subject to  $g = x_1 + x_2 - 4 \leq 0$

$L = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 + u(x_1 + x_2 - 4 + s^2)$ ; the KKT necessary conditions are

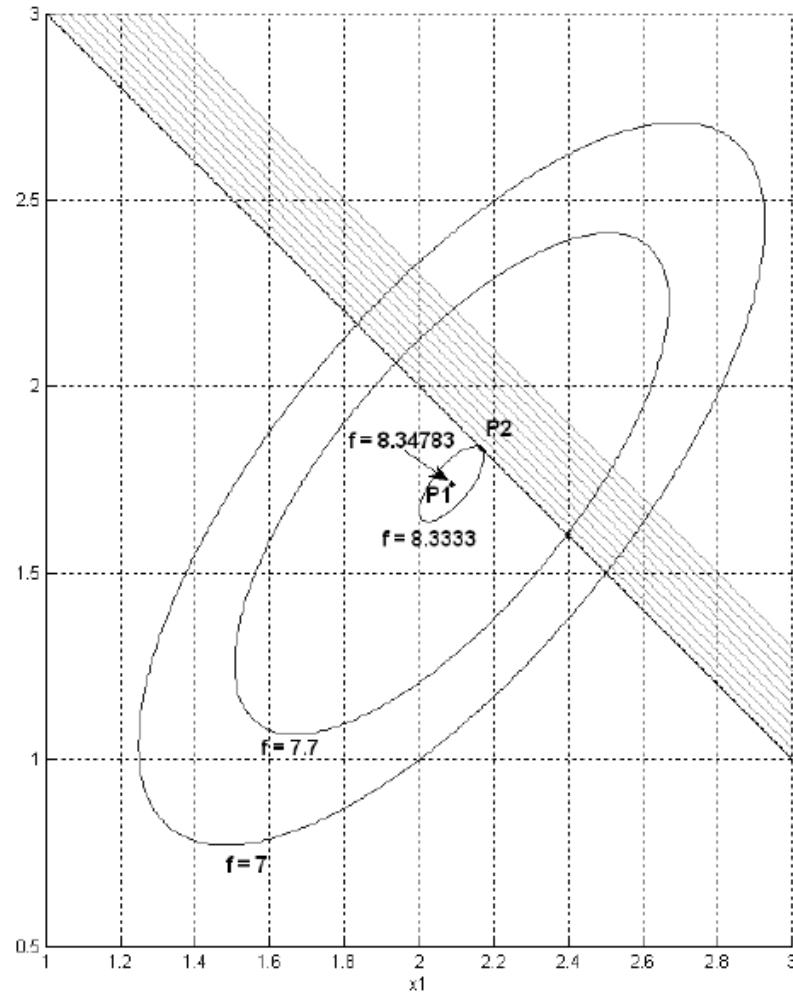
$$\partial L / \partial x_1 = -8x_1 + 5x_2 + 8 + u = 0; \quad \partial L / \partial x_2 = -6x_2 + 5x_1 + u = 0;$$

$$\partial L / \partial u = x_1 + x_2 - 4 + s^2 = 0; \quad \partial L / \partial s = 2us = 0$$

**Case 1.**  $u = 0$ ; gives a KKT point as  $(48/23, 40/23)$ ;  $F(\mathbf{x}^*) = -192/23 = 8.348$ .

**Case 2.**  $s = 0$ ; gives a KKT point as  $(13/6, 11/6)$ ;  $u^* = 1/6$ ,  $F(\mathbf{x}^*) = -8.33333$ .

**Optimum point P (2.086, 1.73)**



## **MATLAB Code**

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;
h1=x1-x2-2;
g1=-x1-x2+4;
cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.56')
hold on
cv2=[0 0.01];
const2=contour(x1,x2,h1,cv2,'k');
cv2=[0:0.03:0.3];
const2=contour(x1,x2,g1,cv2,'g');
cv2=[0 0.001];
const2=contour(x1,x2,g1,cv2,'k');
fv=[2 4 6];
fs=contour(x1,x2,f,fv,'b');
a=[3];
b=[1];
plot(a,b,'.k');
grid
hold off
```

4. Minimize  $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$   
 subject to  $x_1 + x_2 \geq 4$   
 $x_1 - x_2 - 2 = 0$

### Solution

Minimize  $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$ ; subject to  $h = x_1 - x_2 - 2 = 0$ ;  $g = -x_1 - x_2 + 4 \leq 0$ .

$$L = (x_1 - 1)^2 + (x_2 - 1)^2 + v(x_1 - x_2 - 2) + u(-x_1 - x_2 + 4 + s^2)$$

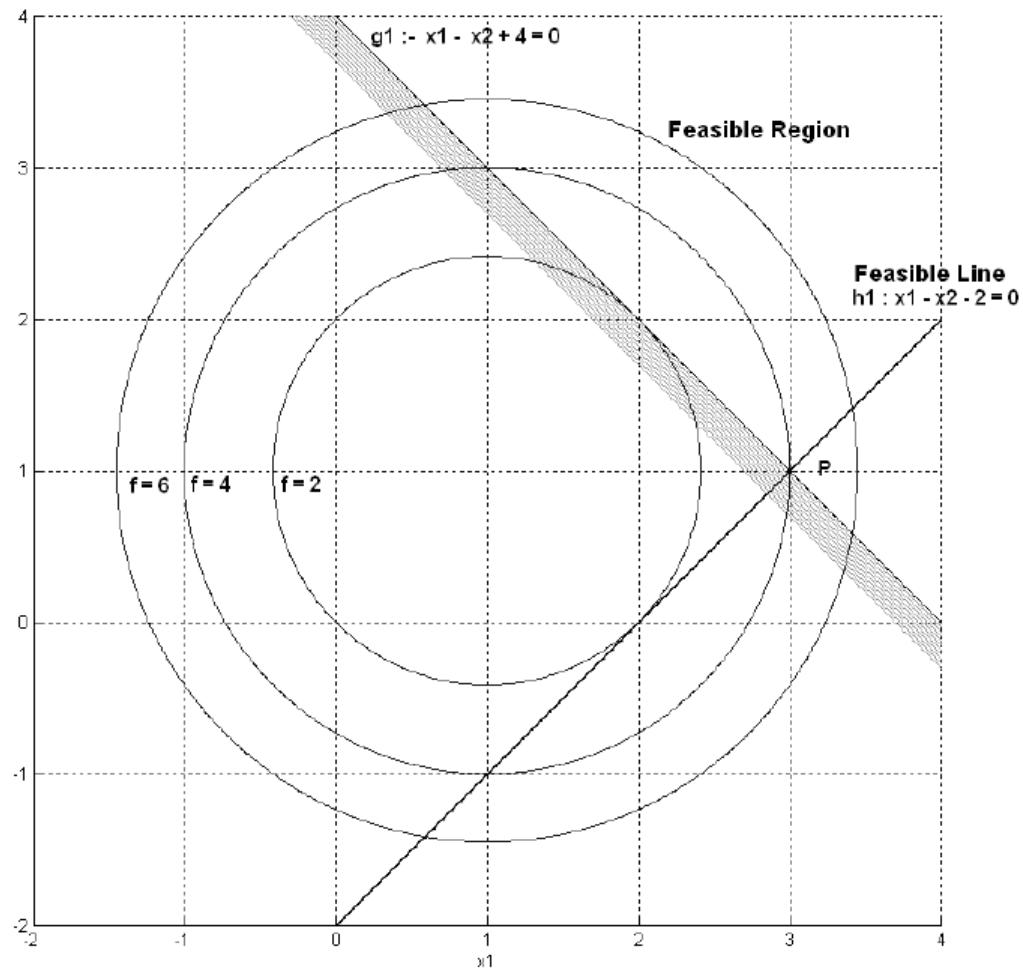
$$\partial L / \partial x_1 = 2(x_1 - 1) + v - u = 0; \quad \partial L / \partial x_2 = 2(x_2 - 1) - v - u = 0$$

$$h = x_1 - x_2 - 2 = 0; \quad -x_1 - x_2 + 4 + s^2 = 0; \quad us = 0, \quad u \geq 0.$$

**Case 1.**  $u = 0$ ; no candidate minimum.

**Case 2.**  $s = 0$ ; gives  $(3, 1)$  as a KKT point with  $v = -2, u = 2, f = 4$ .

**Optimum point P (3, 1)**



### **MATLAB Code**

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;
h1=x1-x2-2;
g1=-x1-x2+4;
cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.57')
hold on
cv2=[0 0.01];
const2=contour(x1,x2,h1,cv2,'k');
cv2=[0:0.03:0.3];
const2=contour(x1,x2,g1,cv2,'g');
cv2=[0 0.001];
const2=contour(x1,x2,g1,cv2,'k');
fv=[2 4 6];
fs=contour(x1,x2,f,fv,'b');
a=[3];
b=[1];
plot(a,b,'.k');
grid
hold off
```

- 5.
- Minimize  $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$   
 subject to  $x_1 + x_2 = 4$   
 $x_1 - x_2 - 2 \geq 0$

## Solution

Minimize  $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$ ; subject to  $h = x_1 - x_2 - 2 = 0$ ;  $g = -x_1 - x_2 + 4 \leq 0$ .

$$L = (x_1 - 1)^2 + (x_2 - 1)^2 + v(x_1 - x_2 - 2) + u(-x_1 - x_2 + 4 + s^2)$$

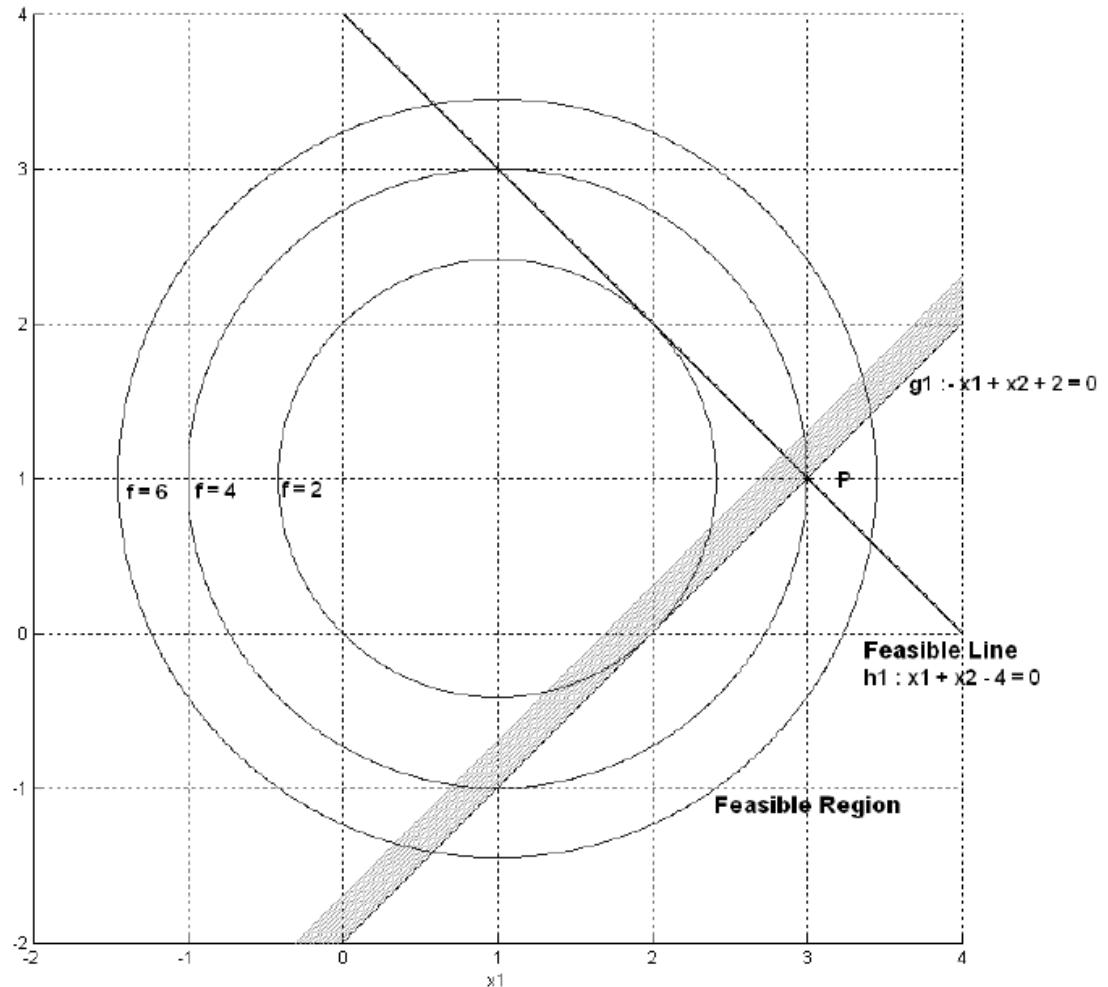
$$\partial L / \partial x_1 = 2(x_1 - 1) + v - u = 0; \quad \partial L / \partial x_2 = 2(x_2 - 1) - v - u = 0$$

$$h = x_1 - x_2 - 2 = 0; \quad -x_1 - x_2 + 4 + s^2 = 0; \quad us = 0, \quad u \geq 0.$$

**Case 1.**  $u = 0$ ; no candidate minimum.

**Case 2.**  $s = 0$ ; gives  $(3, 1)$  as a KKT point with  $v = -2$ ,  $u = 2$ ,  $f = 4$ .

### optimum point P (3, 1)



## **MATLAB Code**

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;
h1=x1+x2-4;
g1=-x1+x2+2;
cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'), ylabel('x2')
title('Exercise 4.58')
hold on
cv2=[0 0.01];
const2=contour(x1,x2,h1,cv2,'k');
cv2=[0:0.03:0.3];
const2=contour(x1,x2,g1,cv2,'g');
cv2=[0 0.001];
const2=contour(x1,x2,g1,cv2,'k');
fv=[2 4 6];
fs=contour(x1,x2,f,fv,'b');
a=[3];
b=[1];
plot(a,b,'.k');
grid
hold off
```

6.

$$\text{Minimize } f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$$

$$\text{subject to } x_1 + x_2 \geq 4$$

$$x_1 - x_2 \geq 2$$

## Solution

Minimize  $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$ ; subject to  $g_1 = -x_1 - x_2 + 4 \leq 0$ ;  $g_2 = -x_1 + x_2 + 2 \leq 0$ .

$$L = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(-x_1 - x_2 + 4 + s_1^2) + u_2(-x_1 + x_2 + 2 + s_2^2)$$

$$\partial L / \partial x_1 = 2(x_1 - 1) - u_1 - u_2 = 0; \quad \partial L / \partial x_2 = 2(x_2 - 1) - u_1 + u_2 = 0$$

$$-x_1 - x_2 + 4 + s_1^2 = 0; \quad -x_1 + x_2 + 2 + s_2^2 = 0; \quad u_1 s_1 = 0, u_2 s_2 = 0, \quad u_1, u_2 \geq 0.$$

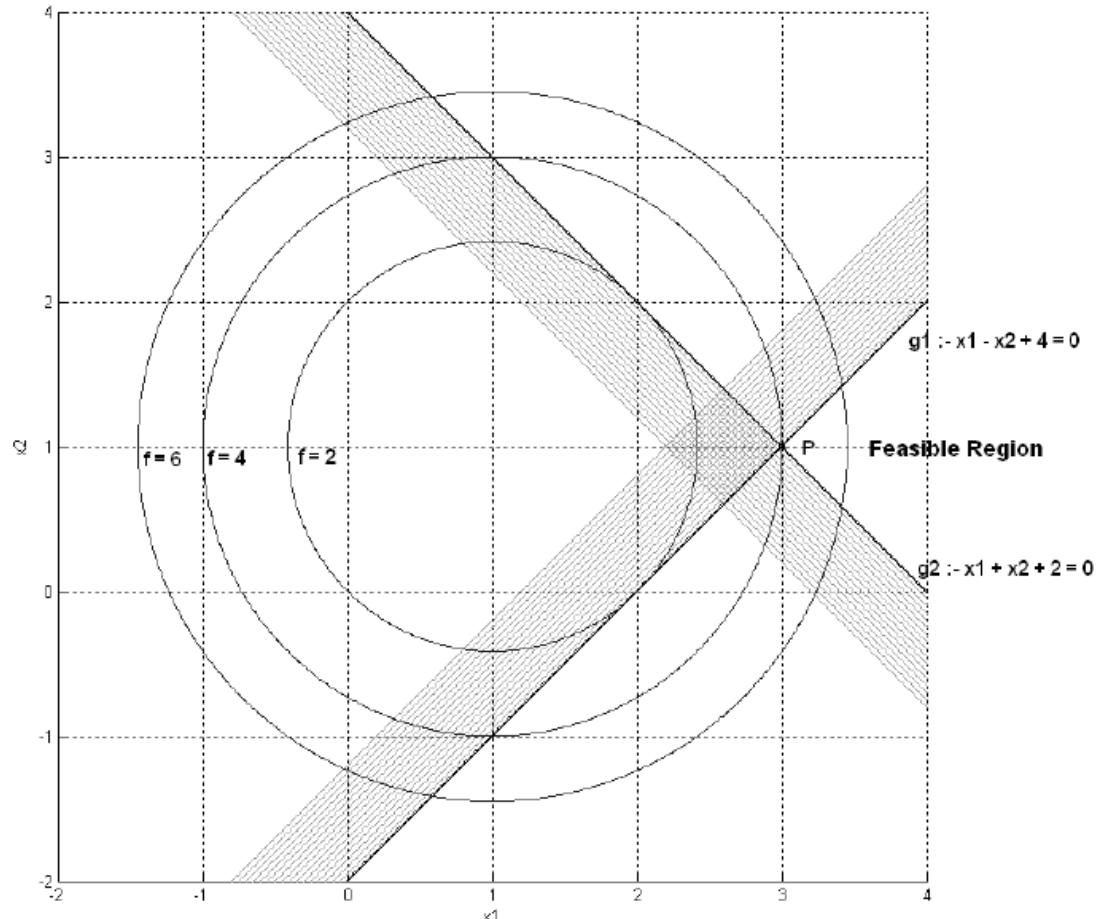
**Case 1.**  $u_1 = 0, u_2 = 0$ ; no candidate minimum.

**Case 2.**  $u_1 = 0, s_2 = 0$ ; no candidate minimum.

**Case 3.**  $s_1 = 0, u_2 = 0$ ; no candidate minimum.

**Case 4.**  $s_1 = 0, s_2 = 0$ ; gives  $(3, 1)$  as a KKT point with  $u_1 = 2, u_2 = 2, f = 4$ .

## optimum point P (3, 1)



## **MATLAB Code**

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;
g1=-x1-x2+4;
g2=-x1+x2+2;
cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'), ylabel('x2')
title('Exercise 4.59')
hold on
cv1=[0:0.05:0.8];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x1,x2,g1,cv1,'k');
cv2=[0:0.05:0.8];
const2=contour(x1,x2,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x1,x2,g2,cv2,'k');
fv=[2];
fs=contour(x1,x2,f,fv,'b');
a=[3];
b=[1];
plot(a,b,'.k');
grid
hold off
```

7.

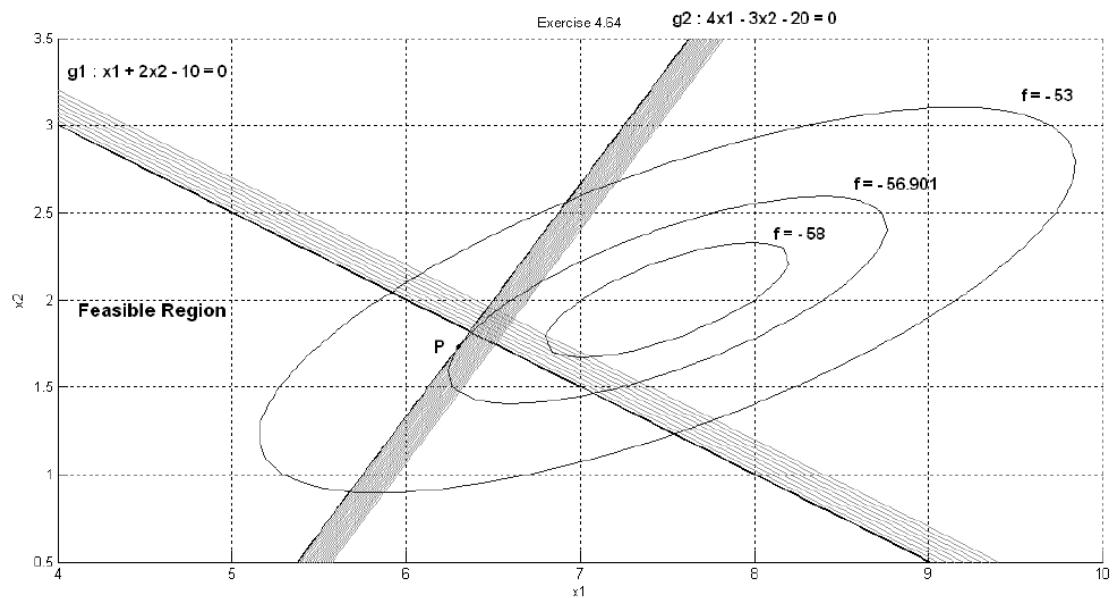
$$\begin{aligned} \text{Minimize } f(x_1, x_2) &= 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2 \\ \text{subject to } x_1 + 2x_2 &\leq 10 \\ 4x_1 - 3x_2 &\leq 20; x_i \geq 0; i = 1, 2 \end{aligned}$$

## Solution

Minimize  $f(\mathbf{x}) = 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2$ , subject to  $g_1 = x_1 + 2x_2 - 10 \leq 0$ ,  
 $g_2 = 4x_1 - 3x_2 - 20 \leq 0$ ,  $g_3 = -x_1 \leq 0$ ,  $g_4 = -x_2 \leq 0$ .

There are 16 cases, but only the case  $u_1 = u_3 = u_4 = 0$ ,  $s_2 = 0$  yields a solution:  
 $(6.3, 1.733)$ ,  $u_2 = 0.8$ ,  $f = -56.901$ .

### Optimum point P (6.3,1.733)



## **MATLAB Code**

```
clear all
axis equal
[x1,x2]=meshgrid(4:0.1:10, 0.5:0.1:3.5);
f=2*x1.^2-6*x1.*x2+9*x2.^2-18*x1+9*x2;
g1=x1+2*x2-10;
g2=4*x1-3*x2-20;
g3=-x1;
g4=-x2;
cla reset
axis equal
axis ([4 10 0.5 3.5])
xlabel('x1'),ylabel('x2')
title('Exercise 4.64')
hold on
cv1=[0:0.05:0.4];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x1,x2,g1,cv1,'k');
cv2=[0:0.05:0.8];
const2=contour(x1,x2,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x1,x2,g2,cv2,'k');
cv3=[0:0.03:0.3];
const3=contour(x1,x2,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(x1,x2,g3,cv3,'k');
cv4=[0:0.03:0.3];
const4=contour(x1,x2,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(x1,x2,g4,cv4,'k');
fv=[-58 -56.901 -53];
fs=contour(x1,x2,f,fv,'b');
a=[6.3];
b=[1.733];
plot(a,b,'.k');
grid
hold off
```

8.

$$\text{Minimize } f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$$

$$\text{subject to } x_1 + x_2 - 4 \leq 0$$

$$x_1 - x_2 - 2 \leq 0$$

## Solution

$$\text{Minimize } f(\mathbf{x}) = (x_1 - 1)^2 + (x_2 - 1)^2; \text{ subject to } g_1 = x_1 + x_2 - 4 \leq 0; g_2 = x_1 - x_2 - 2 \leq 0.$$

$$L = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(x_1 + x_2 - 4 + s_1^2) + u_2(x_1 - x_2 - 2 + s_2^2)$$

$$\partial L / \partial x_1 = 2(x_1 - 1) + u_1 + u_2 = 0; \quad \partial L / \partial x_2 = 2(x_2 - 1) + u_1 - u_2 = 0$$

$$x_1 + x_2 - 4 + s_1^2 = 0; \quad x_1 - x_2 - 2 + s_2^2 = 0;$$

$$u_1 s_1 = 0, \quad u_2 s_2 = 0; \quad u_1 \geq 0, \quad u_2 \geq 0$$

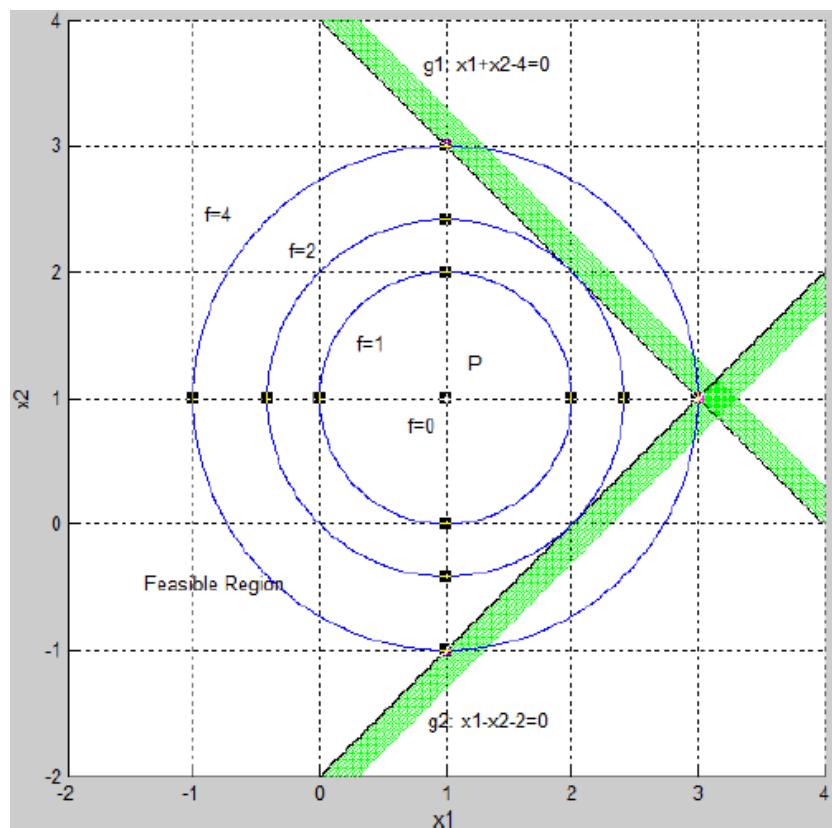
**Case 1.**  $u_1 = 0, u_2 = 0$ ; gives  $(1, 1)$  as a KKT point,  $f = 0$

**Case 2.**  $u_1 = 0, s_2 = 0$ ; no candidate minimum.

**Case 3.**  $s_1 = 0, u_2 = 0$ ; no candidate minimum.

**Case 4.**  $s_1 = 0, s_2 = 0$ ; no candidate minimum.

## Optimum Point (1,1)



### **MATLAB Code**

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;
g1=x1+x2-4;
g2=x1-x2-2;
cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.66')
hold on
cv1=[0:0.03:0.3];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');
cv2=[0:0.03:0.3];
const2=contour(x1,x2,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x1,x2,g2,cv2,'k');
fv=[0 1 2 4];
fs=contour(x1,x2,f,fv,'b');
a=[1];
b=[1];
plot(a,b,'.k');
grid
hold off
```

9.

$$\text{Minimize } f(x_1, x_2) = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4 \\ \text{subject to } x_1^2 + x_2^2 + 2x_1 \geq 16$$

### Solution

$$\text{Minimize } f(x_1, x_2) = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4, \text{ subject to } g_1 = 16 - (x_1^2 + x_2^2 + 2x_1) \leq 0.$$

$$L = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4 + u_1(16 - x_1^2 - x_2^2 - 2x_1 + s_1^2)$$

$$\partial L / \partial x_1 = 18x_1 - 18x_2 - 2u_1x_1 - 2u_1 = 0; \quad \partial L / \partial x_2 = -18x_1 + 26x_2 - 2u_1x_2 = 0$$

$$x_1 + x_2 - 4 + s_1^2 = 0; \quad 2 - x_1 + s_2^2 = 0; \quad u_1s_1 = 0, \quad u_2s_2 = 0, \quad u_1 \geq 0, \quad u_2 \geq 0$$

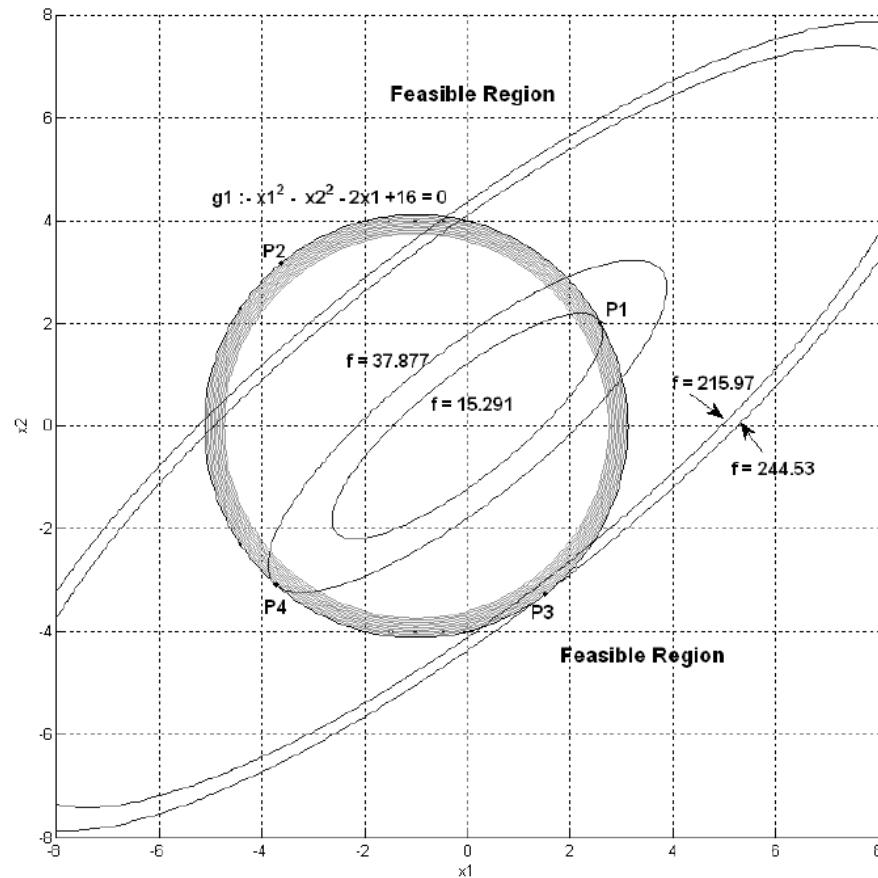
$$-x_1^2 - x_2^2 - 2x_1 + 16 + s_1^2 = 0; \quad u_1s_1 = 0, \quad u_1 \geq 0$$

**Case 1.**  $u_1 = 0$ ; no candidate minimum ( $s_1^2 < 0$ ).

**Case 2.**  $s_1 = 0$ ; Solving the nonlinear system of equations, we get the following KKT points:

$$(2.5945, 2.0198), \quad u_1 = 1.4390, \quad f = 15.291; \quad (-3.630, 3.1754), \quad u_1 = 23.2885, \quad f = 215.97; \\ (1.5088, -3.2720), \quad u_1 = 17.1503, \quad f = 244.53; \quad (-3.7322, -3.0879), \quad u_1 = 2.1222, \quad f = 37.877.$$

**Optimum Point P1(2.945, 2.0198) with objective function = 15.291**



## **MATLAB Code**

```
clear all
axis equal
[x1,x2]=meshgrid(-8:0.1:8, -8:0.1:8);
f=9*x1.^2-18*x1.*x2+13*x2.^2-4;
g1=-x1.^2-x2.^2-2*x1+16;
cla reset
axis equal
axis ([-8 8 -8 8])
xlabel('x1'), ylabel('x2')
title('Exercise 4.68')
hold on
cv1=[0:0.3:3];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');
fv=[15.291 37.877 215.97 244.53 ];
fs=contour(x1,x2,f,fv,'b');
a=[2.5945 -3.630 1.5088 -3.7322];
b=[2.0198 3.1754 -3.2720 -3.0879];
plot(a,b,'.k');
grid
hold off
```

10.

Minimize  $f(x, y) = (x - 4)^2 + (y - 6)^2$   
subject to  $x + y \leq 12$   
 $x \leq 6$   
 $x, y \geq 0$

### Solution

Minimize  $f(x, y) = (x - 4)^2 + (y - 6)^2$ ; subject to  $g_1 = x + y - 12 \leq 0$ ;

$$g_2 = x - 6 \leq 0; g_3 = -x \leq 0; g_4 = -y \leq 0;$$

$$\begin{aligned} L = & ((x-4)^2 + (y-6)^2) + u_1(x+y-12+s_1^2) + u_2(x-6+s_2^2) \\ & + u_3(-x+s_3^2) + u_4(-y+s_4^2) \end{aligned}$$

$$\partial L / \partial x = 2(x-4) + u_1 + u_2 - u_3 = 0; \quad \partial L / \partial y = 2(y-6) + u_1 - u_4 = 0;$$

$$x + y - 12 + s_1^2 = 0; \quad x - 6 + s_2^2 = 0; \quad -x + s_3^2 = 0; \quad -y + s_4^2 = 0;$$

$$u_i s_i = 0; \quad u_i \geq 0; \quad i = 1 \text{ to } 4 \text{ (there are 16 cases).}$$

**Case 1.**  $u_1 = u_2 = u_3 = u_4 = 0$ ; gives  $(4, 6)$  as a KKT point ;  $f = 0$ .

**Case 2.**  $u_1 = u_2 = u_3 = 0, s_4 = 0$ ; gives no candidate point.

**Case 3.**  $u_1 = u_2 = u_4 = 0, s_3 = 0$ ; gives no candidate point.

**Case 4.**  $u_1 = u_3 = u_4 = 0, s_2 = 0$ ; gives no candidate point.

**Case 5.**  $u_2 = u_3 = u_4 = 0, s_1 = 0$ ; gives no candidate point.

**Case 6.**  $u_1 = u_2 = 0, s_3 = s_4 = 0$ ; gives no candidate point.

**Case 7.**  $u_1 = u_3 = 0, s_2 = s_4 = 0$ ; gives no candidate point.

**Case 8.**  $u_1 = u_4 = 0, s_2 = s_3 = 0$ ; gives no candidate point.

**Case 9.**  $u_2 = u_3 = 0, s_1 = s_4 = 0$ ; gives no candidate point.

**Case 10.**  $u_2 = u_4 = 0, s_1 = s_3 = 0$ ; gives no candidate point.

**Case 11.**  $u_3 = u_4 = 0, s_1 = s_2 = 0$ ; gives no candidate point.

**Case 12.**  $u_1 = 0, s_2 = s_3 = s_4 = 0$ ; gives no candidate point.

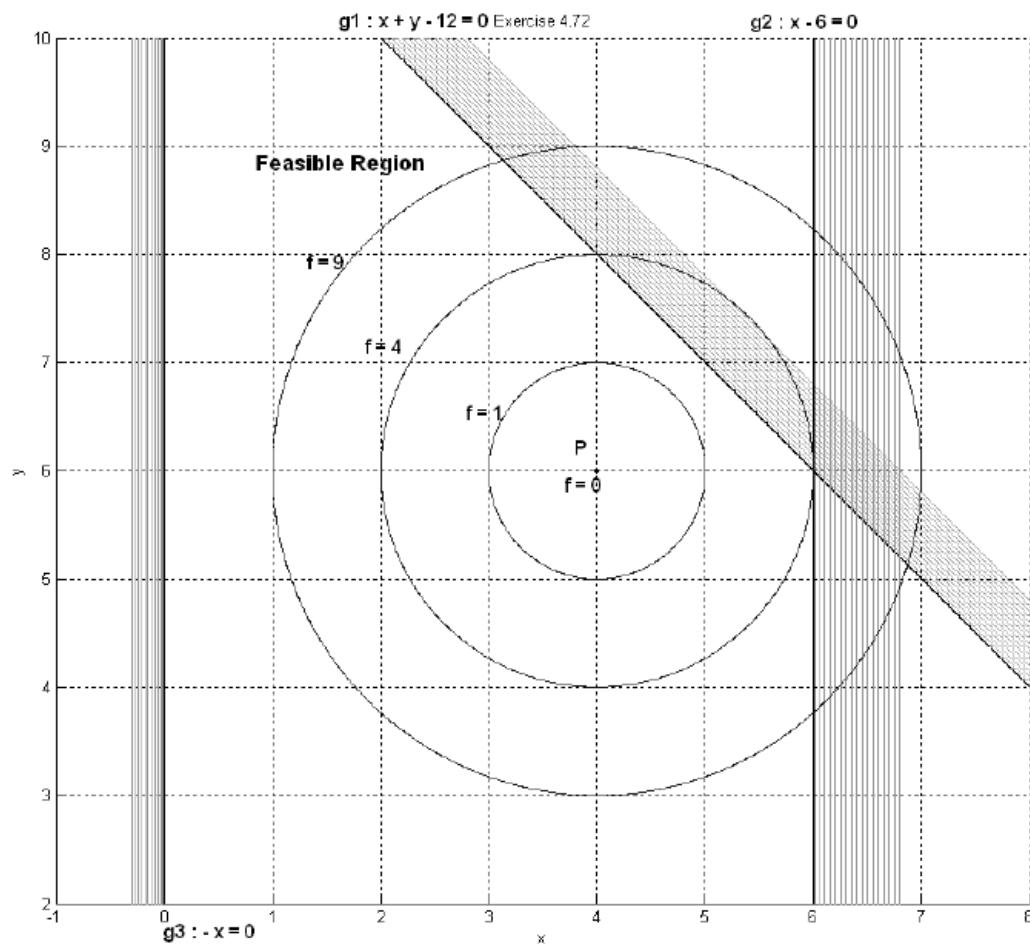
**Case 13.**  $u_2 = 0, s_1 = s_3 = s_4 = 0$ ; gives no candidate point.

**Case 14.**  $u_3 = 0, s_1 = s_2 = s_4 = 0$ ; gives no candidate point.

**Case 15.**  $u_4 = 0, s_1 = s_2 = s_3 = 0$ ; gives no candidate point.

**Case 16.**  $s_1 = s_2 = s_3 = s_4 = 0$ ; gives no candidate point.

**Optimum Point (4,6)**



### **MATLAB Code**

```
clear all
axis equal
[x,y]=meshgrid(-1:0.01:8, 2:0.01:10);
f=(x-4).^2+(y-6).^2;
g1=x+y-12;
g2=x-6;
g3=-x;
g4=-y;
cla reset
axis equal
axis ([-1 8 2 10])
xlabel('x'), ylabel('y')
title('Exercise 4.72')
hold on
cv1=[0:0.05:0.8];
const1=contour(x,y,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x,y,g1,cv1,'k');
cv2=[0:0.05:0.8];
const2=contour(x,y,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x,y,g2,cv2,'k');
cv3=[0:0.03:0.3];
const3=contour(x,y,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(x,y,g3,cv3,'k');
cv4=[0:0.03:0.3];
const4=contour(x,y,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(x,y,g4,cv4,'k');
fv=[0 1 4 9];
fs=contour(x,y,f,fv,'b');
a=[4];
b=[6];
plot(a,b,'.k');
grid
hold off
```

11.

Maximize  $\tilde{F}(x, y) = (x - 4)^2 + (y - 6)^2$   
 subject to  $x + y \leq 12$   
 $6 \geq x$   
 $x, y \geq 0$

## Solution

Minimize  $f(x, y) = -(x - 4)^2 - (y - 6)^2$ ; subject to  $g_1 = x + y - 12 \leq 0$ ;

$$g_2 = x - 6 \leq 0; g_3 = -x \leq 0; g_4 = -y \leq 0;$$

$$\begin{aligned} L = & \left( -(x-4)^2 - (y-6)^2 \right) + u_1(x+y-12+s_1^2) + u_2(x-6+s_2^2) \\ & + u_3(-x+s_3^2) + u_4(-y+s_4^2) \end{aligned}$$

$$\partial L / \partial x = -2(x-4) + u_1 + u_2 - u_3 = 0;$$

$$\partial L / \partial y = -2(y-6) + u_1 - u_4 = 0;$$

$$x + y - 12 + s_1^2 = 0;$$

$$x - 6 + s_2^2 = 0;$$

$$-x + s_3^2 = 0;$$

$$-y + s_4^2 = 0;$$

$$u_i s_i = 0; u_i \geq 0; i = 1 \text{ to } 4 \text{ (there are 16 cases).}$$

**Case 1.**  $u_1 = u_2 = u_3 = u_4 = 0$ ; gives  $(4, 6)$  as a KKT point;  $F = 0$ .

**Case 2.**  $u_2 = u_3 = u_4 = 0, s_1 = 0$ ; gives  $(5, 7)$  as a KKT point with  $u_1 = 2; F = 2$ .

**Case 3.**  $u_1 = u_3 = u_4 = 0, s_2 = 0$ ; gives  $(6, 6)$  as a KKT point with  $u_2 = 4; F = 4$ . Note that for this case,  $s_1$  is also 0. Therefore this is an abnormal case where both  $u_1$  and  $s_1$  are zero.

**Case 4.**  $u_1 = u_2 = u_4 = 0, s_3 = 0$ ; gives  $(0, 6)$  as a KKT point with  $u_3 = 8; F = 16$ .

**Case 5.**  $u_1 = u_2 = u_3 = 0, s_4 = 0$ ; gives  $(4, 0)$  as a KKT point with  $u_4 = 12; F = 36$ .

**Case 6.**  $u_3 = u_4 = 0, s_1 = s_2 = 0$ ; gives  $(6, 6)$  as a KKT point with  $u_1 = 0, u_2 = 4; F = 4$ . Note that for this case,  $s_1$  is also 0. Therefore this is an abnormal case where both  $u_1$  and  $s_1$  are zero.

**Case 7.**  $u_1 = u_4 = 0, s_2 = s_3 = 0$ ; gives no candidate point.

**Case 8.**  $u_1 = u_2 = 0, s_3 = s_4 = 0$ ; gives  $(0, 0)$  as a KKT point with  $u_3 = 8$  and  $u_4 = 12; F = 52$ .

**Case 9.**  $u_2 = u_3 = 0, s_1 = s_4 = 0$ ; gives no candidate point.

**Case 10.**  $u_2 = u_4 = 0, s_1 = s_3 = 0$ ; gives  $(0, 12)$  as a KKT point with  $u_1 = 12, u_3 = 20; F = 52$ .

**Case 11.**  $u_1 = u_3 = 0, s_2 = s_4 = 0$ ; gives  $(6, 0)$  as a KKT point with  $u_2 = 4, u_4 = 12; F = 40$ .

**Case 12.**  $u_1 = 0, s_2 = s_3 = s_4 = 0$ ; gives no candidate point.

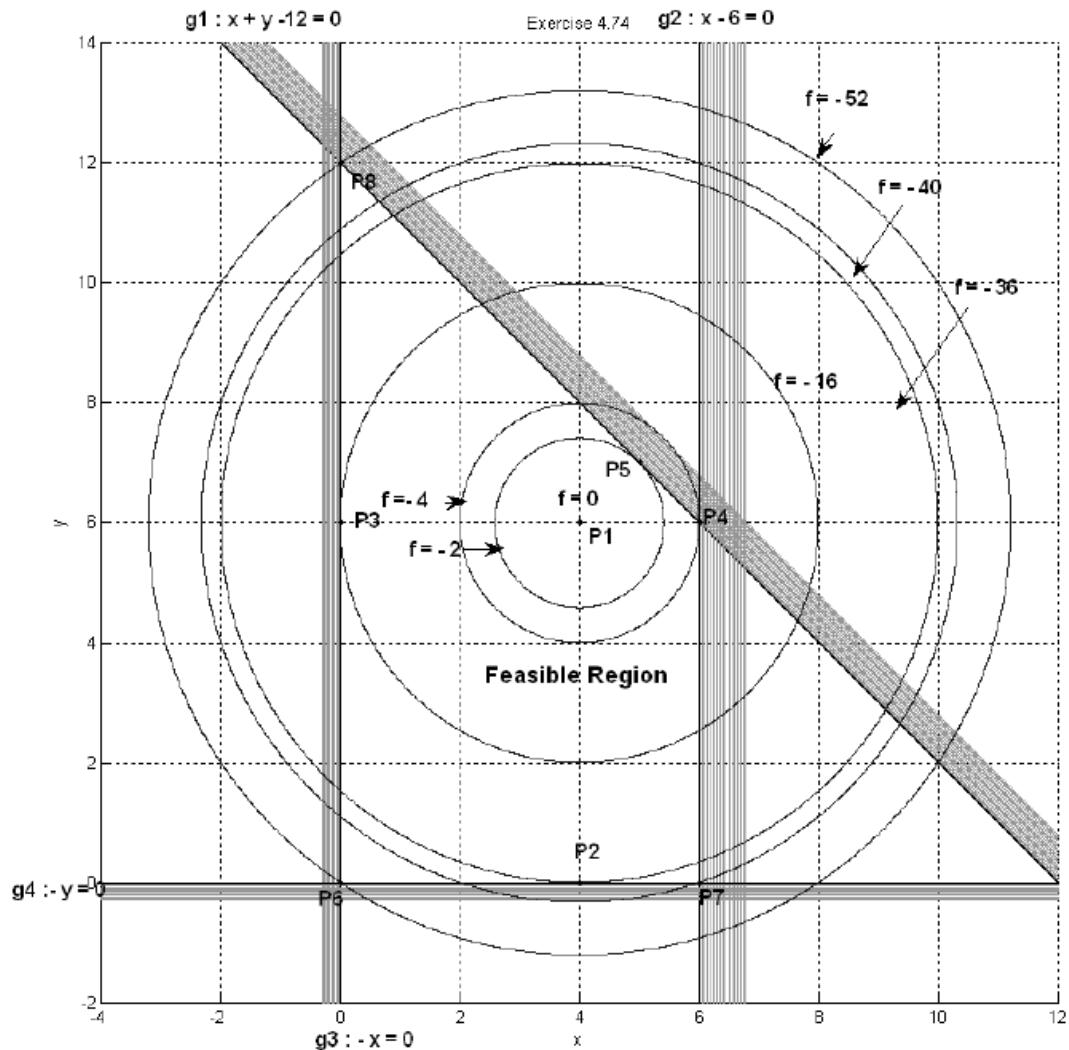
**Case 13.**  $u_2 = 0, s_1 = s_3 = s_4 = 0$ ; gives no candidate point.

**Case 14.**  $u_3 = 0, s_1 = s_2 = s_4 = 0$ ; gives no candidate point.

**Case 15.**  $u_4 = 0, s_1 = s_2 = s_3 = 0$ ; gives no candidate point.

**Case 16.**  $s_1 = s_2 = s_3 = s_4 = 0$ ; gives no candidate point.

### Optimum Point (4,6)



## **MATLAB Code**

```
clear all
axis equal
[x,y]=meshgrid(-4:0.01:12, -2:0.01:14);
f=(-1)*((x-4).^2+(y-6).^2);
g1=x+y-12;
g2=x-6;
g3=-x;
g4=-y;
cla reset
axis equal
axis ([-4 12 -2 14])
xlabel('x'), ylabel('y')
title('Exercise 4.74')
hold on
cv1=[0:0.05:0.8];
const1=contour(x,y,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x,y,g1,cv1,'k');
cv2=[0:0.05:0.8];
const2=contour(x,y,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x,y,g2,cv2,'k');
cv3=[0:0.03:0.3];
const3=contour(x,y,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(x,y,g3,cv3,'k');
cv4=[0:0.03:0.3];
const4=contour(x,y,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(x,y,g4,cv4,'k');
fv=[-52 -40 -36 -16 -4 -2 0];
fs=contour(x,y,f,fv,'b');
a=[4 6 0 6 5 0 6 0 6 4];
b=[6 0 6 6 7 0 0 12 6 0];
plot(a,b,'.k');
grid
hold off
```

12.

$$\text{Maximize } F(r, t) = (r - 3)^2 + (t - 2)^2$$

$$\text{subject to } 10 \geq r + t$$

$$t \leq 5$$

$$r, t \geq 0$$

### Solution

$$\text{Minimize } f(r, t) = -(r - 3)^2 - (t - 2)^2$$

$$\text{subject to } g_1 = r + t - 10 \leq 0;$$

$$g_2 = t - 5 \leq 0; \quad g_3 = -r \leq 0; \quad g_4 = -t \leq 0;$$

$$L = \left( -(r - 3)^2 - (t - 2)^2 \right) + u_1(r + t - 10 + s_1^2) + u_2(t - 5 + s_2^2) + u_3(-r + s_3^2) + u_4(-t + s_4^2)$$

$$\partial L / \partial r = -2(r - 3) + u_1 - u_3 = 0 \quad (\text{a})$$

$$\partial L / \partial t = -2(t - 2) + u_1 + u_2 - u_4 = 0 \quad (\text{b})$$

$$r + t - 10 + s_1^2 = 0; \quad t - 5 + s_2^2 = 0; \quad -r + s_3^2 = 0; \quad -t + s_4^2 = 0; \quad (\text{c})$$

$u_i s_i = 0; u_i \geq 0; i = 1 \text{ to } 4$  (there are 16 cases).

**Case 1.**  $u_1 = u_2 = u_3 = u_4 = 0$ ;

Equations (a) and (b) give  $(3, 2)$  as a KKT point;  $F = 0$ . Equations (c) give all  $s_i^2 > 0, i = 1 \text{ to } 4$ .

Therefore it is a feasible point

**Case 2.**  $u_2 = u_3 = u_4 = 0, s_1 = 0$ ;

Equations (a) to (c) give  $(5.5, 4.5)$  as a KKT point with  $u_1 = 5; F = 12.5$ .

**Case 3.**  $u_1 = u_3 = u_4 = 0, s_2 = 0$ ;

Equations (a) to (c) give  $(3, 5)$  as a KKT point with  $u_2 = 6; F = 9$ .

**Case 4.**  $u_1 = u_2 = u_4 = 0, s_3 = 0$ ;

Equations (b) and (c) give  $(0, 2)$  as a KKT point with  $u_3 = 6$  from Eq. (a);  $F = 9$ .

**Case 5.**  $u_1 = u_2 = u_3 = 0, s_4 = 0$ ;

Equations (a) and (c) give  $(3, 0)$  as a KKT point with  $u_4 = 4$  from Eq. (b);  $F = 4$ .

**Case 6.**  $u_3 = u_4 = 0, s_1 = s_2 = 0$ ;

Equations (a) and (c) give  $(5, 5)$  as a KKT point with  $u_1 = 4, u_2 = 2; F = 13$ .

**Case 7.**  $u_1 = u_4 = 0, s_2 = s_3 = 0$ ;

Equations (a) and (c) give  $(0, 5)$  as a KKT point with  $u_2 = 6, u_3 = 6; F = 18$ .

**Case 8.**  $u_1 = u_2 = 0, s_3 = s_4 = 0$ ;

Equations (a) and (c) give  $(0, 0)$  as a KKT point with  $u_3 = 6, u_4 = 4; F = 13$ .

**Case 9.**  $u_2 = u_3 = 0, s_1 = s_4 = 0;$

Equations (a) and (c) give  $(10, 0)$  as a KKT point with  $u_1 = 14, u_4 = 18; F = 53.$

**Case 10.**  $u_2 = u_4 = 0, s_1 = s_3 = 0;$

Equations (a) and (c) give  $(0, 10);$  not a KKT point since  $u_1 = 16, u_2 = 22$  and  $s_2^2 = -5.$

**Case 11.**  $u_1 = u_3 = 0, s_2 = s_4 = 0;$

Equations (c) give  $t = 5$  and  $t = 0$  which is an inconsistency.

**Case 12.**  $u_4 = 0, s_1 = s_2 = s_3 = 0;$  gives no candidate point.

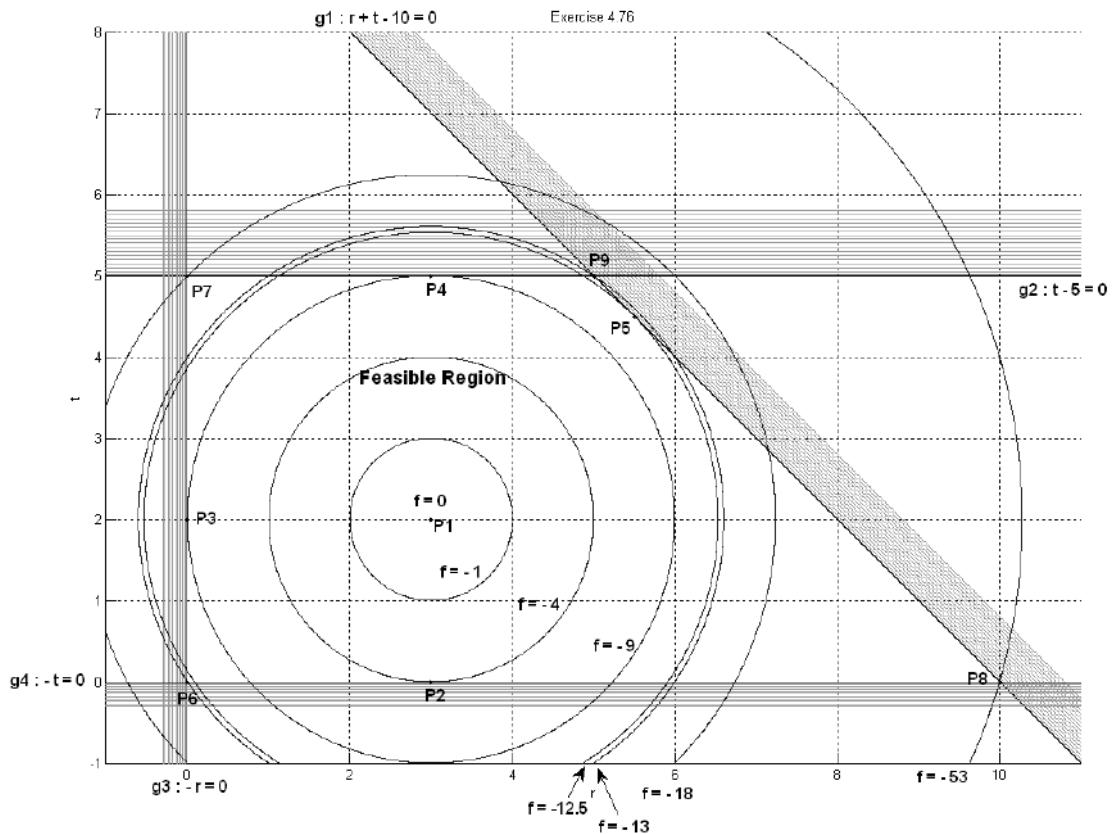
**Case 13.**  $u_1 = 0, s_2 = s_3 = s_4 = 0;$  gives no candidate point.

**Case 14.**  $u_2 = 0, s_1 = s_3 = s_4 = 0;$  gives no candidate point.

**Case 14.**  $u_3 = 0, s_1 = s_2 = s_4 = 0;$  gives no candidate point.

**Case 16.**  $s_1 = s_2 = s_3 = s_4 = 0;$  gives no candidate point.

Optimum Point  $(3,2)$



## **MATLAB Code**

```
clear all
axis equal
[r,t]=meshgrid(-1:0.01:11, -1:0.01:8);
f=(-1)*((r-3).^2+(t-2).^2);
g1=r+t-10;
g2=t-5;
g3=-r;
g4=-t;
cla reset
axis equal
axis ([-1 11 -1 8])
xlabel('r'), ylabel('t')
title('Exercise 4.76')
hold on
cv1=[0:0.05:0.8];
const1=contour(r,t,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(r,t,g1,cv1,'k');
cv2=[0:0.05:0.8];
const2=contour(r,t,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(r,t,g2,cv2,'k');
cv3=[0:0.03:0.3];
const3=contour(r,t,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(r,t,g3,cv3,'k');
cv4=[0:0.03:0.3];
const4=contour(r,t,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(r,t,g4,cv4,'k');
fv=[0 -1 -4 -9 -12.5 -13 -18 -53];
fs=contour(r,t,f,fv,'b');
a=[3 3 0 3 5.5 0 0 10 5];
b=[2 0 2 5 4.5 0 5 0 5];
plot(a,b,'.k');
grid
hold off
```

13.

An engineering design problem is formulated as

$$\text{Minimize } f(x_1, x_2) = x_1^2 + 320x_1x_2$$

$$\text{subject to } \frac{1}{100}(x_1 - 60x_2) \leq 0$$

$$1 - \frac{1}{3600}x_1(x_1 - x_2) \leq 0$$

$$x_1, x_2 \geq 0$$

Write KKT necessary conditions and solve for the candidate minimum designs. Verify the solutions graphically. Interpret the KKT conditions on the graph for the problem.

### **Solution**

$$\text{Minimize } f(\mathbf{x}) = x_1^2 + 320x_1x_2, \text{ subject to } g_1 = x_1/60x_2 - 1 \leq 0;$$

$$g_2 = 1 - x_1(x_1 - x_2)/3600 \leq 0; \quad g_3 = -x_1 \leq 0; \quad g_4 = -x_2 \leq 0$$

$$L = (x_1^2 + 320x_1x_2) + u_1(x_1/60x_2 - 1 + s_1^2) + u_2(1 - x_1(x_1 - x_2)/3600 + s_2^2) + u_3(-x_1 + s_3^2) + u_4(-x_2 + s_4^2)$$

$$\partial L/\partial x_1 = 2x_1 + 320x_2 + u_1/60x_2 - u_2(2x_1 - x_2)/3600 - u_3 = 0$$

$$\partial L/\partial x_2 = 320x_1 - u_1x_1/60x_2^2 + u_2x_1/3600 - u_4 = 0;$$

$$g_i + s_i^2 = 0, \quad u_i s_i = 0, \quad u_i \geq 0; \quad i = 1 \text{ to } 4$$

There are 16 cases because there are four inequality constraints. A case which yields the solution is identified as  $s_1, s_2 = 0; u_3, u_4 = 0$ . The solution is

$$x_1 = 60.50634, \quad x_2 = 1.008439, \quad u_1 = 19529, \quad u_2 = 229.9, \quad f = 23,186.4.$$

All the KKT conditions are satisfied. The solution can be verified graphically. It shows that the point obtained using KKT conditions is indeed a minimum point.

### **Optimum Point (60.5,1.008)**