SPC 307 - Aerodynamics
Sheet 5
Dynamics of an incompressible, inviscid flow field

1. As illustrated in Fig. 1, a tornado can be approximated by a free vortex of strength Γ for $r > R_c$, where R_c is the radius of the core. Velocity measurements at points A and B indicate that V_A = 125 ft/s and V_B = 60 ft/s. Determine the distance from point A to the center of the tornado. Why can the free vortex model not be used to approximate the tornado throughout the flow field (r ≥ 0)?

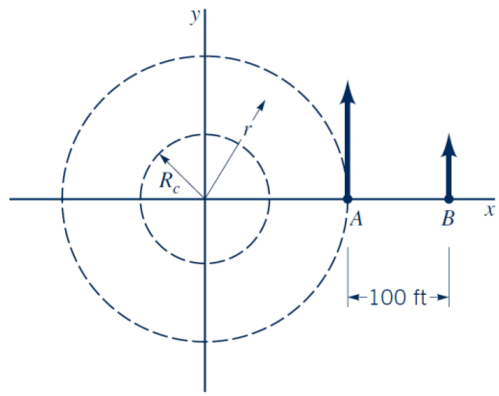


Fig.1

For a free vortex

$$v_\theta = \frac{K}{r} \qquad (\text{Eq. } 6.86)$$

Thus, at r_A, $v_\theta = 125 \frac{ft}{s}$, so that $K = 125\, r_A$

and at r_B, $v_\theta = 60 \frac{ft}{s}$, so that $K = 60\, r_B$.

Therefore,

$$125\, r_A = 60\, r_B$$

and since

$$r_B - r_A = 100 \text{ ft}$$

it follows that

$$125\, r_A = 60\,(100 + r_A)$$

or

$$\underline{\underline{r_A = 92.3 \text{ ft}}}$$

The free vortex cannot be used to approximate a tornado throughout the flow field since at $r=0$ the velocity becomes infinite.

2. The streamlines in a particular two-dimensional flow field are all concentric circles, as shown in Fig. 2. The velocity is given by the equation $v_\theta = \omega r$ where ω is the angular velocity of the rotating mass of fluid. Determine the circulation around the path ABCD.

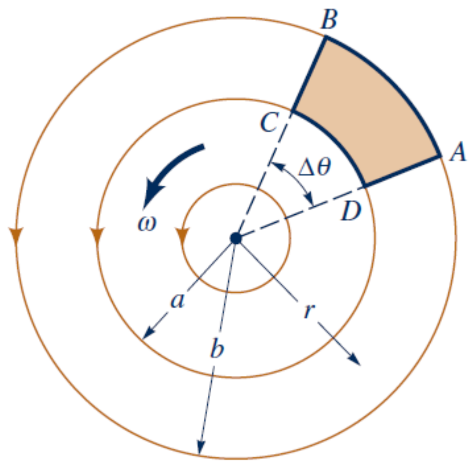


Fig.2

$$\Gamma = \oint_{ABCD} \vec{V} \cdot d\vec{s}$$

$$= \int_{AB} v_\theta \, b \, d\theta + \int_{BC} v_r \, dr + \int_{CD} v_\theta \, a \, d\theta + \int_{DA} v_r \, dr \qquad (1$$

Since $v_r = 0$ and $v_\theta = \omega r$, Eq. (1) becomes

$$\Gamma = \int_{\theta_1}^{\theta_2} \omega b^2 \, d\theta + 0 + \int_{\theta_2}^{\theta_1} \omega a^2 \, d\theta + 0$$

$$= \omega b^2 (\theta_2 - \theta_1) + \omega a^2 (\theta_1 - \theta_2)$$

or

$$\Gamma = \omega (\theta_2 - \theta_1)(b^2 - a^2) = \underline{\underline{\omega \Delta\theta (b^2 - a^2)}}$$

3. When water discharges from a tank through an opening in its bottom, a vortex may form with a curved surface profile, as shown in Fig. 3. Assume that the velocity distribution in the vortex is the same as that for a free vortex. At the same time the water is being discharged from the tank at point A, it is desired to discharge a small quantity of water through the pipe B. As the discharge through A is increased, the strength of the vortex, as indicated by its circulation, is increased. Determine the maximum strength that the vortex can have in order that no air is sucked in at B. Express your answer in terms of the circulation. Assume that the fluid level in the tank at a large distance from the opening at A remains constant and viscous effects are negligible.

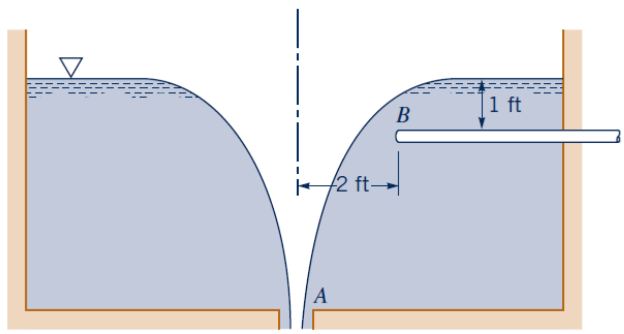


Fig. 3.

From Example 6.6,

$$z_s = -\frac{\Gamma^2}{8\pi^2 r^2 g}$$

Air will be sucked into pipe when $z_s = -1\ ft$ for $r = 2\ ft$.

Thus,

$$\Gamma^2 = -8\pi^2 r^2 g\, z_s = -8\pi^2 (2ft)^2 \left(32.2\ \frac{ft}{s^2}\right)(-1\ ft)$$

or

$$|\Gamma| = \underline{\underline{101\ \frac{ft^2}{s}}}$$

4. The Water flows over a flat surface at 4 ft/s, as shown in Fig. 4. A pump draws off water through a narrow slit at a volume rate of 0.1 ft3/s per foot length of the slit. Assume that the fluid is incompressible and inviscid and can be represented by the combination of a uniform flow and a sink. Locate the stagnation point on the wall (point A) and determine the equation for the stagnation streamline. How far above the surface, surface, H, must the fluid be so that it does not get sucked into the slit?

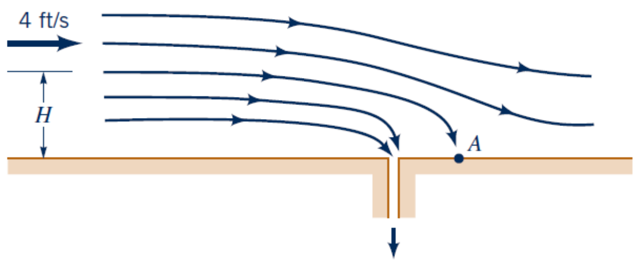


Fig. 4.

$$\psi = \psi_{\substack{\text{uniform}\\ \text{flow}}} + \psi_{\text{sink}} = U r \sin\theta - \frac{m}{2\pi}\theta \qquad (1$$

Thus,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\cos\theta - \frac{m}{2\pi r} \qquad (2$$

and

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U\sin\theta$$

Along the wall $v_\theta = 0$, and the stagnation point occurs where $v_r = 0$, so that from Eq. (2)

$$0 = U\cos(0°) - \frac{m}{2\pi r_s}$$

and therefore

$$r_s = \frac{m}{2\pi U}$$

For $U = 4\,\frac{ft}{s}$ and $m = 0.2\,\frac{ft^2}{s}$ (note that a source strength of $0.2\,\frac{ft^2}{s}$ must be used to obtain $0.1\,\frac{ft^2}{s}$ through slit which is only one half of a "full" sink). Thus,

$$r_s = \frac{0.2\,\frac{ft^2}{s}}{2\pi\left(4\,\frac{ft}{s}\right)} = 0.00796\ ft$$

and the stagnation point is on the wall <u>0.00796 ft to the right of slit</u>.

The value of ψ at the stagnation point ($r = 0.00796$ ft, $\theta = 0°$) is zero (Eq. 1) so that the equation of the stagnation streamline is

$$0 = U r \sin\theta - \frac{m}{2\pi}\theta$$

or

$$r\sin\theta = \frac{m}{2\pi U}\theta$$

Since $y = r\sin\theta$ the equation of the stagnation streamline can be written as

$$\underline{\underline{y = \frac{m}{2\pi U}\theta}}$$

Fluid above the stagnation streamline will not be sucked into slit. The maximum distance, H, for the stagnation streamline occurs as $\theta \rightarrow \pi$ so that

$$H = \frac{m\pi}{2\pi U} = \frac{0.2\ \frac{ft^2}{s}}{2\left(4\ \frac{ft}{s}\right)} = \underline{\underline{0.0250\ ft}}$$

(Note: All the fluid below the stagnation streamline must pass through the slit. Thus, from conservation of mass

$$HU = \text{flow into slit}$$

or

$$H = \frac{0.1\ \frac{ft^2}{s}}{4\ \frac{ft}{s}} = 0.0250\ ft$$

which checks with the answer above.)

5. The Two sources, one of strength m and the other with strength 3m, are located on the x axis as shown in Fig. 5. Determine the location of the stagnation point in the flow produced by these sources.

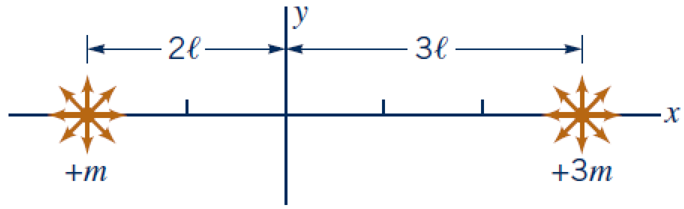
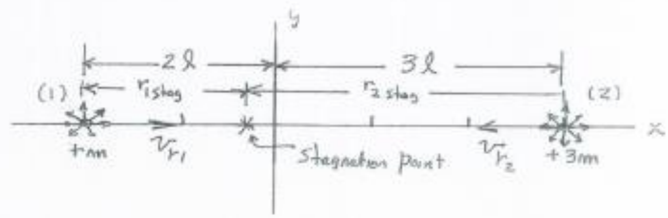


Fig. 5.

Since the flow from each source is in the radial direction, it is only along the x-axis that the two radial components can cancel and create a stagnation point.



For source (1)
$$v_{r_1} = \frac{m}{2\pi r_1}$$

and for source (2)
$$v_{r2} = \frac{3m}{2\pi r_2}$$

The stagnation point occurs where $v_{r_1} = v_{r2}$ so that
$$\frac{m}{2\pi r_{1stag}} = \frac{3m}{2\pi r_{2stag}}$$

and
$$\frac{r_{2stag}}{r_{1stag}} = 3$$

Also,
$$r_{1stag} + r_{2stag} = 2\ell + 3\ell = 5\ell$$

so that
$$r_{1stag} + 3r_{1stag} = 5\ell$$
$$r_{1stag} = \frac{5}{4}\ell$$

Thus,
$$x_{stag} = -\left(2\ell - \frac{5}{4}\ell\right) = \underline{\underline{-0.75\ell}}$$

6. The velocity potential for a spiral vortex flow is given by

$$\phi = (\Gamma/2\pi)\theta - (\mathrm{m}/2\pi)\ln r$$

where Γ and m are constants. Show that the angle, α, between the velocity vector and the radial direction is constant throughout the flow field (see Fig. 6).

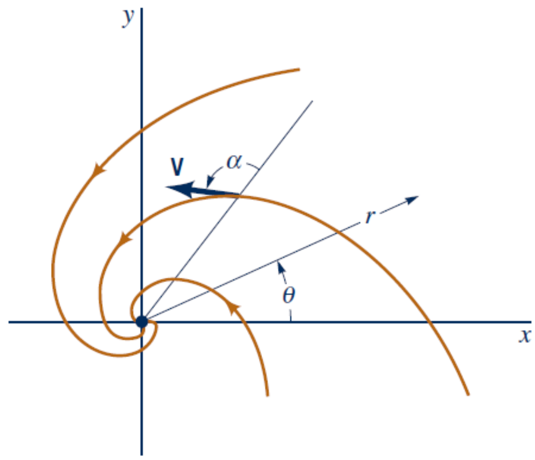


Fig. 6.

For the velocity potential given,

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{m}{2\pi r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$

Since $\quad \vec{V} \cdot \hat{e}_r = |\vec{V}| \cos \alpha$

and $\quad \vec{V} = v_r \hat{e}_r + v_\theta \hat{e}_\theta$

then

$$\cos \alpha = \frac{\vec{V} \cdot \hat{e}_r}{|\vec{V}|} = \frac{v_r}{\sqrt{v_r^2 + v_\theta^2}}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{v_\theta}{v_r}\right)^2}} = \frac{1}{\sqrt{1 + \frac{\left(\frac{\Gamma}{2\pi r}\right)^2}{\left(-\frac{m}{2\pi r}\right)^2}}}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{\Gamma}{m}\right)^2}}$$

Thus, for a given Γ and m the angle α is a constant.

7. The combination of a uniform flow and a source can be used to describe flow around a streamlined body called a half-body. Assume that a certain body has the shape of a half-body with a thickness of 0.5 m. If this body is placed in an airstream moving at 15 m/s, what source strength is required to simulate flow around the body?

The width of half-body $= 2\pi b$ (See Fig. 6.24)

so that

$$b = \frac{(0.5\,m)}{2\pi}$$

From Eq. 6.99

$$b = \frac{m}{2\pi U}$$

where m is the source strength, and therefore

$$m = 2\pi U b = 2\pi\left(15\ \frac{m}{s}\right)\left(\frac{0.5\,m}{2\pi}\right)$$

$$= \underline{7.50\ \frac{m^2}{s}}$$

8. The One end of a pond has a shoreline that resembles a half-body as shown in Fig. 7. A vertical porous pipe is located near the end of the pond so that water can be pumped out. When water is pumped at the rate of 0.06 m³/s through a 3-m-long pipe, what will be the velocity at point A? Hint: Consider the flow inside a half-body.

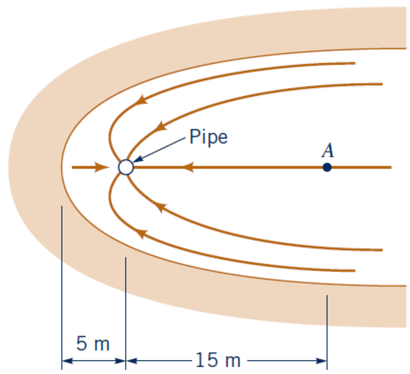


Fig. 7.

For a half-body,

$$\psi = U r \sin\theta + \frac{m}{2\pi}\theta \qquad \text{(Eq. 6.97)}$$

so that

$$v_\theta = -\frac{\partial \psi}{\partial r} = U \sin\theta$$

and

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\cos\theta + \frac{m}{2\pi r}$$

Thus, at point A, $\theta = 0$, $r = 15\,m$ and

$$v_\theta = 0$$

$$v_r = V_A = U + \frac{m}{2\pi(15)} \qquad (1)$$

For a flowrate of $0.06\,\frac{m^3}{s}$ in a 3-m long pipe, the source strength is $\frac{0.06}{3}\,\frac{m^2}{s}$. Since

$$b = \frac{m}{2\pi U} \qquad \text{(Eq. 6.99)}$$

then with $b = 5m$

$$U = \frac{m}{2\pi b} = \frac{\left(\frac{0.06}{3}\,\frac{m^2}{s}\right)}{2\pi(5m)} = 6.37 \times 10^{-4}\,\frac{m}{s}$$

From Eq. (1)

$$V_A = 6.37 \times 10^{-4}\,\frac{m}{s} + \frac{\left(\frac{0.06}{3}\,\frac{m^2}{s}\right)}{2\pi(15m)}$$

$$= 8.49 \times 10^{-4}\,\frac{m}{s}$$

9. Rankine oval is formed by combining a source–sink pair, each having a strength of 36 ft^2/s and separated by a distance of 12 ft along the x axis, with a uniform velocity of 10 ft/s (in the positive x direction). Determine the length and thickness of the oval.

$$\frac{\ell}{a} = \left[\frac{m}{\pi U a} + 1\right]^{1/2} \qquad (\text{Eq. } 6.107)$$

$$\frac{h}{a} = \frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right] \tan\left[2\left(\frac{\pi U a}{m}\right)\frac{h}{a}\right] \qquad (\text{Eq. } 6.109)$$

For $m = 36\ \frac{ft^2}{s}$, $a = 6\ ft$, and $U = 10\ \frac{ft}{s}$,

$$\frac{\pi U a}{m} = \frac{\pi\left(10\frac{ft}{s}\right)(6\ ft)}{36\ \frac{ft^2}{s}} = 5.24$$

Thus, length $= 2\ell$ and from Eq. 6.107

$$\underline{\underline{\text{length}}} = 2\,(6\ ft)\left[\frac{1}{5.24} + 1\right]^{1/2} = \underline{\underline{13.1\ ft}}$$

The thickness, $2h$, can be determined from Eq. 6.109 by trial and error. Assume value for h/a and compare with right hand side of Eq. 6.109. (See table below.)

$\frac{h}{a}$	$\frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right] \tan\left[2\,(5.24)\frac{h}{a}\right]$
0.250	0.269
0.251	0.262
0.252	0.256
0.253	0.250 ← use

Thus, $\frac{h}{a} \approx 0.253$

and $\underline{\underline{\text{thickness}}} = 2h = 2\,(6\ ft)(0.253) = \underline{\underline{3.04\ ft}}$

10. An ideal fluid flows past an infinitely long, semicircular "hump" located along a plane boundary, as shown in Fig. 8. Far from the hump the velocity field is uniform, and the pressure is p_0. (a) Determine expressions for the maximum and minimum values of the pressure along the hump, and indicate where these points are located. Express your answer in terms of r, U, and p_0. (b) If the solid surface is the streamline, determine the equation of the streamline passing through the point $\theta = \pi/2$, r = 2a.

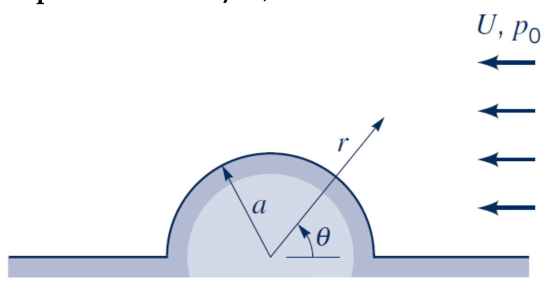


Fig. 8.

(a) On the surface of the hump,

$$p_s = p_0 + \frac{1}{2}\rho U^2 (1 - 4\sin^2\theta) \qquad (Eq.\ 6.116)$$

The maximum pressure occurs where $\sin\theta = 0$, or at $\theta = 0, \pi$, and at these points

$$\underline{\underline{p_s(max) = p_0 + \frac{1}{2}\rho U^2}} \qquad \underline{\underline{(at\ \theta = 0\ and\ \pi)}}$$

The minimum pressure occurs where $\sin\theta = 1$, or at $\theta = \frac{\pi}{2}$, and at this point

$$\underline{\underline{p_s(min) = p_0 - \frac{3}{2}\rho U^2}} \qquad \underline{\underline{(at\ \theta = \frac{\pi}{2})}}$$

(b) For uniform flow in the <u>negative</u> x-direction,

$$\psi = -Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta$$

(refer to discussion associated with the derivation of Eq. 6.112).

At $\theta = \frac{\pi}{2}$, $r = 2a$

$$\psi = -2aU\left(1 - \frac{a^2}{(2a)^2}\right)\sin\frac{\pi}{2} = -\frac{3}{2}aU$$

and thus the equation of the streamline passing through this point is

$$-\frac{3}{2}aU = -Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta$$

or

$$\frac{2}{3}\frac{r}{a}\left(1 - \frac{a^2}{r^2}\right)\sin\theta = 1$$

11. Water flows around a 6-ft-diameter bridge pier with a velocity of 12 ft/s. Estimate the force (per unit length) that the water exerts on the pier. Assume that the flow can be approximated as an ideal fluid flow around the front half of the cylinder, but due to flow separation, the average pressure on the rear half is constant and approximately equal to 1/2 the pressure at point A (see Fig. 9).

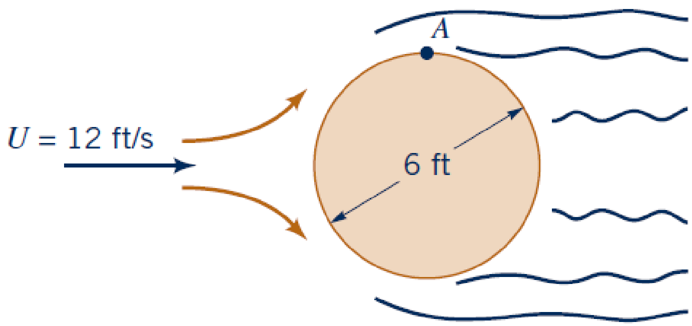


Fig. 9.

From Fig. 6.28 it follows that the drag on a section (between $\theta=0$ and $\theta=\alpha$) of a circular cylinder is given by the equation

$$\text{Drag} = F_x = -\int_0^{\alpha} p_s \cos\theta \, a \, d\theta$$

For the force on the front half of the cylinder (per unit length)

$$F_{x_1} = -2\int_{\pi/2}^{\pi} p_s \cos\theta \, a \, d\theta \qquad (1)$$

and due to symmetry $F_y = 0$. From Eq. 6.116

$$p_s = p_0 + \frac{1}{2}\rho U^2 \left(1 - 4\sin^2\theta\right) \qquad (\text{Eq. } 6.116)$$

and since we are only interested in the force due to the flowing fluid we will let $p_0 = 0$. Thus, from Eq. (1)

$$F_{x_1} = -2\int_{\pi/2}^{\pi} \frac{1}{2}\rho U^2 \left(1 - 4\sin^2\theta\right)\cos\theta \, a \, d\theta \qquad (2)$$

Since $$\int_{\pi/2}^{\pi} \cos\theta \, d\theta = \sin\theta\Big]_{\pi/2}^{\pi} = -1$$

and $$\int_{\pi/2}^{\pi} \sin^2\theta \cos\theta \, d\theta = \frac{\sin^3\theta}{3}\Big]_{\pi/2}^{\pi} = -\frac{1}{3}$$

It follows from Eq.(2) that

$$F_{x_1} = -\frac{\rho U^2 a}{3}$$

Note that the negative sign indicates that the water is actually "pulling" on the cylinder (front half) in the upstream direction. However, when the effect of the rear half of the cylinder is taken into account (in a real fluid) there will be a net drag in the direction of flow.

The pressure at the top of the cylinder (point A) is given by

$$p_s = p_0 + \frac{1}{2}\rho U^2 (1 - 4\sin^2\theta) \qquad \text{(Eq. 6. 116)}$$

and with $\theta = \pi/2$

$$p_A = p_0 - \frac{3}{2}\rho U^2$$

Since $p_0 = 0$

$$p_A = -\frac{3}{2}\rho U^2$$

Note that the negative pressure will give a positive F_x and

$$F_{x_2} = -\frac{p_A}{2} \times \text{projected area} = -\frac{p_A}{2} \times 2a(1)$$

So that

$$F_{x_2} = \frac{3}{4}\rho U^2 (2a)(1) = \frac{3}{2}\rho U^2 a$$

Thus,

$$F_x = F_{x_1} + F_{x_2}$$

$$= -\frac{\rho U^2 a}{3} + \frac{3\rho U^2 a}{2}$$

$$= \frac{7}{6}\rho U^2 a$$

With the data given,

$$F_x = \frac{7}{6}\left(1.94\ \frac{\text{slugs}}{\text{ft}^3}\right)\left(12\ \frac{\text{ft}}{\text{s}}\right)^2 (3\ \text{ft}) = \underline{\underline{978\ \frac{\text{lb}}{\text{ft}}}}$$

12. Consider the steady potential flow around the circular cylinder shown in Fig. 10. On a plot show the variation of the magnitude of the dimensionless fluid velocity, V/U, along the positive y axis. At what distance, y/a (along the y axis), is the velocity within 1% of the free-stream velocity?

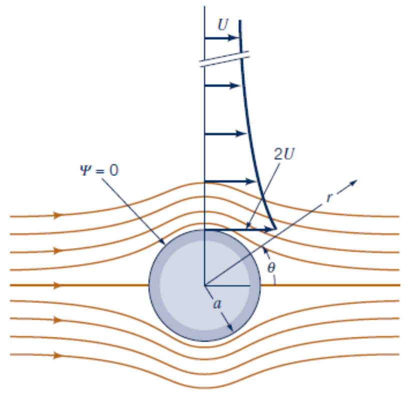


Fig. 10.

Along the y-axis $v_r = 0$ so that the magnitude of the velocity, V, is equal to $|v_\theta|$. Since

$$v_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta \qquad \text{(Eq. 6.115)}$$

it follows that along the positive y-axis ($\theta = \frac{\pi}{2}$, $r = y$)

$$V = |v_\theta| = U\left(1 + \frac{a^2}{y^2}\right)$$

or

$$\frac{V}{U} = 1 + \frac{a^2}{y^2} = 1 + \frac{1}{\left(\frac{y}{a}\right)^2} \qquad (1)$$

Tabulated data and a plot of the data are given below. It can be seen from these results that for

$$\underline{\frac{y}{a} \geq 10}$$

the velocity V is within 1% of the free-stream velocity U.

y/a	V/U
1.00	2.000
2.00	1.250
3.00	1.111
4.00	1.063
5.00	1.040
6.00	1.028
7.00	1.020
8.00	1.016
9.00	1.012
10.00	1.010

Calculated from Eq. (1)