SPC 407
Supersonic & Hypersonic Fluid Dynamics

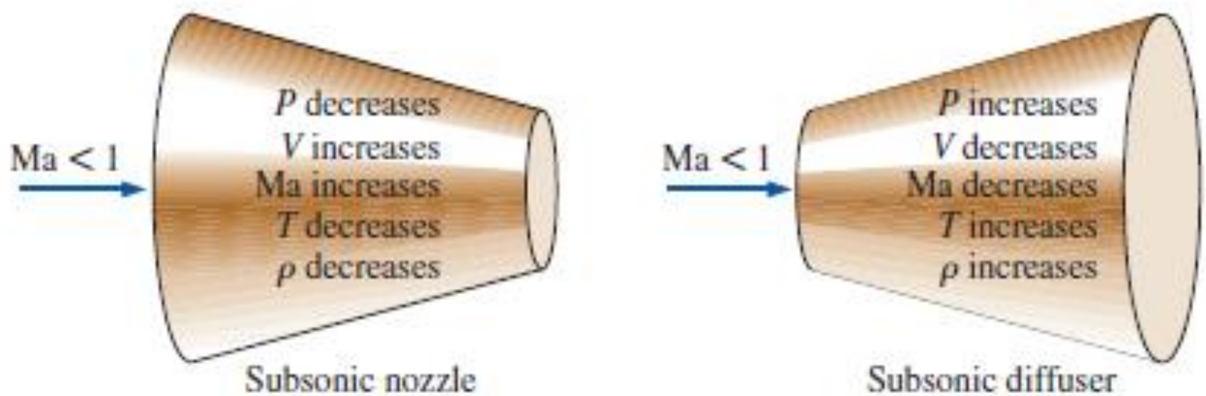
Lecture 3

October 16, 2016

Ahmed Mohamed Nagib Elmekawy, PhD, P.E.

Governing Equations

- Flow inside Nozzle and Diffuser



(a) Subsonic flow

Governing Equations

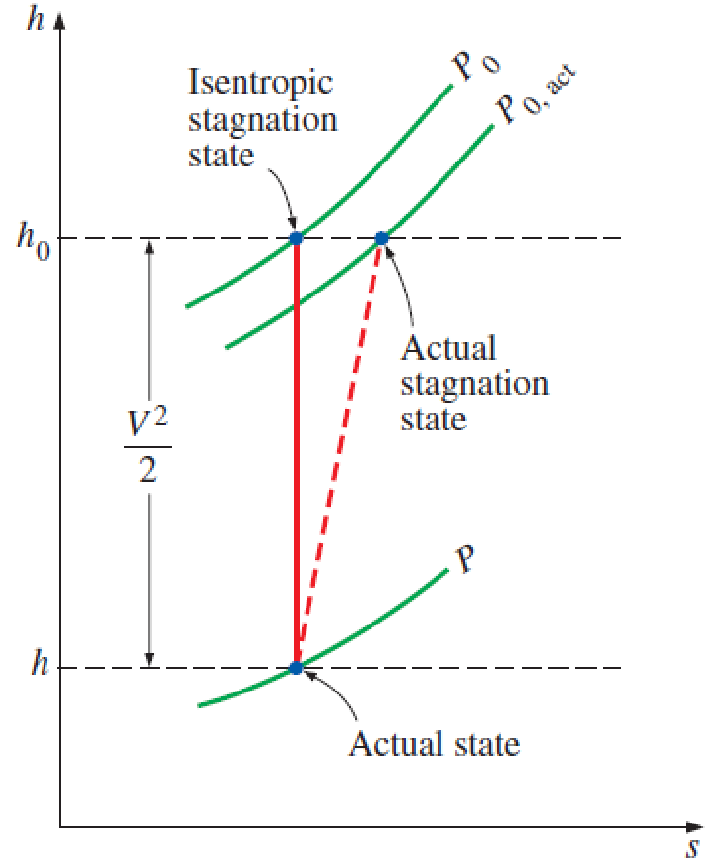
- Enthalpy

$$h = u + P/\rho$$

- Stagnation Properties
  - Stagnation Enthalpy

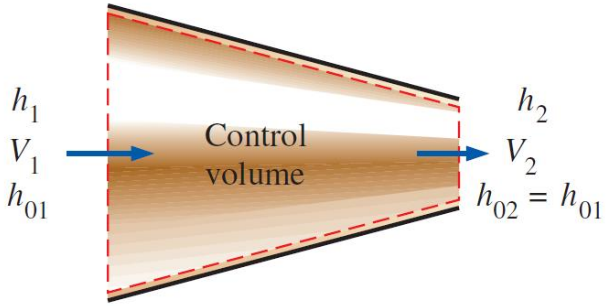
$$h_0 = h + \frac{V^2}{2} \qquad \text{(kJ/kg)}$$

  - Stagnation Pressure P_o
  - Stagnation Temperature T_o



Governing Equations

Consider the steady flow of a fluid through a duct such as a nozzle, diffuser, or some other flow passage where the flow takes place adiabatically and with no shaft or electrical work, as shown in Fig



Assuming the fluid experiences little or no change in its elevation

$$\dot{E}_{\text{in}} = \dot{E}_{\text{out}}$$

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

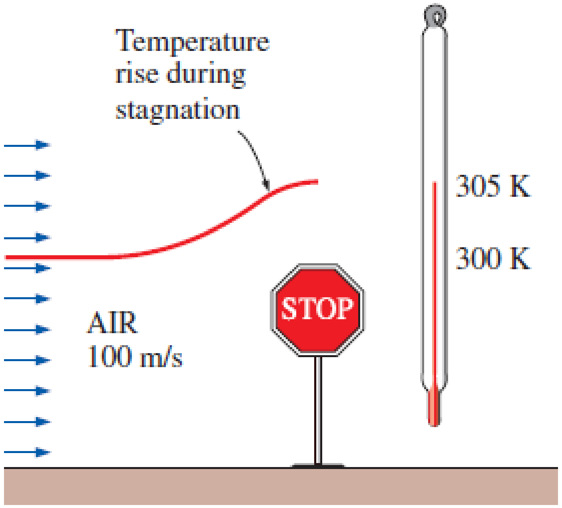
$$h_{01} = h_{02}$$

Governing Equations

When the fluid is approximated as an *ideal gas* with constant specific heats, its enthalpy can be replaced by

$$c_p T_0 = c_p T + \frac{V^2}{2}$$

$$T_0 = T + \frac{V^2}{2c_p}$$



For ideal gases with constant specific heats

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{k/(k-1)}$$

Governing Equations

Then the energy balance for a single-stream, steady-flow device can be expressed as

$$\dot{E}_{\text{in}} = \dot{E}_{\text{out}}$$

$$q_{\text{in}} + w_{\text{in}} + (h_{01} + gz_1) = q_{\text{out}} + w_{\text{out}} + (h_{02} + gz_2)$$

where

$$h_o = C_p T_o$$

$$(q_{\text{in}} - q_{\text{out}}) + (w_{\text{in}} - w_{\text{out}}) = c_p(T_{02} - T_{01}) + g(z_2 - z_1)$$

Speed of sound

An important parameter in the study of compressible flow is the **speed of sound** c
For an ideal gas it simplifies to

$$c = \sqrt{\frac{dP}{d\rho}} = \sqrt{kRT}$$

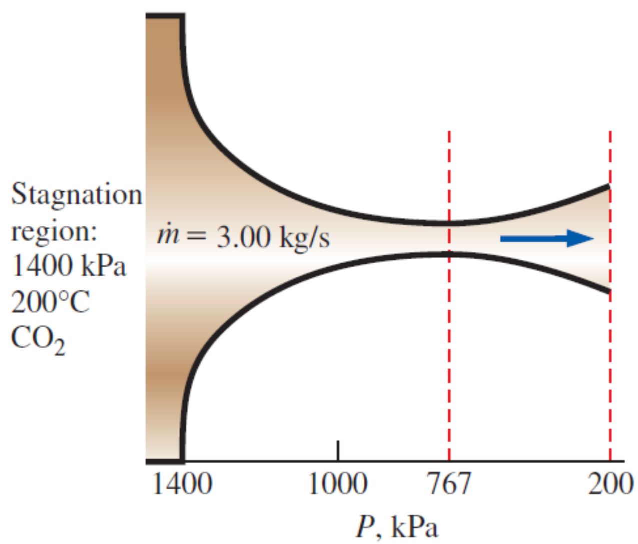
where k is the specific heat ratio of the gas and R is the specific gas constant. The ratio of the speed of the flow to the speed of sound is the dimensionless Mach number M

$$M = \frac{V}{c}$$

During fluid flow through many devices such as nozzles, diffusers, and turbine blade passages, flow quantities vary primarily in the flow direction only, and the flow can be approximated as one-dimensional isentropic flow with good accuracy.

Example Gas Flow through a Converging–Diverging Duct

Carbon dioxide flows steadily through a varying cross-sectional area duct such as a nozzle shown in Fig. 12–6 at a mass flow rate of 3.00 kg/s. The carbon dioxide enters the duct at a pressure of 1400 kPa and 200°C with a low velocity, and it expands in the nozzle to an exit pressure of 200 kPa. The duct is designed so that the flow can be approximated as isentropic. Determine the density, velocity, flow area, and Mach number at each location along the duct that corresponds to an overall pressure drop of 200 kPa.



Example Gas Flow through a Converging–Diverging Duct

SOLUTION Carbon dioxide enters a varying cross-sectional area duct at specified conditions. The flow properties are to be determined along the duct.

Assumptions **1** Carbon dioxide is an ideal gas with constant specific heats at room temperature. **2** Flow through the duct is steady, one-dimensional, and isentropic.

Properties For simplicity we use $c_p = 0.846$ kJ/kg·K and $k = 1.289$ throughout the calculations, which are the constant-pressure specific heat and specific heat ratio values of carbon dioxide at room temperature. The gas constant of carbon dioxide is $R = 0.1889$ kJ/kg·K.

Analysis We note that the inlet temperature is nearly equal to the stagnation temperature since the inlet velocity is small. The flow is isentropic, and thus the stagnation temperature and pressure throughout the duct remain constant. Therefore,

$$T_0 \cong T_1 = 200°\mathrm{C} = 473\ \mathrm{K}$$

and

$$P_0 \cong P_1 = 1400\ \mathrm{kPa}$$

Example Gas Flow through a Converging–Diverging Duct

To illustrate the solution procedure, we calculate the desired properties at the location where the pressure is 1200 kPa, the first location that corresponds to a pressure drop of 200 kPa.

From Eq. 12–5,

$$T = T_0\left(\frac{P}{P_0}\right)^{(k-1)/k} = (473\ \text{K})\left(\frac{1200\ \text{kPa}}{1400\ \text{kPa}}\right)^{(1.289-1)/1.289} = 457\ \text{K}$$

From Eq. 12–4,

$$V = \sqrt{2c_p(T_0 - T)}$$

$$= \sqrt{2(0.846\ \text{kJ/kg}\cdot\text{K})(473\ \text{K} - 457\ \text{K})\left(\frac{1000\ \text{m}^2/\text{s}^3}{1\ \text{kJ/kg}}\right)}$$

$$= 164.5\ \text{m/s} \cong \mathbf{164\ m/s}$$

From the ideal-gas relation,

$$\rho = \frac{P}{RT} = \frac{1200\ \text{kPa}}{(0.1889\ \text{kPa}\cdot\text{m}^3/\text{kg}\cdot\text{K})(457\ \text{K})} = \mathbf{13.9\ kg/m^3}$$

Example Gas Flow through a Converging–Diverging Duct

From the mass flow rate relation,

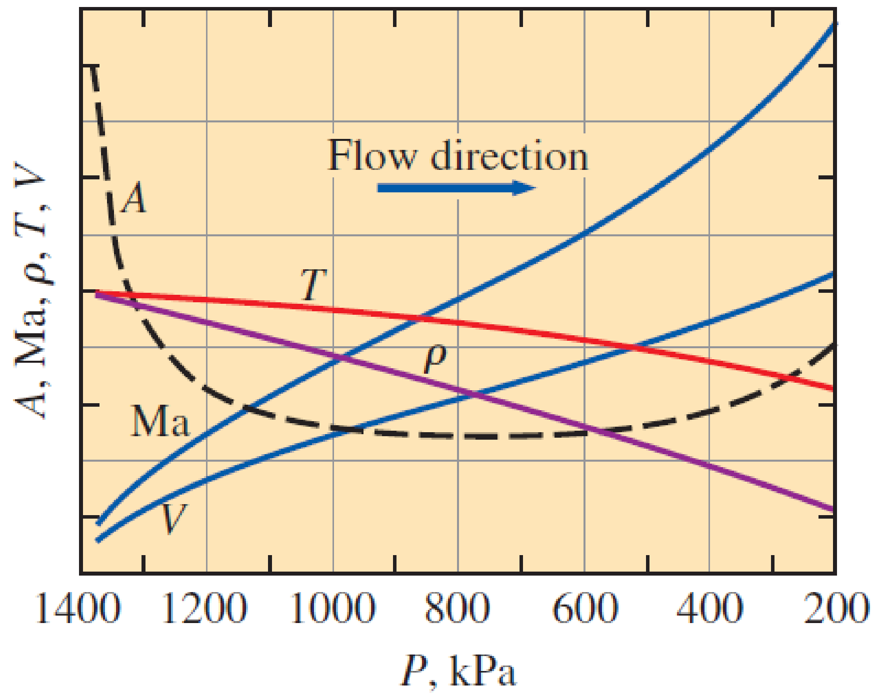
$$A = \frac{\dot{m}}{\rho V} = \frac{3.00 \text{ kg/s}}{(13.9 \text{ kg/m}^3)(164.5 \text{ m/s})} = 13.1 \times 10^{-4} \text{ m}^2 = \mathbf{13.1 \text{ cm}^2}$$

From Eqs. 12–11 and 12–12,

$$c = \sqrt{kRT} = \sqrt{(1.289)(0.1889 \text{ kJ/kg·K})(457 \text{ K})\left(\frac{1000 \text{ m}^2/\text{s}^2}{1 \text{ kJ/kg}}\right)} = 333.6 \text{ m/s}$$

$$\text{Ma} = \frac{V}{c} = \frac{164.5 \text{ m/s}}{333.6 \text{ m/s}} = \mathbf{0.493}$$

Example Gas Flow through a Converging–Diverging Duct



Variation of fluid properties in flow direction in the duct described in Example 12–2 for $\dot{m}$ = 3 kg/s = constant

P, kPa	T, K	V, m/s	ρ, kg/m^3	c, m/s	A, cm^2	Ma
1400	473	0	15.7	339.4	∞	0
1200	457	164.5	13.9	333.6	13.1	0.493
1000	439	240.7	12.1	326.9	10.3	0.736
800	417	306.6	10.1	318.8	9.64	0.962
767*	413	317.2	9.82	317.2	9.63	1.000
600	391	371.4	8.12	308.7	10.0	1.203
400	357	441.9	5.93	295.0	11.5	1.498
200	306	530.9	3.46	272.9	16.3	1.946

Variation of Fluid Velocity with Flow Area

We will study couplings among the velocity, density, and flow areas for isentropic duct flow.

Consider the mass balance for a steady-flow process:

$$\dot{m} = \rho A V$$

Differentiating and dividing the resultant equation by the mass flow rate, we obtain

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

Consider the conservation of Energy for a steady isentropic flow process, and by Neglecting the potential energy, the energy balance for an isentropic flow with no work interactions is expressed:

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

$$h + \frac{V^2}{2} = const.$$

Differentiate $$dh + V\,dV = 0$$

Variation of Fluid Velocity with Flow Area

The Second law of thermodynamic:

$$T\,ds = dh - v\,dP$$

For isentropic flow $ds = 0$,

$$dh = v\,dP = \frac{1}{\rho}\,dP$$

Substitute in Eq $dh + V\,dV = 0$

$$\frac{dP}{\rho} + V\,dV = 0$$

Combine with Eq and rearrange

$$\frac{dA}{A} = \frac{dP}{\rho}\left(\frac{1}{V^2} - \frac{d\rho}{dP}\right)$$

$$\frac{dA}{A} = \frac{dP}{\rho V^2}\left(1 - V^2\frac{d\rho}{dP}\right)$$

$$c = \sqrt{\frac{dP}{d\rho}}$$

Variation of Fluid Velocity with Flow Area

$$\frac{dA}{A} = \frac{dP}{\rho V^2}(1 - M^2)$$

This is an important relation for isentropic flow in ducts since it describes the variation of pressure with flow area. We note that *A*, r, and *V* are positive quantities. For *subsonic* flow (M <1), the term $1 - M^2$ is positive; and thus *dA* and *dP* must have the same sign. That is, the pressure of the fluid must increase as the flow area of the duct increases and must decrease as the flow area of the duct decreases. Thus, at subsonic velocities, the pressure decreases in converging ducts (subsonic nozzles) and increases in diverging ducts (subsonic diffusers).

In *supersonic* flow (M >1), the term $1 - M^2$ is negative, and thus *dA* and *dP* must have opposite signs. That is, the pressure of the fluid must increase as the flow area of the duct decreases and must decrease as the flow area of the duct increases. Thus, at supersonic velocities, the pressure decreases in diverging ducts (supersonic nozzles) and increases in converging ducts (supersonic diffusers).

Variation of Fluid Velocity with Flow Area

Another important relation for the isentropic flow of a fluid is obtained by substituting $\rho V = -\frac{dP}{dV}$

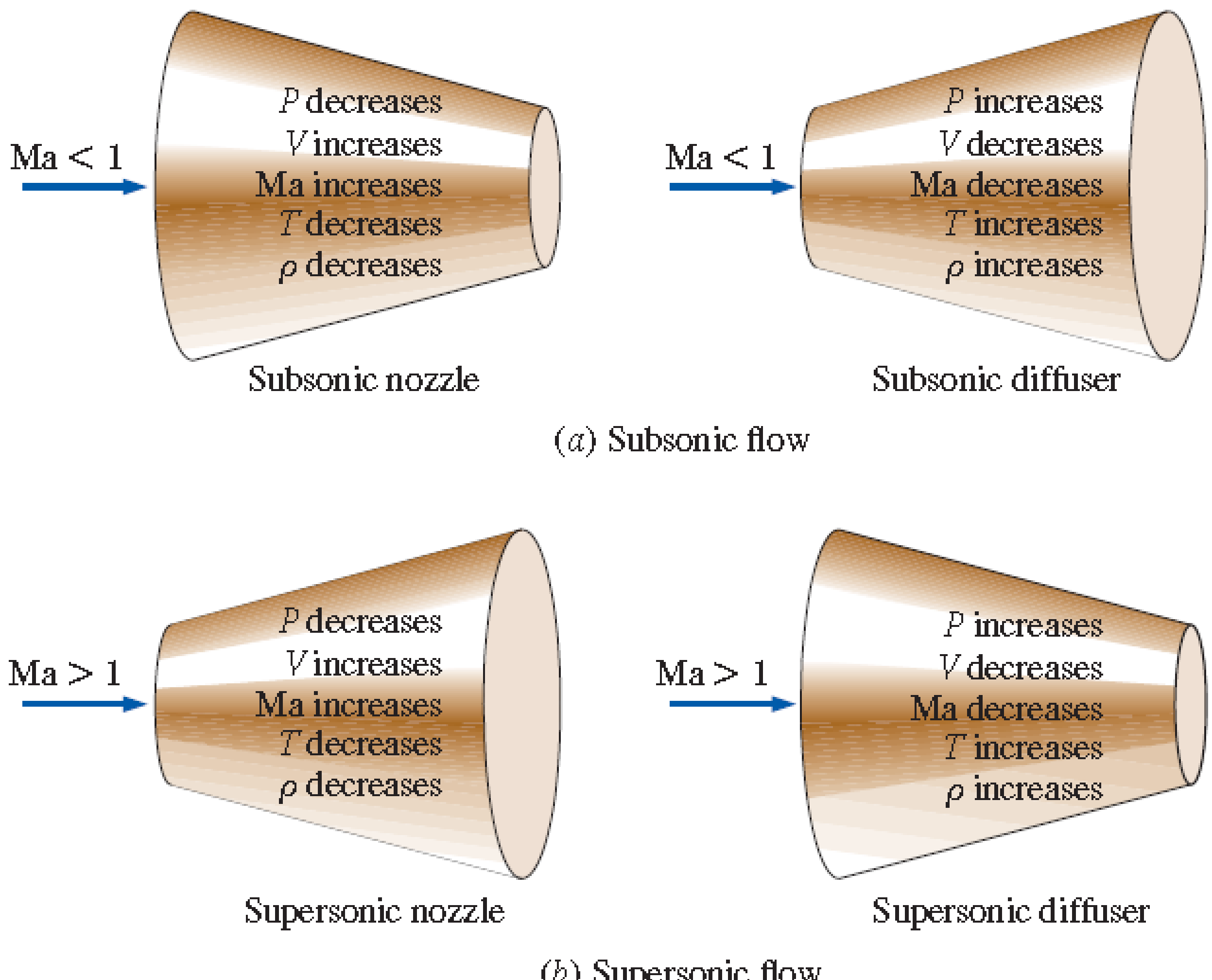
$$\frac{dA}{A} = -\frac{dV}{V}(1 - M^2)$$

This equation governs the shape of a nozzle or a diffuser in subsonic or supersonic isentropic flow. Noting that *A* and *V* are positive quantities, we conclude the following:

For subsonic flow (Ma < 1), $\frac{dA}{dV}$<0

For supersonic flow (M > 1), $\frac{dA}{dV}$<0

Property Relations for Isentropic Flow of Ideal Gases



(a) Subsonic flow

(b) Supersonic flow

The temperature T of an ideal gas anywhere in the flow is related to the stagnation temperature T_0 through Eq. 12–4:

$$T_0 = T + \frac{V^2}{2c_p}$$

or

$$\frac{T_0}{T} = 1 + \frac{V^2}{2c_pT}$$

Noting that $c_p = kR/(k - 1)$, $c^2 = kRT$, and $\mathrm{Ma} = V/c$, we see that

$$\frac{V^2}{2c_pT} = \frac{V^2}{2[kR/(k-1)]T} = \left(\frac{k-1}{2}\right)\frac{V^2}{c^2} = \left(\frac{k-1}{2}\right)\mathrm{Ma}^2$$

Substitution yields

$$\frac{T_0}{T} = 1 + \left(\frac{k-1}{2}\right)\mathrm{Ma}^2 \tag{12–18}$$

which is the desired relation between T_0 and T.

The ratio of the stagnation to static pressure is obtained by substituting Eq. 12–18 into Eq. 12–5:

$$\frac{P_0}{P} = \left[1 + \left(\frac{k-1}{2}\right)\mathrm{Ma}^2\right]^{k/(k-1)} \tag{12–19}$$

The ratio of the stagnation to static density is obtained by substituting Eq. 12–18 into Eq. 12–6:

$$\frac{\rho_0}{\rho} = \left[1 + \left(\frac{k-1}{2}\right)\mathrm{Ma}^2\right]^{1/(k-1)} \tag{12–20}$$

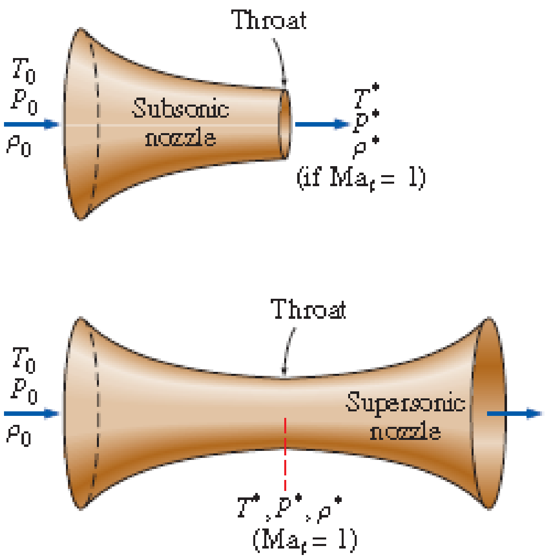


FIGURE 12–12
When $\mathrm{Ma}_t = 1$, the properties at the nozzle throat are the critical properties.

Numerical values of T/T_0, P/P_0, and ρ/ρ_0 are listed versus the Mach number in Table A–13 for $k = 1.4$, which are very useful for practical compressible flow calculations involving air.

The properties of a fluid at a location where the Mach number is unity (the throat) are called **critical properties**, and the ratios in Eqs. (12–18) through (12–20) are called **critical ratios** when Ma = 1 (Fig. 12–12). It is standard practice in the analysis of compressible flow to let the superscript asterisk (*) represent the critical values. Setting Ma = 1 in Eqs. 12–18 through 12–20 yields

$$\frac{T^*}{T_0} = \frac{2}{k+1} \qquad \textbf{(12–21)}$$

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)} \qquad \textbf{(12–22)}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{1/(k-1)} \qquad \textbf{(12–23)}$$

These ratios are evaluated for various values of k and are listed in Table 12–2. The critical properties of compressible flow should not be confused with the thermodynamic properties of substances at the *critical point* (such as the critical temperature T_c and critical pressure P_c).

$$\text{Ma}^* = \text{Ma}\sqrt{\frac{k+1}{2+(k-1)\text{Ma}^2}}$$

$$\frac{A}{A^*} = \frac{1}{\text{Ma}}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}\text{Ma}^2\right)\right]^{0.5(k+1)/(k-1)}$$

$$\frac{P}{P_0} = \left(1+\frac{k-1}{2}\text{Ma}^2\right)^{-k/(k-1)}$$

$$\frac{\rho}{\rho_0} = \left(1+\frac{k-1}{2}\text{Ma}^2\right)^{-1/(k-1)}$$

$$\frac{T}{T_0} = \left(1+\frac{k-1}{2}\text{Ma}^2\right)^{-1}$$

TABLE A–13

One-dimensional isentropic compressible flow functions for an ideal gas with $k = 1.4$

Ma	Ma*	A/A^*	P/P_0	ρ/ρ_0	T/T_0
0	0	∞	1.0000	1.0000	1.0000
0.1	0.1094	5.8218	0.9930	0.9950	0.9980
0.2	0.2182	2.9635	0.9725	0.9803	0.9921
0.3	0.3257	2.0351	0.9395	0.9564	0.9823
0.4	0.4313	1.5901	0.8956	0.9243	0.9690
0.5	0.5345	1.3398	0.8430	0.8852	0.9524
0.6	0.6348	1.1882	0.7840	0.8405	0.9328
0.7	0.7318	1.0944	0.7209	0.7916	0.9107
0.8	0.8251	1.0382	0.6560	0.7400	0.8865
0.9	0.9146	1.0089	0.5913	0.6870	0.8606
1.0	1.0000	1.0000	0.5283	0.6339	0.8333
1.2	1.1583	1.0304	0.4124	0.5311	0.7764
1.4	1.2999	1.1149	0.3142	0.4374	0.7184
1.6	1.4254	1.2502	0.2353	0.3557	0.6614
1.8	1.5360	1.4390	0.1740	0.2868	0.6068
2.0	1.6330	1.6875	0.1278	0.2300	0.5556
2.2	1.7179	2.0050	0.0935	0.1841	0.5081
2.4	1.7922	2.4031	0.0684	0.1472	0.4647
2.6	1.8571	2.8960	0.0501	0.1179	0.4252
2.8	1.9140	3.5001	0.0368	0.0946	0.3894
3.0	1.9640	4.2346	0.0272	0.0760	0.3571
5.0	2.2361	25.000	0.0019	0.0113	0.1667
∞	2.2495	∞	0	0	0

$$\mathrm{Ma}^* = \mathrm{Ma}\sqrt{\frac{k+1}{2+(k-1)\mathrm{Ma}^2}}$$

$$\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)\right]^{0.5(k+1)/(k-1)}$$

$$\frac{P}{P_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{-k/(k-1)}$$

$$\frac{\rho}{\rho_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{-1/(k-1)}$$

$$\frac{T}{T_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{-1}$$

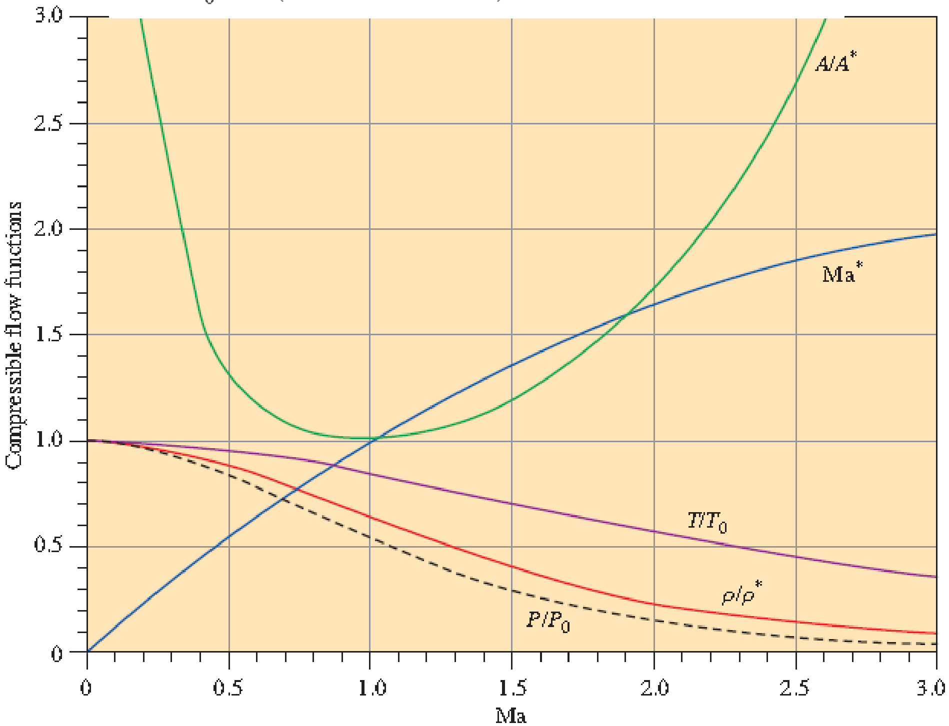


TABLE A–13

One-dimensional isentropic compressible flow functions for an ideal gas with $k = 1.4$

Ma	Ma*	A/A^*	P/P_0	ρ/ρ_0	T/T_0
0	0	∞	1.0000	1.0000	1.0000
0.1	0.1094	5.8218	0.9930	0.9950	0.9980
0.2	0.2182	2.9635	0.9725	0.9803	0.9921
0.3	0.3257	2.0351	0.9395	0.9564	0.9823
0.4	0.4313	1.5901	0.8956	0.9243	0.9690
0.5	0.5345	1.3398	0.8430	0.8852	0.9524
0.6	0.6348	1.1882	0.7840	0.8405	0.9328
0.7	0.7318	1.0944	0.7209	0.7916	0.9107
0.8	0.8251	1.0382	0.6560	0.7400	0.8865
0.9	0.9146	1.0089	0.5913	0.6870	0.8606
1.0	1.0000	1.0000	0.5283	0.6339	0.8333
1.2	1.1583	1.0304	0.4124	0.5311	0.7764
1.4	1.2999	1.1149	0.3142	0.4374	0.7184
1.6	1.4254	1.2502	0.2353	0.3557	0.6614
1.8	1.5360	1.4390	0.1740	0.2868	0.6068
2.0	1.6330	1.6875	0.1278	0.2300	0.5556
2.2	1.7179	2.0050	0.0935	0.1841	0.5081
2.4	1.7922	2.4031	0.0684	0.1472	0.4647
2.6	1.8571	2.8960	0.0501	0.1179	0.4252
2.8	1.9140	3.5001	0.0368	0.0946	0.3894
3.0	1.9640	4.2346	0.0272	0.0760	0.3571
5.0	2.2361	25.000	0.0019	0.0113	0.1667
∞	2.2495	∞	0	0	0

Property Relations for Isentropic Flow of Ideal Gases

TABLE 12–2

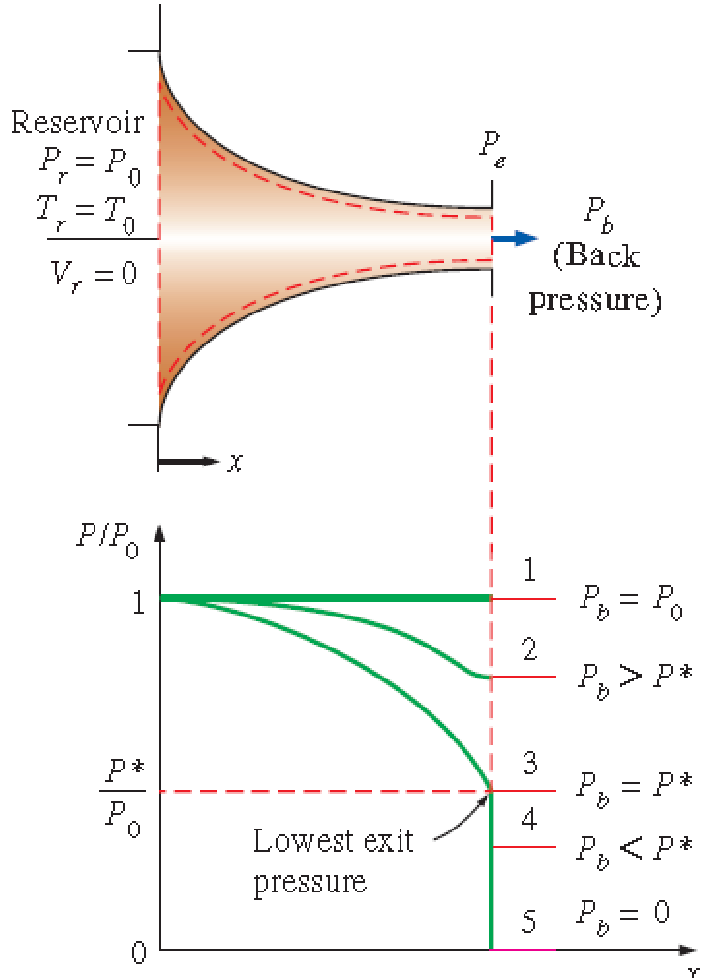
The critical-pressure, critical-temperature, and critical-density ratios for isentropic flow of some ideal gases

	Superheated steam, $k = 1.3$	Hot products of combustion, $k = 1.33$	Air, $k = 1.4$	Monatomic gases, $k = 1.667$
$\frac{P^*}{P_0}$	0.5457	0.5404	0.5283	0.4871
$\frac{T^*}{T_0}$	0.8696	0.8584	0.8333	0.7499
$\frac{\rho^*}{\rho_0}$	0.6276	0.6295	0.6340	0.6495

ISENTROPIC FLOW THROUGH NOZZLES

The effect of back pressure on the pressure distribution along a converging nozzle:

$$P_e = \begin{cases} P_b & \text{for } P_b \geq P^* \\ P^* & \text{for } P_b < P^* \end{cases}$$



ISENTROPIC FLOW THROUGH NOZZLES

Under steady-flow conditions, the mass flow rate through the nozzle is constant and is expressed as

$$\dot{m} = \rho A V = \left(\frac{P}{RT}\right) A\left(\mathrm{Ma}\sqrt{kRT}\right) = PA\mathrm{Ma}\sqrt{\frac{k}{RT}}$$

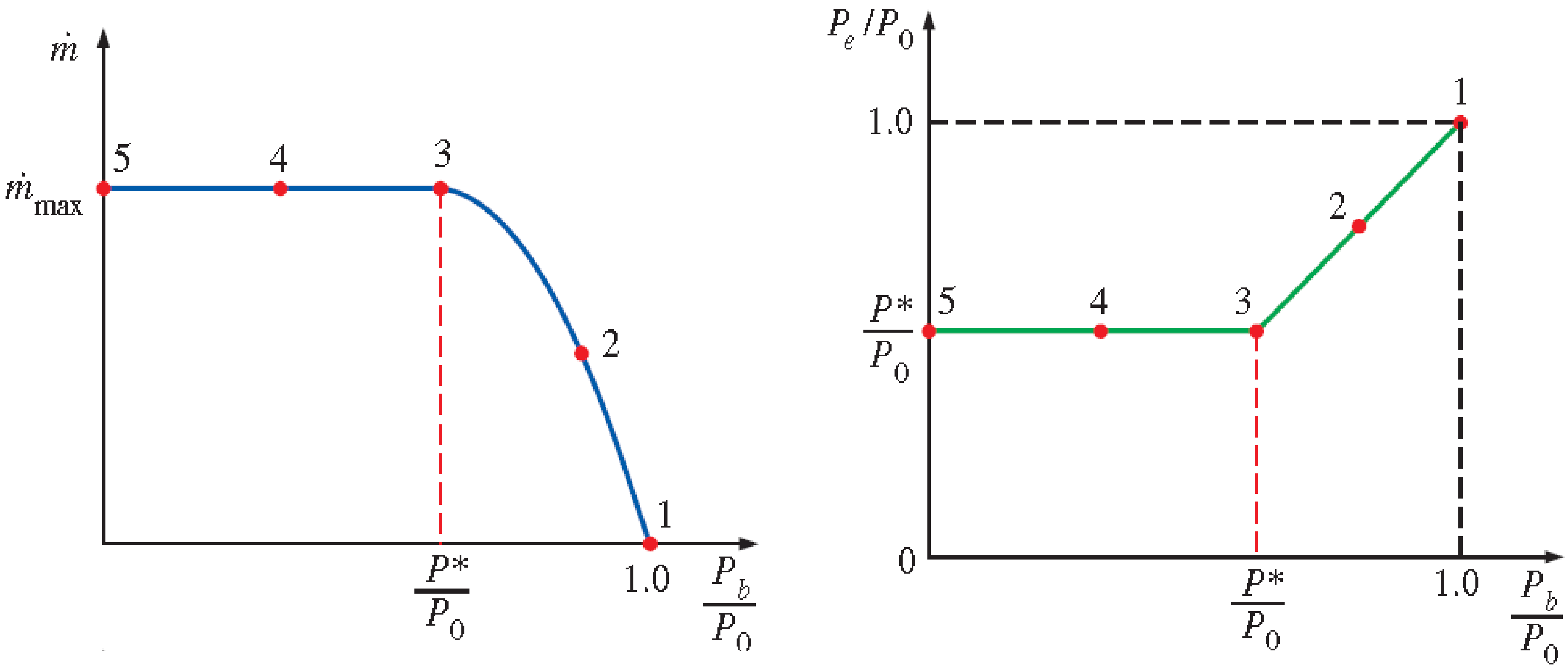
Solving for T from Eq. 12–18 and for P from Eq. 12–19 and substituting,

$$\dot{m} = \frac{A\mathrm{Ma}P_0\sqrt{k/(RT_0)}}{[1 + (k - 1)\mathrm{Ma}^2/2]^{(k+1)/[2(k-1)]}} \tag{12–24}$$

$$\dot{m}_{\max} = A^* P_0\sqrt{\frac{k}{RT_0}}\left(\frac{2}{k + 1}\right)^{(k+1)/[2(k-1)]}$$

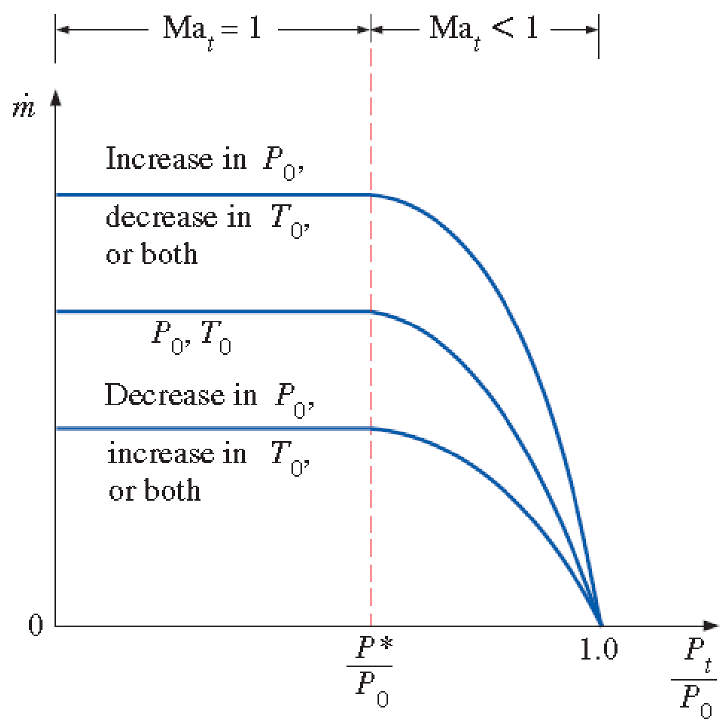
ISENTROPIC FLOW THROUGH NOZZLES

The effect of back pressure P_b on the mass flow rate $\dot{m}$ and the exit pressure P_e of a converging nozzle.



ISENTROPIC FLOW THROUGH NOZZLES

The variation of the mass flow rate through a nozzle with inlet stagnation properties.



A relation for the variation of flow area A through the nozzle relative to throat area A^* can be obtained by combining Eqs. 12–24 and 12–25 for the same mass flow rate and stagnation properties of a particular fluid. This yields

$$\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2}\mathrm{Ma}^2\right)\right]^{(k+1)/[2(k-1)]} \qquad \text{(12–26)}$$

Table **A**–13 gives values of A/A^* as a function of the Mach number for air ($k = 1.4$). There is one value of A/A^* for each value of the Mach number, but there are two possible values of the Mach number for each value of A/A^*—one for subsonic flow and another for supersonic flow.

Another parameter sometimes used in the analysis of one-dimensional isentropic flow of ideal gases is Ma^*, which is the ratio of the local velocity to the speed of sound at the throat:

$$\mathrm{Ma}^* = \frac{V}{c^*} \qquad \text{(12–27)}$$

Equation 12–27 can also be expressed as

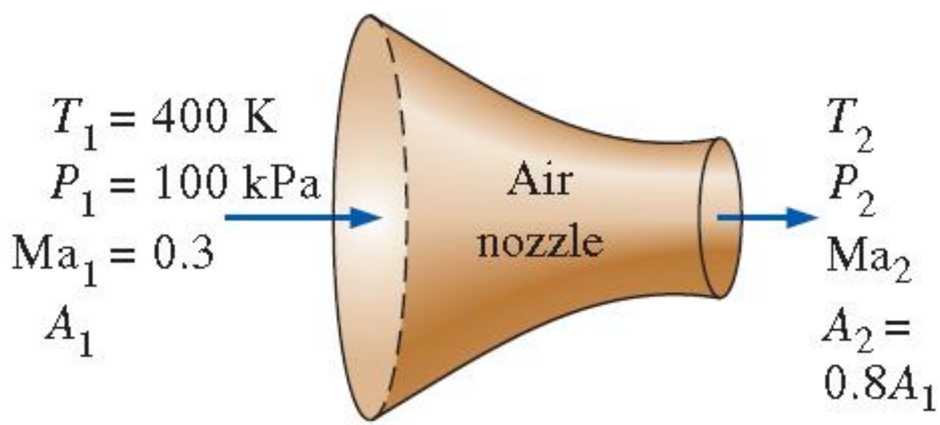
$$\mathrm{Ma}^* = \frac{V}{c}\frac{c}{c^*} = \frac{\mathrm{Ma}\, c}{c^*} = \frac{\mathrm{Ma}\sqrt{kRT}}{\sqrt{kRT^*}} = \mathrm{Ma}\sqrt{\frac{T}{T^*}}$$

$$\mathrm{Ma}^* = \mathrm{Ma}\sqrt{\frac{k+1}{2 + (k-1)\mathrm{Ma}^2}} \qquad \text{(12–28)}$$

Example: Effect of Back Pressure on Mass Flow Rate

EXAMPLE 12–4 Effect of Back Pressure on Mass Flow Rate

Air at 1 MPa and 600°C enters a converging nozzle, shown in Fig. 12–18, with a velocity of 150 m/s. Determine the mass flow rate through the nozzle for a nozzle throat area of 50 cm^2 when the back pressure is (*a*) 0.7 MPa and (*b*) 0.4 MPa.



Example: Effect of Back Pressure on Mass Flow Rate

SOLUTION Air enters a converging nozzle. The mass flow rate of air through the nozzle is to be determined for different back pressures.

Assumptions **1** Air is an ideal gas with constant specific heats at room temperature. **2** Flow through the nozzle is steady, one-dimensional, and isentropic.

Properties The constant pressure specific heat and the specific heat ratio of air are $c_p = 1.005$ kJ/kg·K and k = 1.4.

Analysis We use the subscripts *i* and *t* to represent the properties at the nozzle inlet and the throat, respectively. The stagnation temperature and pressure at the nozzle inlet are determined from Eqs. 12–4 and 12–5:

$$T_{0i} = T_i + \frac{V_i^2}{2c_p} = 873\text{ K} + \frac{(150\text{ m/s})^2}{2(1.005\text{ kJ/kg}\cdot\text{K})}\left(\frac{1\text{ kJ/kg}}{1000\text{ m}^2/\text{s}^2}\right) = 884\text{ K}$$

$$P_{0i} = P_i\left(\frac{T_{0i}}{T_i}\right)^{k/(k-1)} = (1\text{ MPa})\left(\frac{884\text{ K}}{873\text{ K}}\right)^{1.4/(1.4-1)} = 1.045\text{ MPa}$$

These stagnation temperature and pressure values remain constant throughout the nozzle since the flow is assumed to be isentropic. That is,

$$T_0 = T_{0i} = 884\text{ K} \quad \text{and} \quad P_0 = P_{0i} = 1.045\text{ MPa}$$

$$\mathrm{Ma}^* = \mathrm{Ma}\sqrt{\frac{k+1}{2+(k-1)\mathrm{Ma}^2}}$$

$$\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)\right]^{0.5(k+1)/(k-1)}$$

$$\frac{P}{P_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{-k/(k-1)}$$

$$\frac{\rho}{\rho_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{-1/(k-1)}$$

$$\frac{T}{T_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{-1}$$

TABLE A–13

One-dimensional isentropic compressible flow functions for an ideal gas with $k = 1.4$

Ma	Ma*	A/A^*	P/P_0	ρ/ρ_0	T/T_0
0	0	∞	1.0000	1.0000	1.0000
0.1	0.1094	5.8218	0.9930	0.9950	0.9980
0.2	0.2182	2.9635	0.9725	0.9803	0.9921
0.3	0.3257	2.0351	0.9395	0.9564	0.9823
0.4	0.4313	1.5901	0.8956	0.9243	0.9690
0.5	0.5345	1.3398	0.8430	0.8852	0.9524
0.6	0.6348	1.1882	0.7840	0.8405	0.9328
0.7	0.7318	1.0944	0.7209	0.7916	0.9107
0.8	0.8251	1.0382	0.6560	0.7400	0.8865
0.9	0.9146	1.0089	0.5913	0.6870	0.8606
1.0	1.0000	1.0000	0.5283	0.6339	0.8333
1.2	1.1583	1.0304	0.4124	0.5311	0.7764
1.4	1.2999	1.1149	0.3142	0.4374	0.7184
1.6	1.4254	1.2502	0.2353	0.3557	0.6614
1.8	1.5360	1.4390	0.1740	0.2868	0.6068
2.0	1.6330	1.6875	0.1278	0.2300	0.5556
2.2	1.7179	2.0050	0.0935	0.1841	0.5081
2.4	1.7922	2.4031	0.0684	0.1472	0.4647
2.6	1.8571	2.8960	0.0501	0.1179	0.4252
2.8	1.9140	3.5001	0.0368	0.0946	0.3894
3.0	1.9640	4.2346	0.0272	0.0760	0.3571
5.0	2.2361	25.000	0.0019	0.0113	0.1667
∞	2.2495	∞	0	0	0

Example: Effect of Back Pressure on Mass Flow Rate

The critical-pressure ratio is determined from Table 12–2 (or Eq. 12–22) to be $P^*/P_0 = 0.5283$.

(*a*) The back pressure ratio for this case is

$$\frac{P_b}{P_0} = \frac{0.7\text{ MPa}}{1.045\text{ MPa}} = 0.670$$

which is greater than the critical-pressure ratio, 0.5283. Thus the exit plane pressure (or throat pressure P_t) is equal to the back pressure in this case. That is, $P_t = P_b = 0.7$ MPa, and $P_t/P_0 = 0.670$. Therefore, the flow is not choked. From Table A–13 at $P_t/P_0 = 0.670$, we read $\text{Ma}_t = 0.778$ and $T_t/T_0 = 0.892$.

Example: Effect of Back Pressure on Mass Flow Rate

The mass flow rate through the nozzle can be calculated from Eq. 12–24. But it can also be determined in a step-by-step manner as follows:

$$T_t = 0.892T_0 = 0.892(884\text{ K}) = 788.5\text{ K}$$

$$\rho_t = \frac{P_t}{RT_t} = \frac{700\text{ kPa}}{(0.287\text{ kPa}\cdot\text{m}^3/\text{kg}\cdot\text{K})(788.5\text{ K})} = 3.093\text{ kg/m}^3$$

$$V_t = \text{Ma}_t c_t = \text{Ma}_t\sqrt{kRT_t}$$

$$= (0.778)\sqrt{(1.4)(0.287\text{ kJ/kg}\cdot\text{K})(788.5\text{ K})\left(\frac{1000\text{ m}^2/\text{s}^2}{1\text{ kJ/kg}}\right)}$$

$$= 437.9\text{ m/s}$$

Thus,

$$\dot{m} = \rho_t A_t V_t = (3.093\text{ kg/m}^3)(50 \times 10^{-4}\text{ m}^2)(437.9\text{ m/s}) = \mathbf{6.77\text{ kg/s}}$$

Example: Effect of Back Pressure on Mass Flow Rate

(*b*) The back pressure ratio for this case is

$$\frac{P_b}{P_0} = \frac{0.4 \text{ MPa}}{1.045 \text{ MPa}} = 0.383$$

which is less than the critical-pressure ratio, 0.5283. Therefore, sonic conditions exist at the exit plane (throat) of the nozzle, and Ma = 1. The flow is choked in this case, and the mass flow rate through the nozzle is calculated from Eq. 12–25:

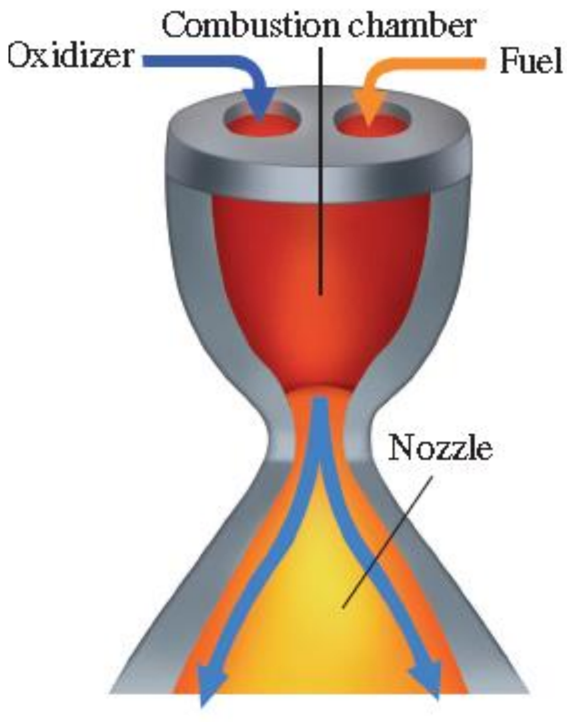
$$\dot{m} = A^* P_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$= (50 \times 10^{-4} \text{ m}^2)(1045 \text{ kPa}) \sqrt{\frac{1.4}{(0.287 \text{ kJ/kg·K})(884 \text{ K})}} \left(\frac{2}{1.4+1}\right)^{2.4/0.8}$$

$$= \mathbf{7.10 \text{ kg/s}}$$

ISENTROPIC FLOW THROUGH NOZZLES

The effect of back pressure on the pressure distribution along a converging divergent nozzle:



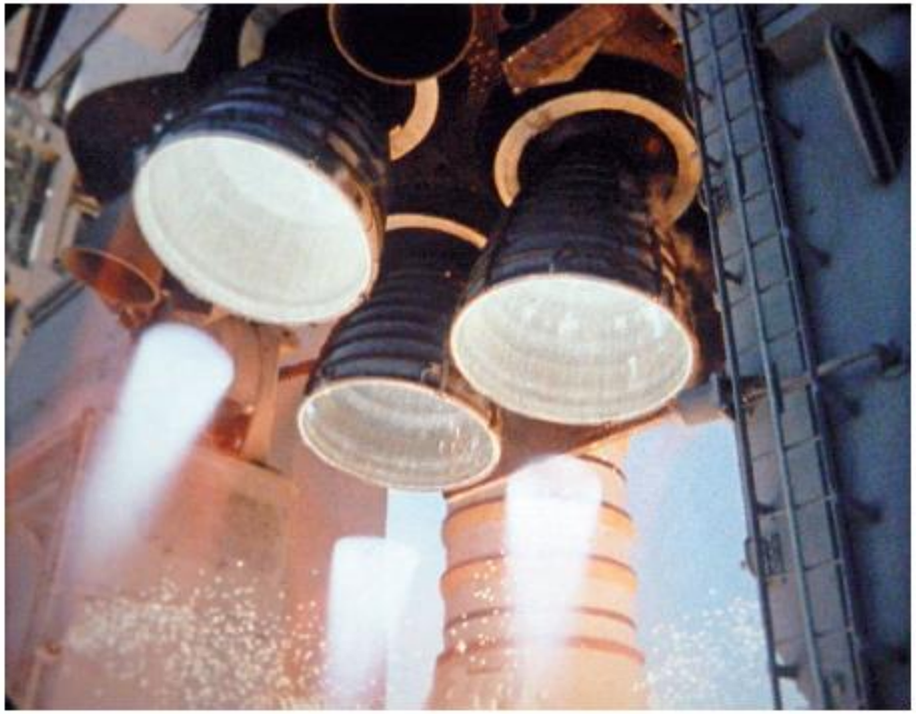
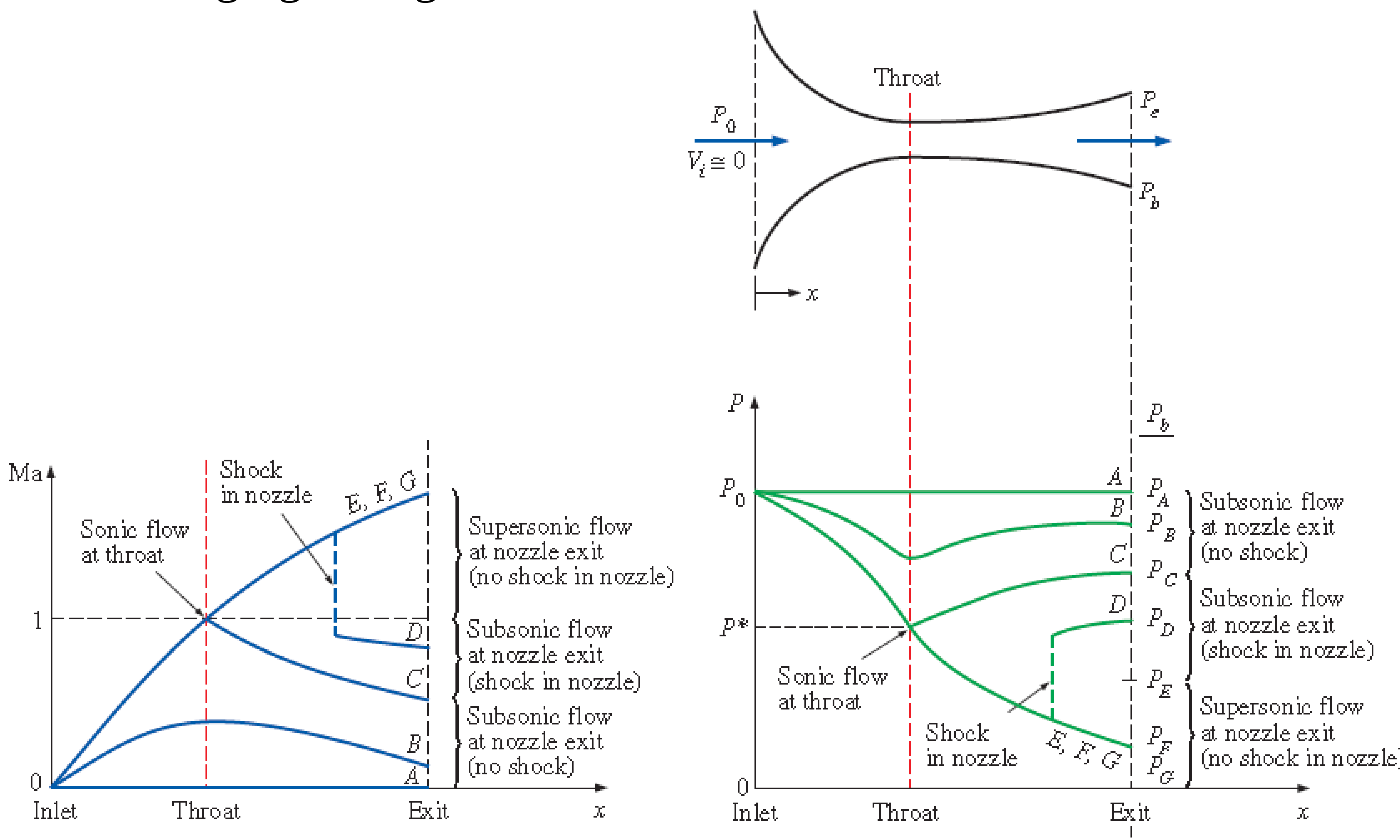


FIGURE 12–20
Converging–diverging nozzles are commonly used in rocket engines to provide high thrust.

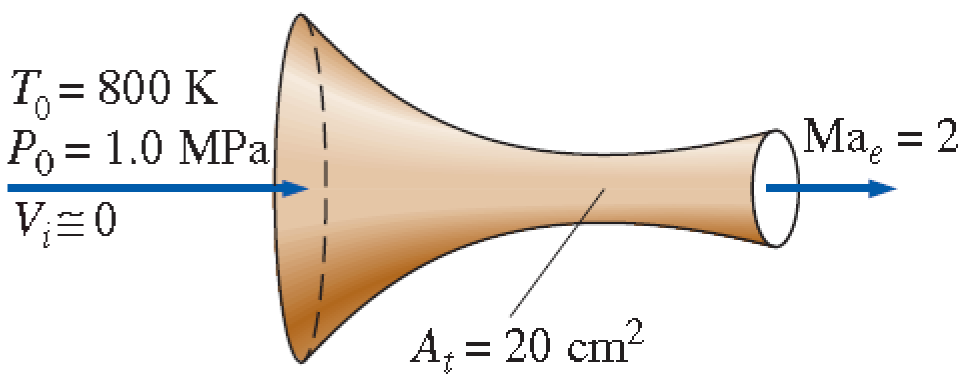
ISENTROPIC FLOW THROUGH NOZZLES

The effect of back pressure on the pressure distribution along a converging divergent nozzle:



Example 12-6 : Airflow through a Converging–Diverging Nozzle

Air enters a converging–diverging nozzle, shown in Fig. 12–22, at 1.0 MPa and 800 K with negligible velocity. The flow is steady, one-dimensional, and isentropic with $k = 1.4$. For an exit Mach number of Ma = 2 and a throat area of 20 cm^2, determine (*a*) the throat conditions, (*b*) the exit plane conditions, including the exit area, and (*c*) the mass flow rate through the nozzle.



Example 12-6 : Airflow through a Converging–Diverging Nozzle

SOLUTION Air flows through a converging–diverging nozzle. The throat and the exit conditions and the mass flow rate are to be determined.

Assumptions **1** Air is an ideal gas with constant specific heats at room temperature. **2** Flow through the nozzle is steady, one-dimensional, and isentropic.

Properties The specific heat ratio of air is given to be $k = 1.4$. The gas constant of air is 0.287 kJ/kg·K.

Analysis The exit Mach number is given to be 2. Therefore, the flow must be sonic at the throat and supersonic in the diverging section of the nozzle. Since the inlet velocity is negligible, the stagnation pressure and stagnation temperature are the same as the inlet temperature and pressure, $P_0 = 1.0$ MPa and $T_0 = 800$ K. Assuming ideal-gas behavior, the stagnation density is

$$\text{Ma}^* = \text{Ma}\sqrt{\frac{k+1}{2+(k-1)\text{Ma}^2}}$$

$$\frac{A}{A^*} = \frac{1}{\text{Ma}}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}\text{Ma}^2\right)\right]^{0.5(k+1)/(k-1)}$$

$$\frac{P}{P_0} = \left(1+\frac{k-1}{2}\text{Ma}^2\right)^{-k/(k-1)}$$

$$\frac{\rho}{\rho_0} = \left(1+\frac{k-1}{2}\text{Ma}^2\right)^{-1/(k-1)}$$

$$\frac{T}{T_0} = \left(1+\frac{k-1}{2}\text{Ma}^2\right)^{-1}$$

TABLE A–13

One-dimensional isentropic compressible flow functions for an ideal gas with $k = 1.4$

Ma	Ma*	A/A^*	P/P_0	ρ/ρ_0	T/T_0
0	0	∞	1.0000	1.0000	1.0000
0.1	0.1094	5.8218	0.9930	0.9950	0.9980
0.2	0.2182	2.9635	0.9725	0.9803	0.9921
0.3	0.3257	2.0351	0.9395	0.9564	0.9823
0.4	0.4313	1.5901	0.8956	0.9243	0.9690
0.5	0.5345	1.3398	0.8430	0.8852	0.9524
0.6	0.6348	1.1882	0.7840	0.8405	0.9328
0.7	0.7318	1.0944	0.7209	0.7916	0.9107
0.8	0.8251	1.0382	0.6560	0.7400	0.8865
0.9	0.9146	1.0089	0.5913	0.6870	0.8606
1.0	1.0000	1.0000	0.5283	0.6339	0.8333
1.2	1.1583	1.0304	0.4124	0.5311	0.7764
1.4	1.2999	1.1149	0.3142	0.4374	0.7184
1.6	1.4254	1.2502	0.2353	0.3557	0.6614
1.8	1.5360	1.4390	0.1740	0.2868	0.6068
2.0	1.6330	1.6875	0.1278	0.2300	0.5556
2.2	1.7179	2.0050	0.0935	0.1841	0.5081
2.4	1.7922	2.4031	0.0684	0.1472	0.4647
2.6	1.8571	2.8960	0.0501	0.1179	0.4252
2.8	1.9140	3.5001	0.0368	0.0946	0.3894
3.0	1.9640	4.2346	0.0272	0.0760	0.3571
5.0	2.2361	25.000	0.0019	0.0113	0.1667
∞	2.2495	∞	0	0	0

Example 12-6 : Airflow through a Converging–Diverging Nozzle

(*a*) At the throat of the nozzle Ma = 1, and from Table A–13 we read

$$\frac{P^*}{P_0} = 0.5283 \qquad \frac{T^*}{T_0} = 0.8333 \qquad \frac{\rho^*}{\rho_0} = 0.6339$$

Thus,

$$P^* = 0.5283P_0 = (0.5283)(1.0\ \text{MPa}) = \mathbf{0.5283\ MPa}$$

$$T^* = 0.8333T_0 = (0.8333)(800\ \text{K}) = \mathbf{666.6\ K}$$

$$\rho^* = 0.6339\rho_0 = (0.6339)(4.355\ \text{kg/m}^3) = \mathbf{2.761\ kg/m^3}$$

Also,

$$V^* = c^* = \sqrt{kRT^*} = \sqrt{(1.4)(0.287\ \text{kJ/kg}\cdot\text{K})(666.6\ \text{K})\left(\frac{1000\ \text{m}^2/\text{s}^2}{1\ \text{kJ/kg}}\right)}$$

$$= \mathbf{517.5\ m/s}$$

Example 12-6 : Airflow through a Converging–Diverging Nozzle

(*b*) Since the flow is isentropic, the properties at the exit plane can also be calculated by using data from Table A–13. For Ma = 2 we read

$$\frac{P_e}{P_0} = 0.1278 \quad \frac{T_e}{T_0} = 0.5556 \quad \frac{\rho_e}{\rho_0} = 0.2300 \quad \mathrm{Ma}_e^* = 1.6330 \quad \frac{A_e}{A^*} = 1.6875$$

Thus,

$$P_e = 0.1278P_0 = (0.1278)(1.0\ \mathrm{MPa}) = \mathbf{0.1278\ MPa}$$

$$T_e = 0.5556T_0 = (0.5556)(800\ \mathrm{K}) = \mathbf{444.5\ K}$$

$$\rho_e = 0.2300\rho_0 = (0.2300)(4.355\ \mathrm{kg/m^3}) = \mathbf{1.002\ kg/m^3}$$

$$A_e = 1.6875A^* = (1.6875)(20\ \mathrm{cm^2}) = \mathbf{33.75\ cm^2}$$

Example 12-6 : Airflow through a Converging–Diverging Nozzle

and

$$V_e = \mathrm{Ma}_e^* c^* = (1.6330)(517.5 \text{ m/s}) = \mathbf{845.1 \text{ m/s}}$$

The nozzle exit velocity could also be determined from $V_e = \mathrm{Ma}_e c_e$, where c_e is the speed of sound at the exit conditions:

$$V_e = \mathrm{Ma}_e c_e = \mathrm{Ma}_e \sqrt{kRT_e} = 2\sqrt{(1.4)(0.287 \text{ kJ/kg}\cdot\text{K})(444.5 \text{ K})\left(\frac{1000 \text{ m}^2/\text{s}^2}{1 \text{ kJ/kg}}\right)}$$

$$= 845.2 \text{ m/s}$$

(c) Since the flow is steady, the mass flow rate of the fluid is the same at all sections of the nozzle. Thus it may be calculated by using properties at any cross section of the nozzle. Using the properties at the throat, we find that the mass flow rate is

$$\dot{m} = \rho^* A^* V^* = (2.761 \text{ kg/m}^3)(20 \times 10^{-4} \text{ m}^2)(517.5 \text{ m/s}) = \mathbf{2.86 \text{ kg/s}}$$

SHOCK WAVES AND EXPANSION WAVES

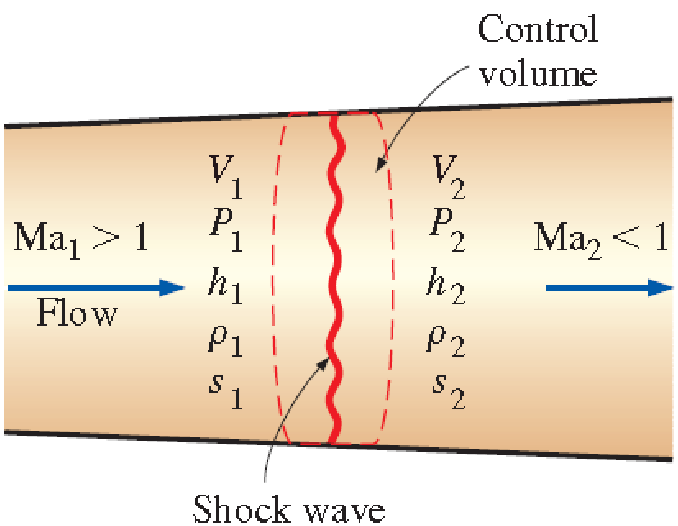


FIGURE 12–23
Control volume for flow across a normal shock wave.

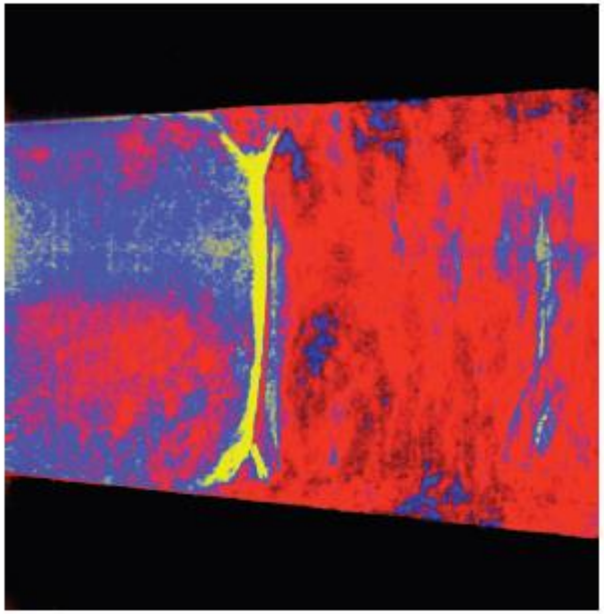


FIGURE 12–24
Schlieren image of a normal shock in a Laval nozzle. The Mach number in the nozzle just upstream (to the left) of the shock wave is about 1.3. Boundary layers distort the shape of the normal shock near the walls and lead to flow separation beneath the shock.