SPC 407

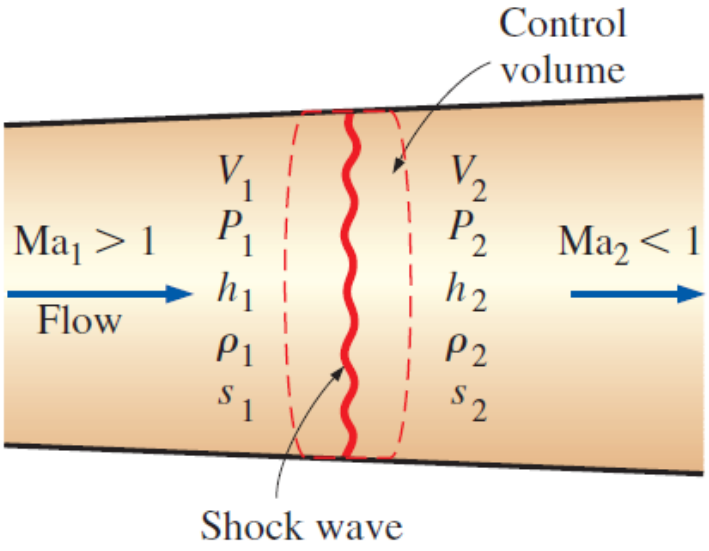
Supersonic & Hypersonic Fluid Dynamics

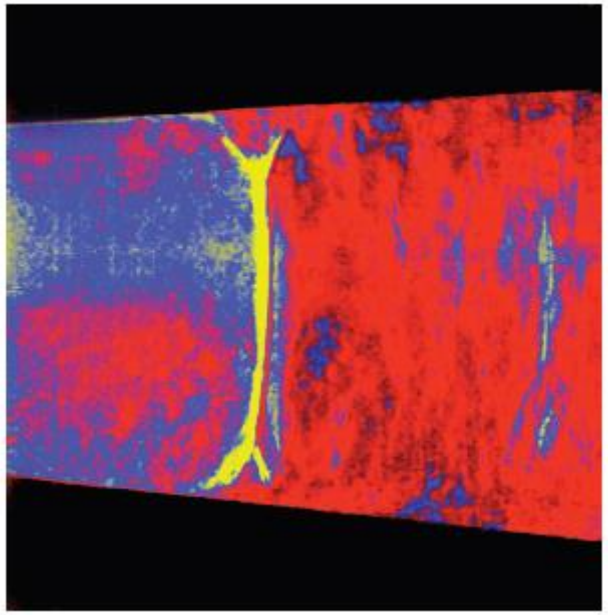
Lecture 5

October 30, 2016

Normal Shockwave

Consider the steady flow of a fluid through a duct such as a nozzle, diffuser, or some other flow passage where the flow takes place adiabatically and with no shaft or electrical work, as shown in Fig





Assuming the fluid experiences little or no change in its elevation

Governing Equations

Linear momentum equation:

$$A(P_1 - P_2) = \dot{m}(V_2 - V_1)$$

Second Law of Thermodynamic

$$s_2 - s_1 \geq 0$$

$$s_2 - s_1 = c_P \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$$

Normal Shockwave Equation Summary and tables

$$T_{01} = T_{02}$$

$$\mathrm{Ma}_2 = \sqrt{\frac{(k-1)\mathrm{Ma}_1^2 + 2}{2k\mathrm{Ma}_1^2 - k + 1}}$$

$$\frac{P_2}{P_1} = \frac{1 + k\mathrm{Ma}_1^2}{1 + k\mathrm{Ma}_2^2} = \frac{2k\mathrm{Ma}_1^2 - k + 1}{k+1}$$

$$\frac{\rho_2}{\rho_1} = \frac{P_2/P_1}{T_2/T_1} = \frac{(k+1)\mathrm{Ma}_1^2}{2 + (k-1)\mathrm{Ma}_1^2} = \frac{V_1}{V_2}$$

$$\frac{T_2}{T_1} = \frac{2 + \mathrm{Ma}_1^2(k-1)}{2 + \mathrm{Ma}_2^2(k-1)}$$

$$\frac{P_{02}}{P_{01}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left[\frac{1 + \mathrm{Ma}_2^2(k-1)/2}{1 + \mathrm{Ma}_1^2(k-1)/2}\right]^{(k+1)/[2(k-1)]}$$

$$\frac{P_{02}}{P_1} = \frac{(1 + k\mathrm{Ma}_1^2)[1 + \mathrm{Ma}_2^2(k-1)/2]^{k/(k-1)}}{1 + k\mathrm{Ma}_2^2}$$

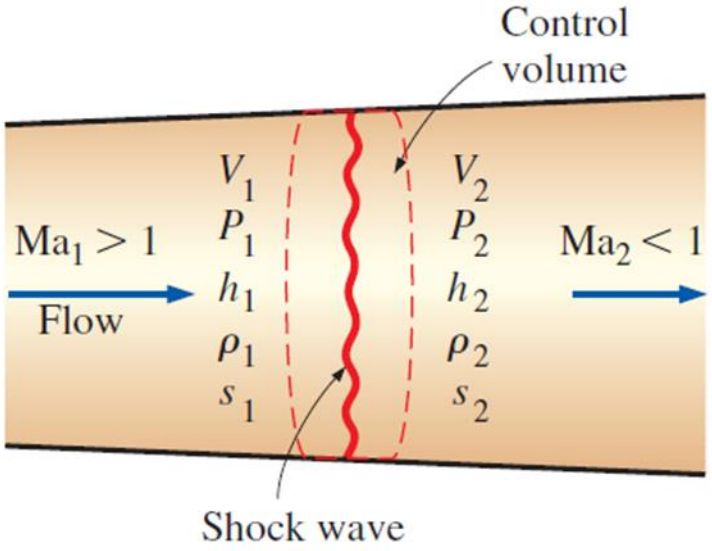


TABLE A–14

One-dimensional normal shock functions for an ideal gas with $k = 1.4$

Ma_1	Ma_2	P_2/P_1	ρ_2/ρ_1	T_2/T_1	P_{02}/P_{01}	P_{02}/P_1
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.8929
1.1	0.9118	1.2450	1.1691	1.0649	0.9989	2.1328
1.2	0.8422	1.5133	1.3416	1.1280	0.9928	2.4075
1.3	0.7860	1.8050	1.5157	1.1909	0.9794	2.7136
1.4	0.7397	2.1200	1.6897	1.2547	0.9582	3.0492
1.5	0.7011	2.4583	1.8621	1.3202	0.9298	3.4133
1.6	0.6684	2.8200	2.0317	1.3880	0.8952	3.8050
1.7	0.6405	3.2050	2.1977	1.4583	0.8557	4.2238
1.8	0.6165	3.6133	2.3592	1.5316	0.8127	4.6695
1.9	0.5956	4.0450	2.5157	1.6079	0.7674	5.1418
2.0	0.5774	4.5000	2.6667	1.6875	0.7209	5.6404
2.1	0.5613	4.9783	2.8119	1.7705	0.6742	6.1654
2.2	0.5471	5.4800	2.9512	1.8569	0.6281	6.7165
2.3	0.5344	6.0050	3.0845	1.9468	0.5833	7.2937
2.4	0.5231	6.5533	3.2119	2.0403	0.5401	7.8969
2.5	0.5130	7.1250	3.3333	2.1375	0.4990	8.5261
2.6	0.5039	7.7200	3.4490	2.2383	0.4601	9.1813
2.7	0.4956	8.3383	3.5590	2.3429	0.4236	9.8624
2.8	0.4882	8.9800	3.6636	2.4512	0.3895	10.5694
2.9	0.4814	9.6450	3.7629	2.5632	0.3577	11.3022
3.0	0.4752	10.3333	3.8571	2.6790	0.3283	12.0610
4.0	0.4350	18.5000	4.5714	4.0469	0.1388	21.0681
5.0	0.4152	29.000	5.0000	5.8000	0.0617	32.6335
∞	0.3780	∞	6.0000	∞	0	∞

Oblique Shockwave

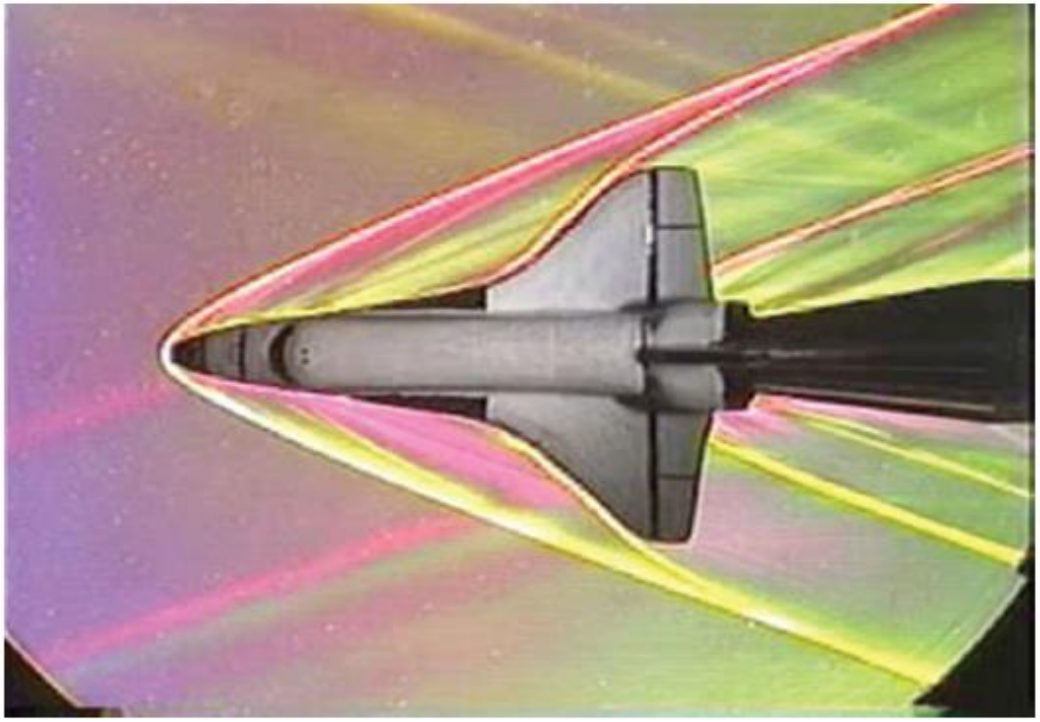


FIGURE 12–32
Schlieren image of a small model of the space shuttle orbiter being tested at Mach 3 in the supersonic wind tunnel of the Penn State Gas Dynamics Lab. Several *oblique shocks* are seen in the air surrounding the spacecraft.

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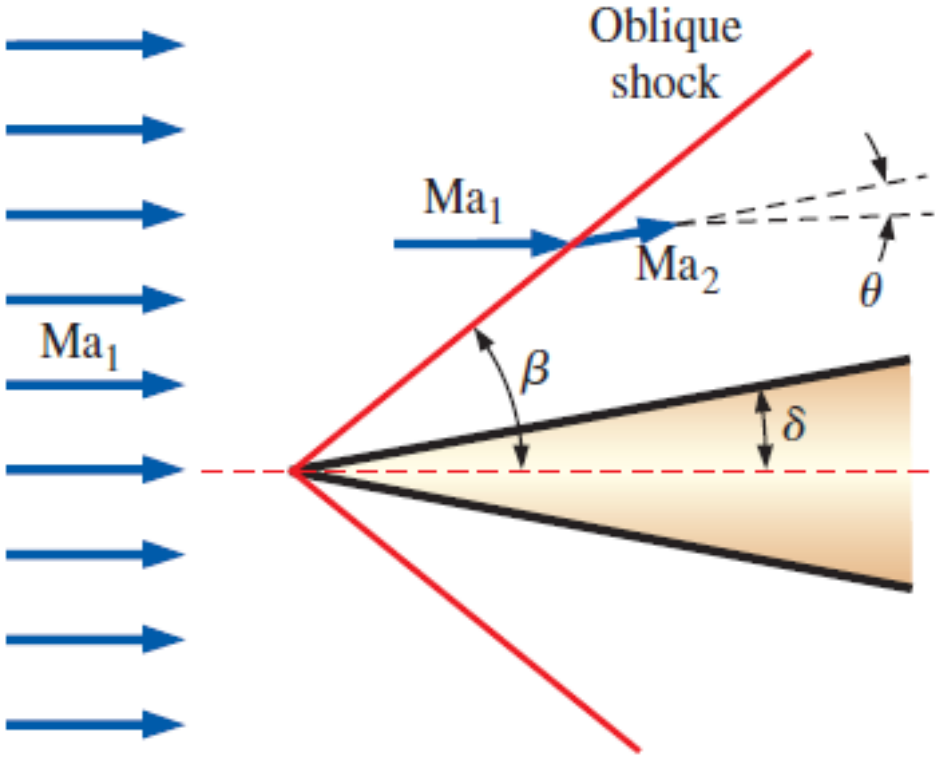
Oblique Shockwave



Oblique Shockwave

Like normal shocks, the Mach number decreases across an oblique shock, and oblique shocks are possible only if the upstream flow is supersonic.
However, unlike normal shocks, in which the downstream Mach number is always subsonic, M_{a2} downstream of an oblique shock can be subsonic, sonic, or supersonic, depending on the upstream Mach number Ma1 and the turning angle.

Oblique Shockwave

We analyze a straight oblique shock in Fig. by decomposing the velocity vectors upstream and downstream of the shock into normal and tangential components, and considering a small control volume around the shock.

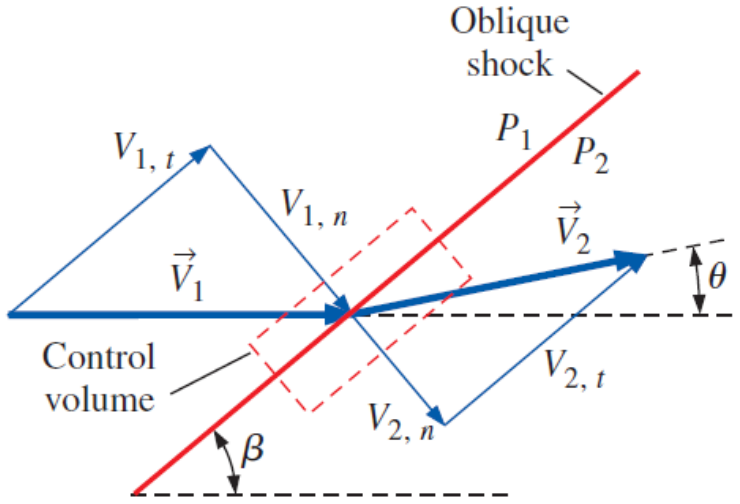


FIGURE 12–34

Velocity vectors through an oblique shock of shock angle β and deflection angle θ.

Governing Equations

Conservation of mass::

$$\rho_1 V_{1,n} A = \rho_2 V_{2,n} A \rightarrow \rho_1 V_{1,n} = \rho_2 V_{2,n}$$

by applying the tangential momentum equation to the control Volume, the tangential component of velocity (parallel to the oblique shock) does not change across the shock

$$V_{1,t} = V_{2,t}$$

Linear momentum equation:

$$A(P_1 - P_2) = \dot{m}(V_2 - V_1)$$

$$P_1 A - P_2 A = \rho V_{2,n} A V_{2,n} - \rho V_{1,n} A V_{1,n} \rightarrow P_1 - P_2 = \rho_2 V_{2,n}^2 - \rho_1 V_{1,n}^2$$

Governing Equations

Conservation of Energy:

$$h_{01} = h_{02} = h_0 \rightarrow h_1 + \frac{1}{2} V_{1,n}^2 + \frac{1}{2} V_{1,t}^2 = h_2 + \frac{1}{2} V_{2,n}^2 + \frac{1}{2} V_{2,t}^2$$

But since $V_{1,t} = V_{2,t}$, this equation reduces to

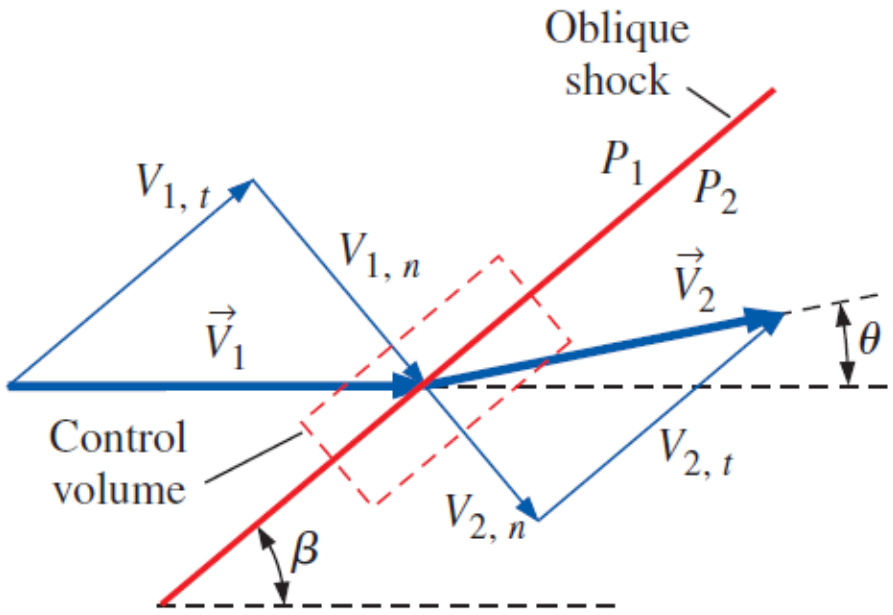
$$h_1 + \frac{1}{2} V_{1,n}^2 = h_2 + \frac{1}{2} V_{2,n}^2$$

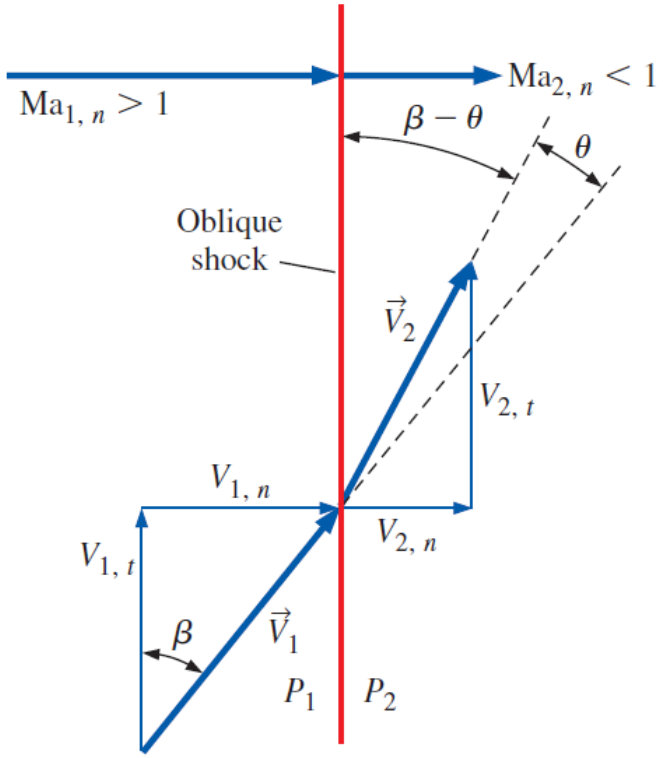
Governing Equations

Careful comparison reveals that the equations for conservation of mass, momentum, and energy across an oblique shock are identical to those across a normal shock, except that they are written in terms of the normal velocity component only. Therefore, the normal shock relations derived previously apply to oblique shocks as well, but must be written in terms of Mach numbers $M_{a1,\,n}$ and $M_{a2,\,n}$ normal to the oblique shock.

Governing Equations

This is most easily visualized by rotating the velocity vectors in Fig. 12–34 by angle $\pi/2$ - β, so that the oblique shock appears to be vertical





Trigonometry yields

$$\mathrm{Ma}_{1,n} = \mathrm{Ma}_1 \sin\beta \quad \text{and} \quad \mathrm{Ma}_{2,n} = \mathrm{Ma}_2 \sin(\beta - \theta)$$

$$\text{where } \mathrm{Ma}_{1,n} = V_{1,n}/c_1 \text{ and } \mathrm{Ma}_{2,n} = V_{2,n}/c_2.$$

Governing Equations

All the equations, shock tables, etc., for normal shocks apply to oblique shocks as well, provided that we use only the normal components of the Mach number.

In fact, you may think of normal shocks as special oblique shocks in which shock angle $\beta = \pi/2$, or 90 degree. We recognize immediately that an oblique shock can exist only if $M_{a1,\,n} > 1$ and $Ma_{2,\,n < 1}$. The normal shock equations appropriate for oblique shocks in an ideal gas are summarized
in terms of $M_{a1,\,n}$.

Governing Equations Summary

$$h_{01} = h_{02} \quad \rightarrow \quad T_{01} = T_{02}$$

$$\mathrm{Ma}_{2,n} = \sqrt{\frac{(k-1)\mathrm{Ma}_{1,n}^2 + 2}{2k\mathrm{Ma}_{1,n}^2 - k + 1}}$$

$$\frac{P_2}{P_1} = \frac{2k\,\mathrm{Ma}_{1,n}^2 - k + 1}{k+1}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_{1,n}}{V_{2,n}} = \frac{(k+1)\mathrm{Ma}_{1,n}^2}{2 + (k-1)\mathrm{Ma}_{1,n}^2}$$

$$\frac{T_2}{T_1} = [2 + (k-1)\mathrm{Ma}_{1,n}^2]\,\frac{2k\,\mathrm{Ma}_{1,n}^2 - k + 1}{(k+1)^2\mathrm{Ma}_{1,n}^2}$$

$$\frac{P_{02}}{P_{01}} = \left[\frac{(k+1)\mathrm{Ma}_{1,n}^2}{2+(k-1)\mathrm{Ma}_{1,n}^2}\right]^{k/(k-1)} \left[\frac{(k+1)}{2k\,\mathrm{Ma}_{1,n}^2 - k + 1}\right]^{1/(k-1)}$$

Governing Equations

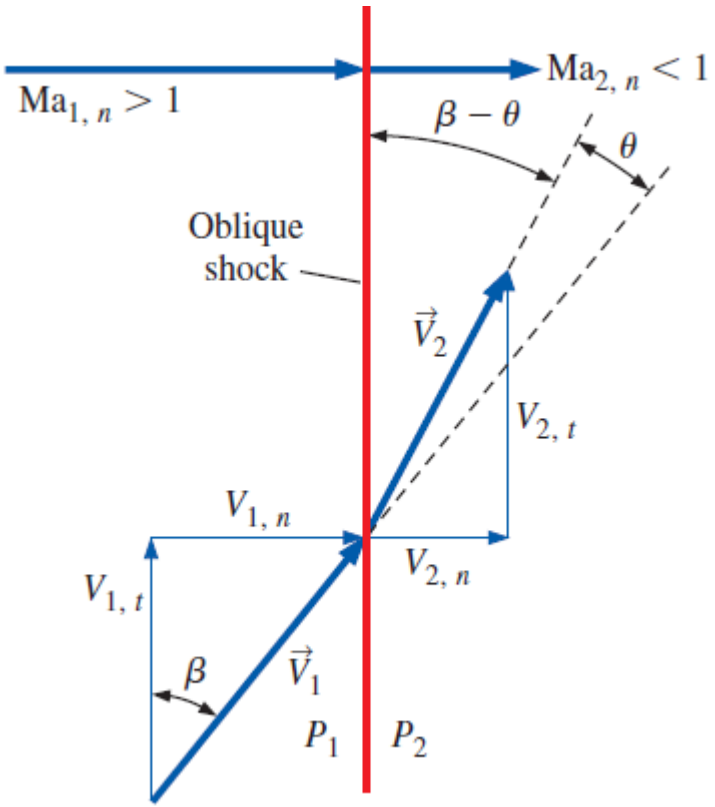
We begin by noting that

$$\tan \beta = V_{1,n}/V_{1,t}$$

$$\tan(\beta - \theta) = V_{2,n}/V_{2,t}$$

Since $V_{1,t} = V_{2,t}$, we combine these two expressions to yield

$$\frac{V_{2,n}}{V_{1,n}} = \frac{\tan(\beta - \theta)}{\tan \beta}$$



By Applying trigonometric identities

$$\cos 2\beta = \cos^2 \beta - \sin^2 \beta \quad \text{and} \quad \tan(\beta - \theta) = \frac{\tan \beta - \tan \theta}{1 + \tan \beta \tan \theta}$$

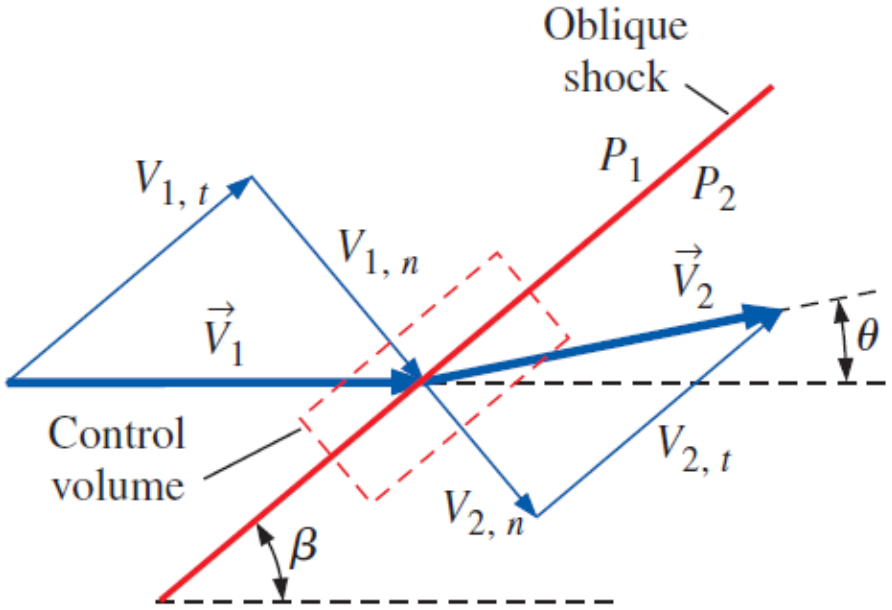
Governing Equations

We get

The θ-β-Ma relationship:

$$\tan\theta = \frac{2\cot\beta(\mathrm{Ma}_1^2\sin^2\beta - 1)}{\mathrm{Ma}_1^2(k + \cos 2\beta) + 2}$$

This Equation provides deflection angle θ as a unique function of shock angle β, specific heat ratio k, and upstream Mach number M_{a1}. For air (k = 1.4), we can plot θ versus β for several values of M_{a1}



Governing Equations

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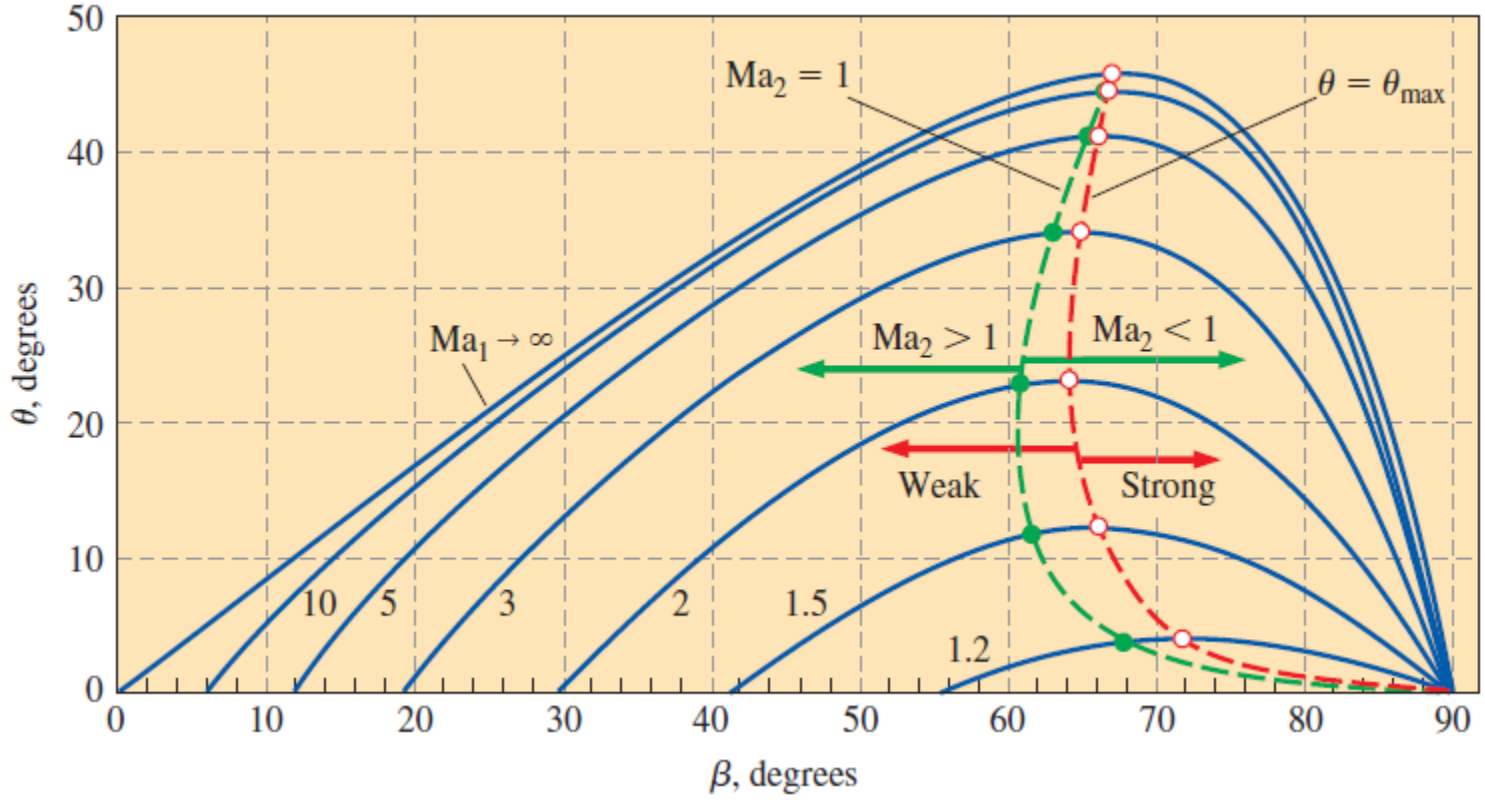

Governing Equations

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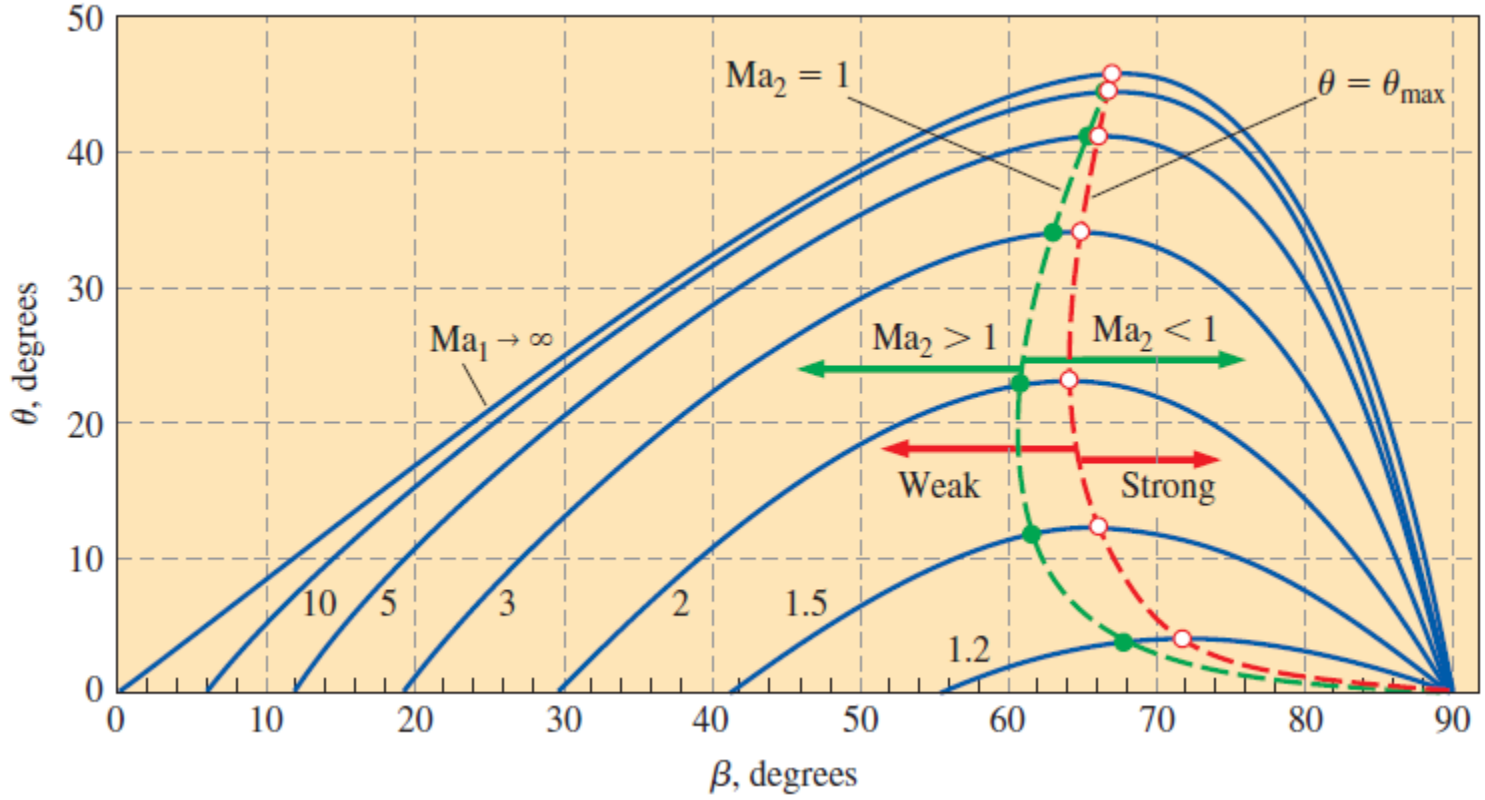
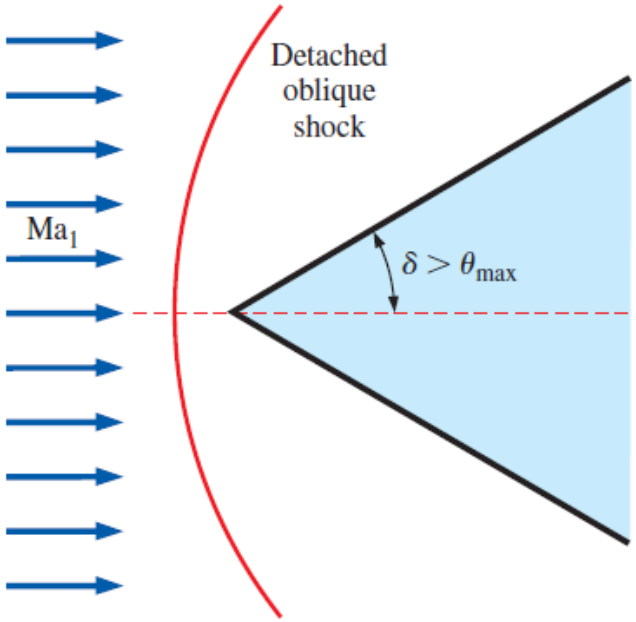
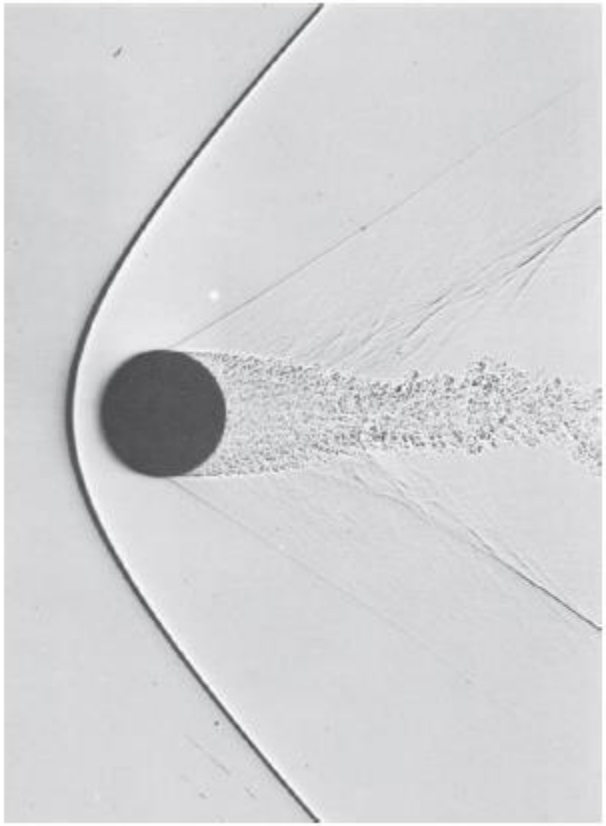


Governing Equations

If the wedge half-angle is greater than θ_{max}, the shock becomes curved and detaches from the nose of the wedge, forming what is called a detached oblique shock or a bow wave



A *detached oblique shock* occurs upstream of a two-dimensional wedge of half-angle δ when δ is greater than the maximum possible deflection angle θ. A shock of this kind is called a *bow wave* because of its resemblance to the water wave that forms at the bow of a ship.



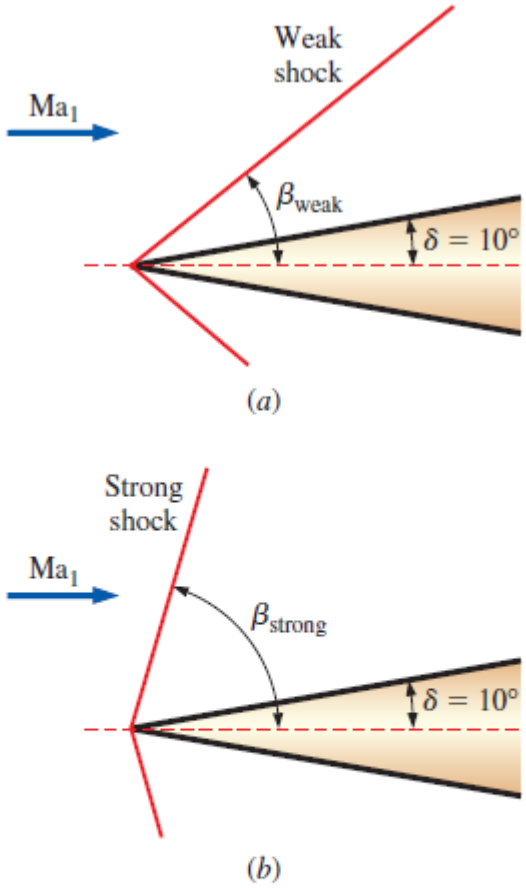


FIGURE 12–44

Two possible oblique shock angles, (*a*) β_{weak} and (*b*) β_{strong}, formed by a two-dimensional wedge of half-angle $\delta = 10°$.

Oblique Shock wave Example

Supersonic air at $Ma_1 = 2.0$ and 75.0 kPa impinges on a two-dimensional wedge of half-angle $\delta = 10°$ (Fig. 12–44). Calculate the two possible oblique shock angles, β_{weak} and β_{strong}, that could be formed by this wedge. For each case, calculate the pressure and Mach number downstream of the oblique shock, compare, and discuss.

Oblique Shock wave Example

SOLUTION We are to calculate the shock angle, Mach number, and pressure downstream of the weak and strong oblique shock formed by a two-dimensional wedge.

Assumptions **1** The flow is steady. **2** The boundary layer on the wedge is very thin.

Properties The fluid is air with $k = 1.4$.

Analysis Because of assumption 2, we approximate the oblique shock deflection angle as equal to the wedge half-angle, i.e., $\theta \cong \delta = 10°$. With $Ma_1 = 2.0$ and $\theta = 10°$, we solve Eq. 12–46 for the two possible values of oblique shock angle β: $\boldsymbol{\beta_{weak}} = \mathbf{39.3°}$ and $\boldsymbol{\beta_{strong}} = \mathbf{83.7°}$. From these values, we use the first part of Eq. 12–44 to calculate upstream normal Mach number $Ma_{1,\,n}$,

Weak shock: $$Ma_{1,n} = Ma_1 \sin\beta \rightarrow Ma_{1,n} = 2.0 \sin 39.3° = 1.267$$

and $$Ma_{2,\,n} = 0.8032$$

Strong shock: $$Ma_{1,n} = Ma_1 \sin\beta \rightarrow Ma_{1,n} = 2.0 \sin 83.7° = 1.988$$

$$Ma_{2,\,n} = 0.5794$$

Oblique Shock wave Example

Weak shock:

$$\frac{P_2}{P_1} = \frac{2k\,\mathrm{Ma}_{1,n}^2 - k + 1}{k + 1} \rightarrow P_2 = (75.0\ \mathrm{kPa})\frac{2(1.4)(1.267)^2 - 1.4 + 1}{1.4 + 1} = \mathbf{128\ kPa}$$

and

Strong shock:

$$\frac{P_2}{P_1} = \frac{2k\,\mathrm{Ma}_{1,n}^2 - k + 1}{k + 1} \rightarrow P_2 = (75.0\ \mathrm{kPa})\frac{2(1.4)(1.988)^2 - 1.4 + 1}{1.4 + 1} = \mathbf{333\ kPa}$$

Finally, we use the second part of Eq. 12–44 to calculate the downstream Mach number,

Weak shock:
$$\mathrm{Ma}_2 = \frac{\mathrm{Ma}_{2,n}}{\sin(\beta - \theta)} = \frac{0.8032}{\sin(39.3^\circ - 10^\circ)} = \mathbf{1.64}$$

and

Strong shock:
$$\mathrm{Ma}_2 = \frac{\mathrm{Ma}_{2,n}}{\sin(\beta - \theta)} = \frac{0.5794}{\sin(83.7^\circ - 10^\circ)} = \mathbf{0.604}$$