SPC 407

Supersonic & Hypersonic Fluid Dynamics

Lecture 8

December 4, 2016

Prandtl–Meyer Expansion Waves

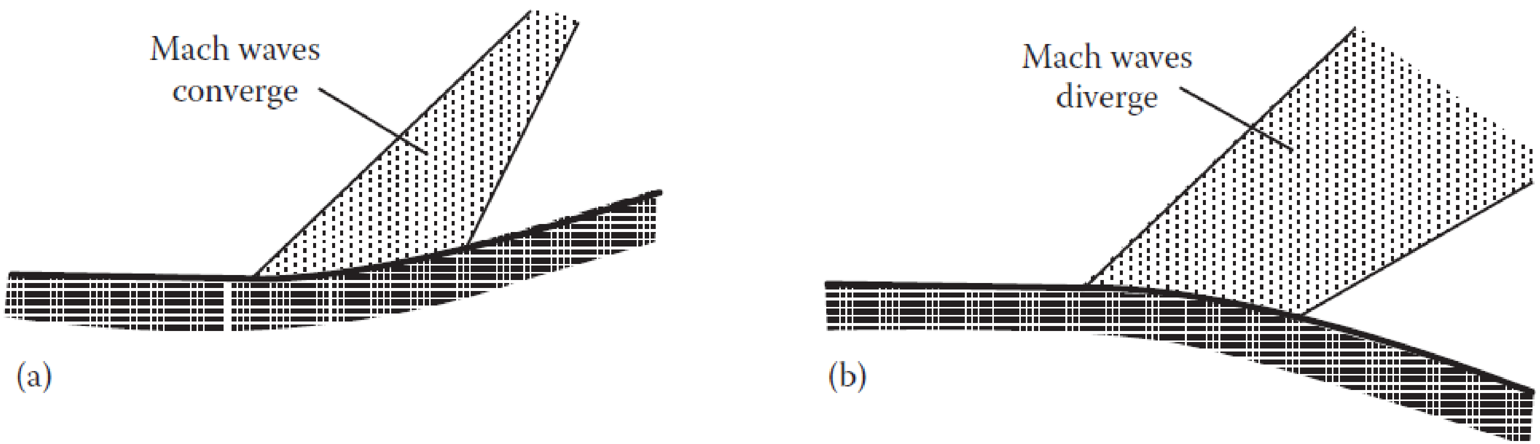


FIGURE 7.6
Mach waves produced in flow around a corner with finite angle: (a) pressure increases through wave system and (b) pressure decreases through wave system.

Prandtl–Meyer Expansion Waves

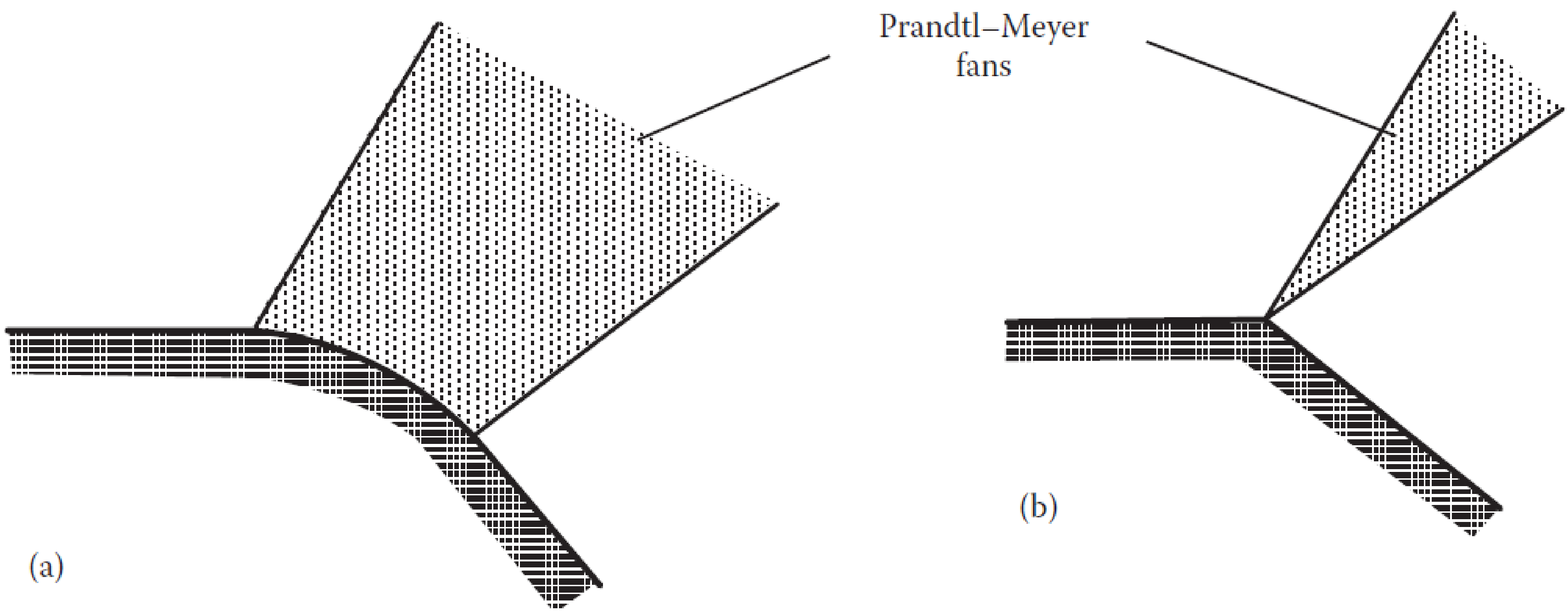


FIGURE 7.7
Prandtl–Meyer flow: (a) centered fan and (b) noncentered fan.

Prandtl–Meyer Expansion Waves

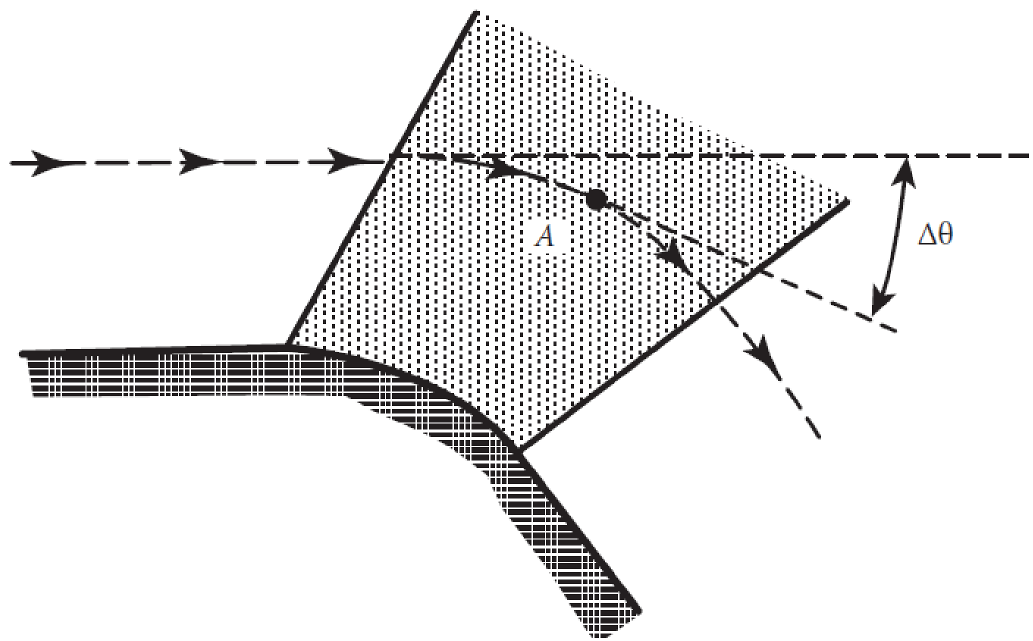
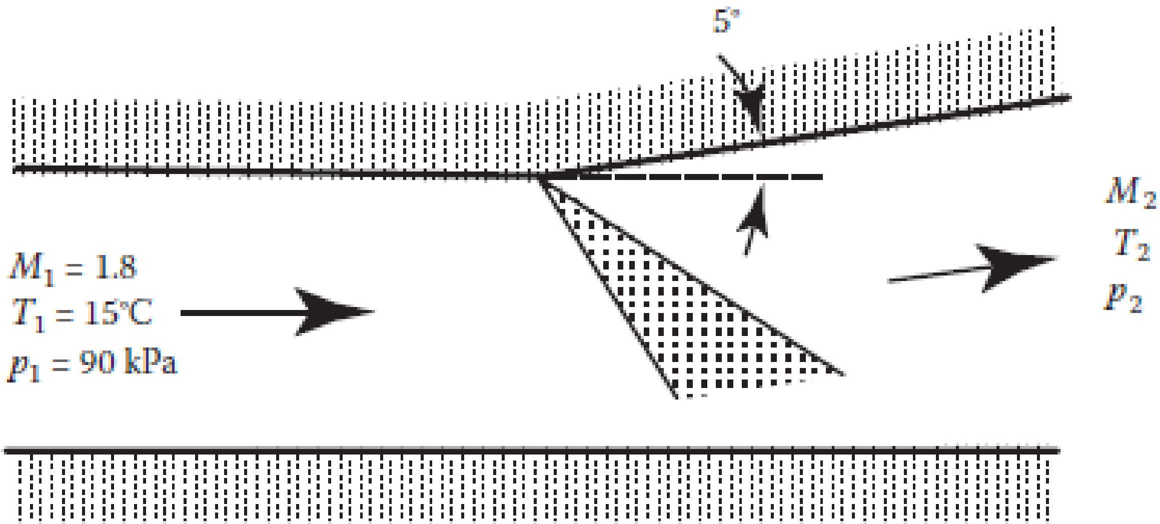
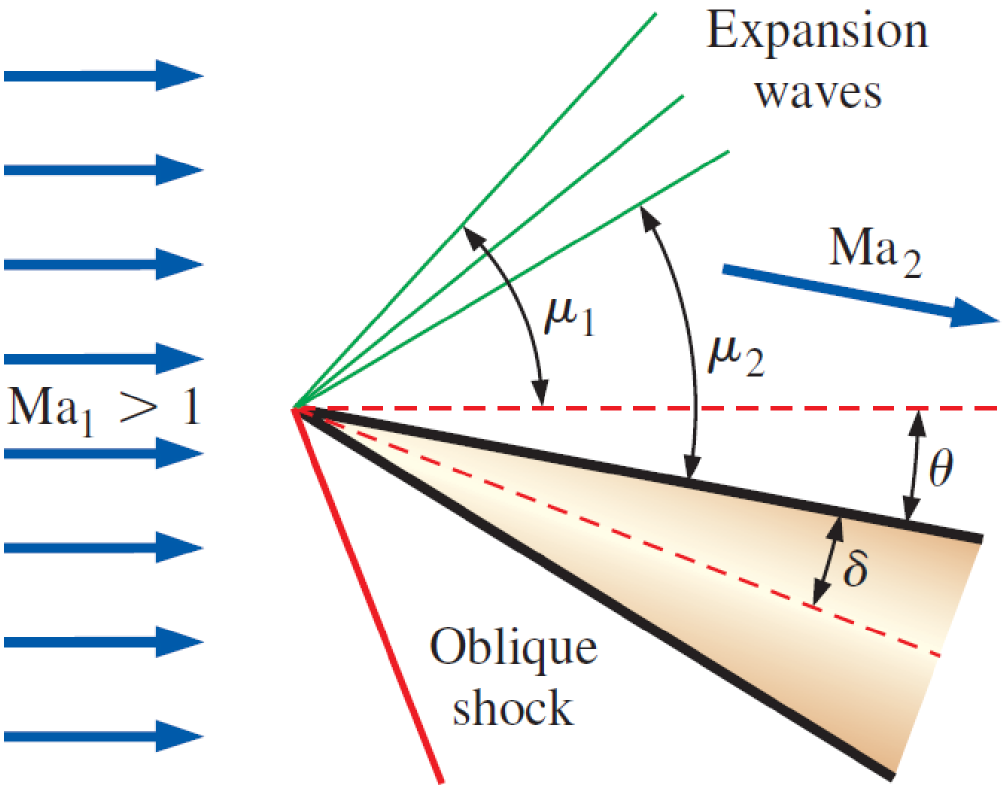


FIGURE 7.8
Prandtl–Meyer flow.

Prandtl–Meyer Expansion Waves

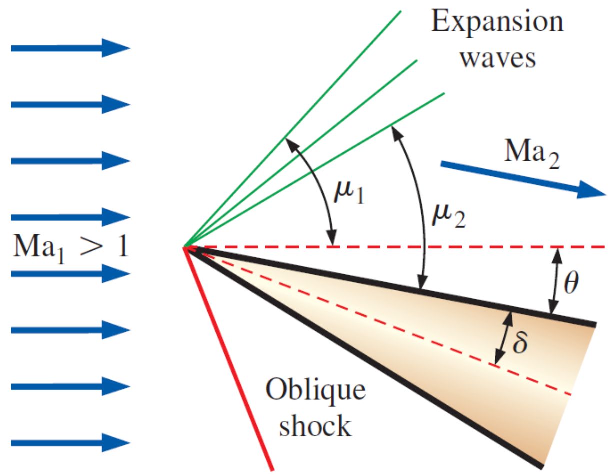


Prandtl–Meyer Expansion Waves



Prandtl–Meyer Expansion Waves

We now address situations where supersonic flow is turned in the *opposite* direction, such as in the upper portion of a two-dimensional wedge at an angle of attack greater than its half-angleδ. We refer to this type of flow as an **expanding flow,** whereas a flow that produces an oblique shock may be called a **compressing flow.**



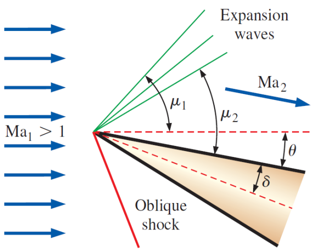
An expansion fan in the upp a two-dimensional wedge at an angle of attack in a supersonic flow. The flow is turned by angle θ, and the Mach number increases across the expansion fan. Mach angles upstream and downstream of the expansion fan are indicated. Only three expansion waves are shown for simplicity, but in fact, there are an infinite number of them. (An oblique shock is also present in the bottom portion of this flow.)

Prandtl–Meyer Expansion Waves

As previously, the flow changes direction to conserve mass. However, unlike a compressing flow, an expanding flow does *not* result in a shock wave. Rather, a continuous expanding region called an **expansion fan** appears, composed of an infinite number of Mach waves called **Prandtl–Meyer expansion waves.** In other words, the flow does not turn suddenly, as through a shock, but *gradually*—each successive Mach wave turns the flow by an infinitesimal amount. Since each individual expansion wave is nearly isentropic, the flow across the entire expansion fan is also nearly isentropic. The Mach number downstream of the expansion *increases* ($Ma_2 > Ma_1$), while pressure, density, and temperature *decrease*, just as they do in the supersonic (expanding) portion of a converging–diverging nozzle.

Prandtl–Meyer Expansion Waves

Prandtl–Meyer expansion waves are inclined at the local Mach angle m, as sketched in Figure. The Mach angle of the first expansion wave is easily determined as μ_1 = $\sin^{-1}$(1/Ma1). Similarly, μ_2 = $\sin^{-1}$(1/Ma2), where we must be careful to measure the angle relative to the *new* direction of flow downstream of the expansion, namely, parallel to the upper wall of the wedge in if we neglect the influence of the boundary layer along the wall. But how do we determine Ma2? It turns out that the turning angle u across the expansion fan can be calculated by integration, making use of the isentropic flow relationships.



Turning angle across an expansion fan: $\theta = \nu(\mathrm{Ma}_2) - \nu(\mathrm{Ma}_1)$ **(12–48)**

where $\nu(\mathrm{Ma})$ is an angle called the **Prandtl–Meyer function** (not to be confused with the kinematic viscosity),

$$\nu(\mathrm{Ma}) = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(\mathrm{Ma}^2-1)}\right) - \tan^{-1}\left(\sqrt{\mathrm{Ma}^2-1}\right) \quad \textbf{(12–49)}$$

Prandtl–Meyer Expansion Waves

To find Ma2 for known values of Ma_1, k, and θ, we calculate n(Ma1) from Eq. 12–49, ν(Ma2) from Eq. 12–48, and then Ma2 from Eq. 12–49, noting that the last step involves solving an implicit equation for Ma2.

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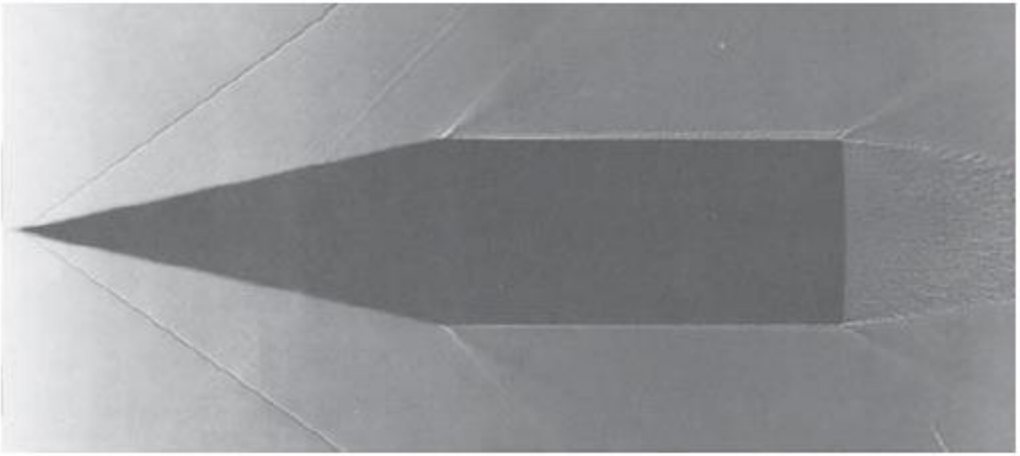
Recognizing that the expansion is isentropic, and hence that T_o, and p_o, are constant through the wave, therefore

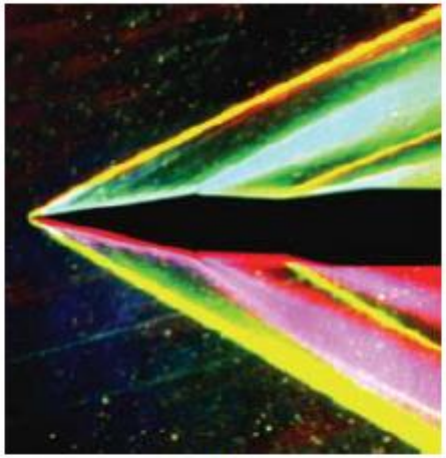
$$\frac{T_1}{T_2} = \frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2} \qquad \frac{p_1}{p_2} = \left[\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\gamma/(\gamma-1)}$$

Prandtl–Meyer Expansion Waves

Prandtl–Meyer expansion fans also occur in axisymmetric supersonic flows, as in the corners and trailing edges of a cone-cylinder (Fig. 12–42)..



(*a*)



(*b*)

FIGURE 12–42

(*a*) A cone-cylinder of 12.5° half-angle in a Mach number 1.84 flow. The boundary layer becomes turbulent shortly downstream of the nose, generating Mach waves that are visible in this shadowgraph. Expansion waves are seen at the corners and at the trailing edge of the cone. (*b*) A similar pattern for Mach 3 flow over an 11° 2-D wedge.

Prandtl–Meyer Expansion Waves Example

EXAMPLE 12–11 Prandtl–Meyer Expansion Wave Calculations

Supersonic air at $Ma_1 = 2.0$ and 230 kPa flows parallel to a flat wall that suddenly expands by $\delta = 10°$ (Fig. 12–45). Ignoring any effects caused by the boundary layer along the wall, calculate downstream Mach number Ma_2 and pressure P_2.

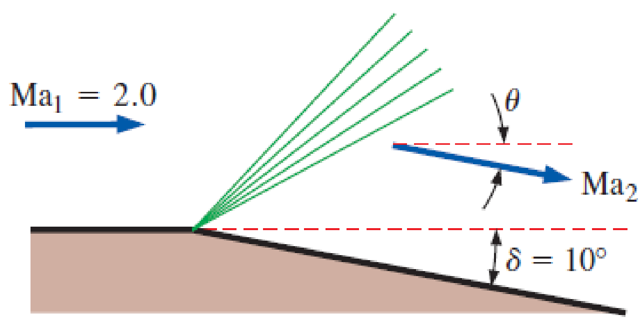


FIGURE 12–45
An expansion fan caused by the sudden expansion of a wall with $\delta = 10°$.

Prandtl–Meyer Expansion Waves Example

SOLUTION We are to calculate the Mach number and pressure downstream of a sudden expansion along a wall.

Assumptions **1** The flow is steady. **2** The boundary layer on the wall is very thin.

Properties The fluid is air with $k = 1.4$.

Analysis Because of assumption 2, we approximate the total deflection angle as equal to the wall expansion angle, i.e., $\theta \cong \delta = 10°$. With $\mathrm{Ma}_1 = 2.0$, we solve Eq. 12–49 for the upstream Prandtl–Meyer function,

$$\nu(\mathrm{Ma}) = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(\mathrm{Ma}^2-1)}\right) - \tan^{-1}\left(\sqrt{\mathrm{Ma}^2-1}\right)$$

$$= \sqrt{\frac{1.4+1}{1.4-1}}\tan^{-1}\left(\sqrt{\frac{1.4-1}{1.4+1}(2.0^2-1)}\right) - \tan^{-1}\left(\sqrt{2.0^2-1}\right) = 26.38°$$

Prandtl–Meyer Expansion Waves Example

Next, we use Eq. 12–48 to calculate the downstream Prandtl–Meyer function,

$$\theta = \nu(\mathrm{Ma}_2) - \nu(\mathrm{Ma}_1) \;\rightarrow\; \nu(\mathrm{Ma}_2) = \theta + \nu(\mathrm{Ma}_1) = 10^\circ + 26.38^\circ = 36.38^\circ$$

Ma_2 is found by solving Eq. 12–49, which is implicit—an equation solver is helpful. We get Ma_2 = **2.38.** There are also compressible flow calculators on the Internet that solve these implicit equations, along with both normal and oblique shock equations; e.g., see www.aoe.vt.edu/~devenpor/aoe3114/calc.html.

We use the isentropic relations to calculate the downstream pressure,

$$P_2 = \frac{P_2/P_0}{P_1/P_0}P_1 = \frac{\left[1 + \left(\dfrac{k-1}{2}\right)\mathrm{Ma}_2^2\right]^{-k/(k-1)}}{\left[1 + \left(\dfrac{k-1}{2}\right)\mathrm{Ma}_1^2\right]^{-k/(k-1)}}(230\ \mathrm{kPa}) = \mathbf{126\ kPa}$$

Since this is an expansion, Mach number increases and pressure decreases, as expected.

Discussion We could also solve for downstream temperature, density, etc., using the appropriate isentropic relations.

Reflected Expansion Wave

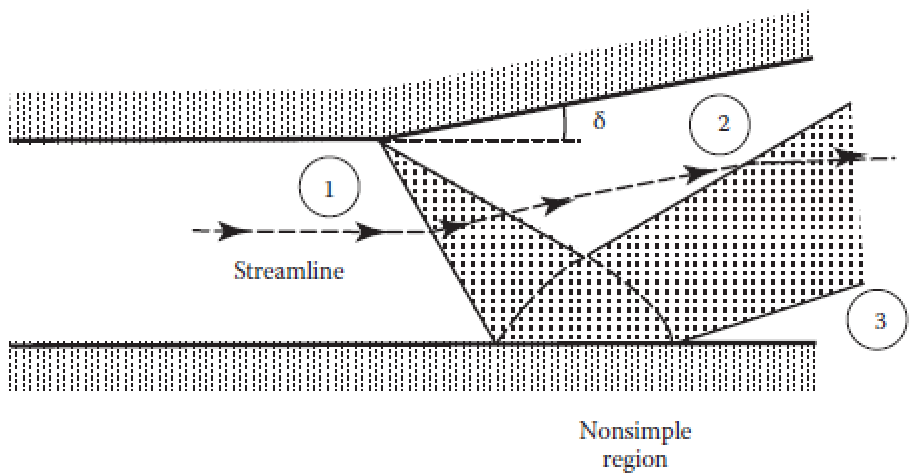


FIGURE 7.12
Reflection of an expansion wave off a flat wall. The regions 1, 2, and 3 used in determining the changes in such a flow are shown in this figure.

Interaction of Expansion Wave

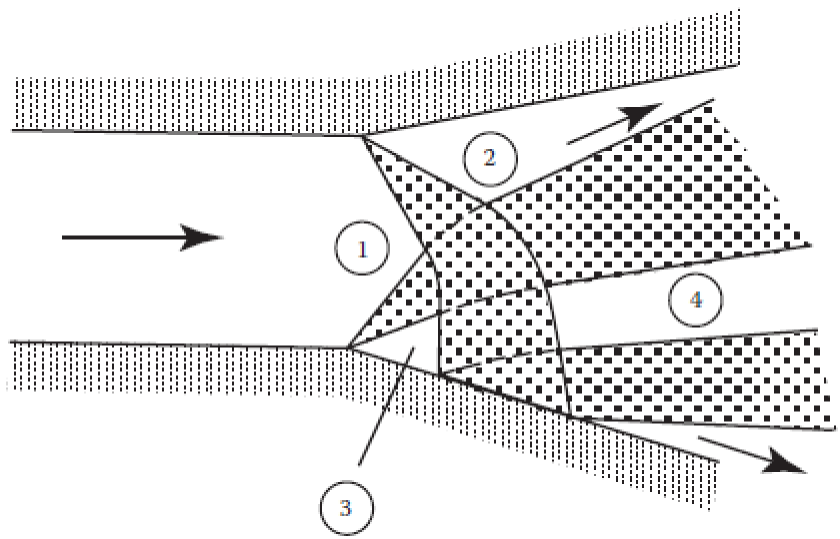
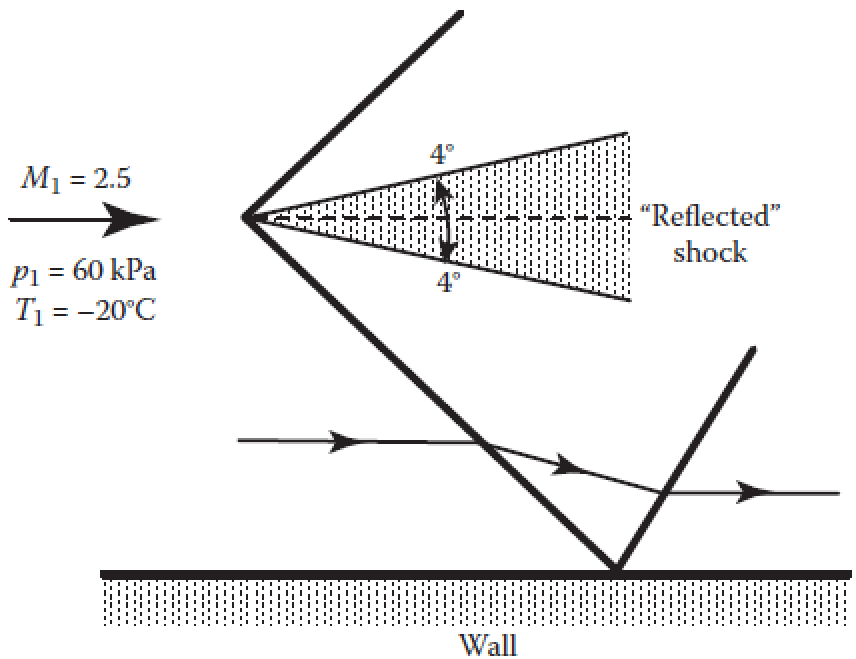


FIGURE 7.13
Interaction of expansion waves

Reflected Oblique Shockwave



Wedge shaped airfoil

Find the lift per meter span for the wedge shaped airfoil shown in Figure. Also, sketch the flow pattern about the airfoil. The Mach number and the pressure ahead of the airfoil are 2.6 and 40 kPa, respectively.

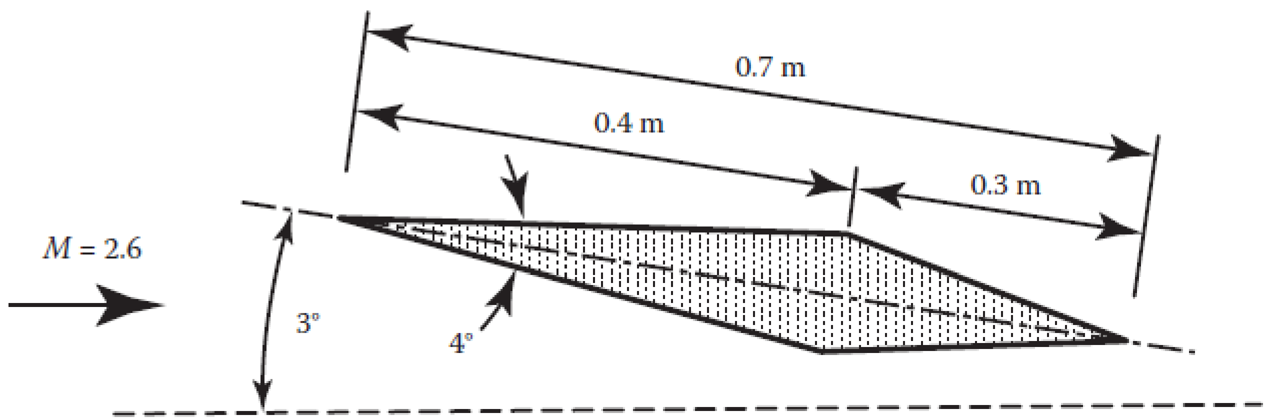
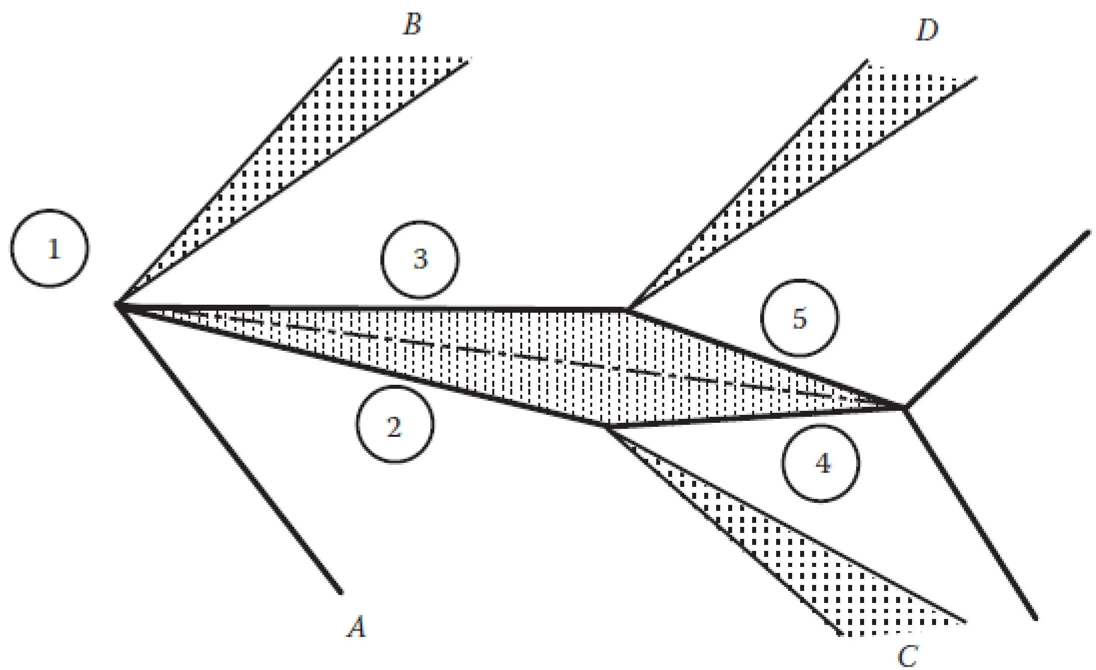


FIGURE E7.5a
Flow situation considered.

Wedge shaped airfoil



Hypersonic Flow

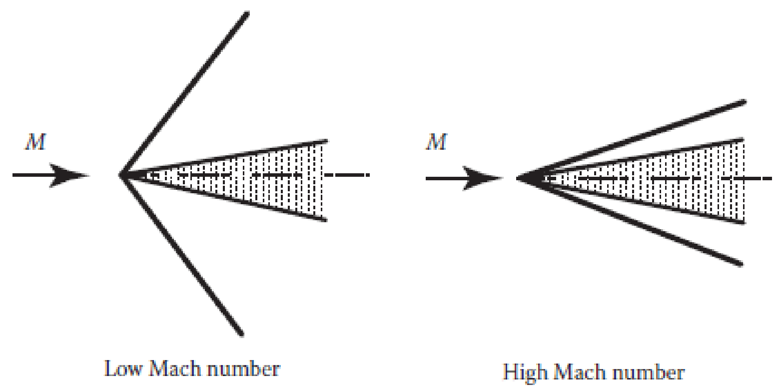
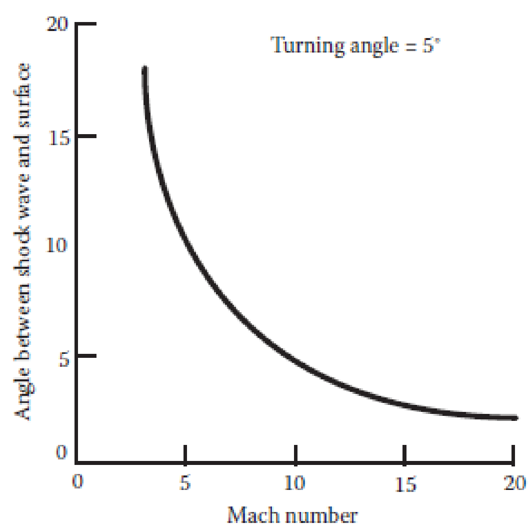


FIGURE 11.4
Shock angle at low and high supersonic Mach number flow over a wedge.



Hypersonic Flow

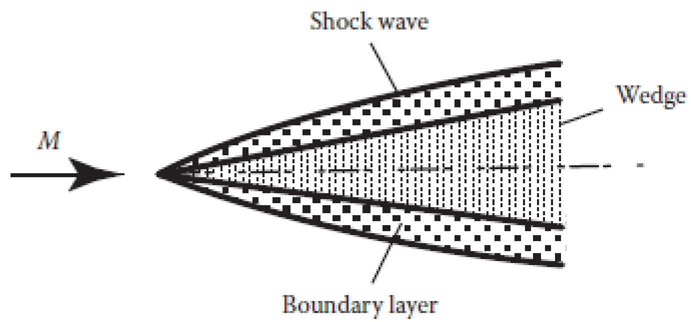


FIGURE 11.6
Interaction between shock wave and boundary layer in hypersonic flow over a wedge.

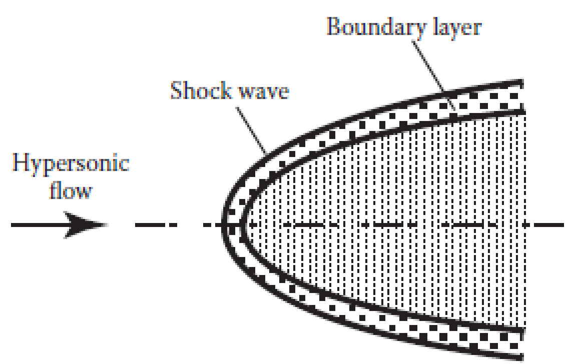


FIGURE 11.7
Interaction between shock wave and boundary layer in hypersonic flow over a curved body.

Hypersonic Flow

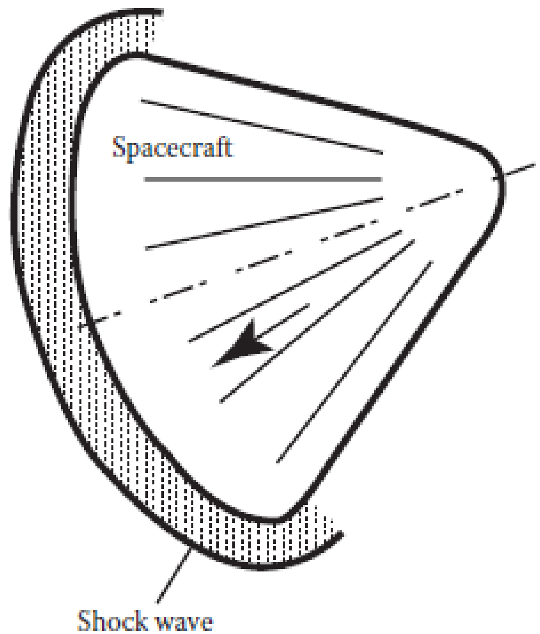


FIGURE 11.9
Flow over early spacecraft during reentry.

High-Temperature Flows

Assumptions that were used till now in this course are

- The specific heats of the gas are constant
- The perfect gas law, $p/\rho = RT$, applies
- There are no changes in the physical nature of the gas in the flow
- The gas is in thermodynamic equilibrium

However, if the temperature in the flow becomes very high, it is possible that some of these assumptions will cease to be valid.