SPC 407
Sheet 3 - Solution
Compressible Flow - Normal Shock wave

1. Consider supersonic flow impinging on the rounded nose of an aircraft. Is the oblique shock that forms in front of the nose an attached or a detached shock? Explain.
   Solution:
   When the wedge half-angle δ is greater than the maximum deflection angle θ_{max}, the shock becomes curved and detaches from the nose of the wedge, forming what is called a detached oblique shock or a bow wave. The numerical value of the shock angle at the nose is $\beta = 90^o$.
   When δ is less than θ_{max}, the oblique shock is attached to the nose.
2. Can a shock wave develop in the converging section of a converging–diverging nozzle? Explain.
   Solution:
   No, because the flow must be supersonic before a shock wave can occur. The flow in the converging section of a nozzle is always subsonic.
   A normal shock (if it is to occur) would occur in the supersonic (diverging) section of the nozzle.
3. Air enters a normal shock at 26 kPa, 230 K, and 815 m/s. Calculate the stagnation pressure and Mach number upstream of the shock, as well as pressure, temperature, velocity, Mach number, and stagnation pressure downstream of the shock. Calculate the entropy change of air across the normal shock wave.
   Solution:
   Assumptions: Air is an ideal gas with constant specific heats and Flow through the nozzle is steady, one-dimensional, and isentropic before the shock occurs.
   Properties: The gas constant of air is R = 0.287 kJ/kg·K. Its specific heat ratio at room temperature is k = 1.4 and c_p=1.005 kJ/kg·K.
   The stagnation temperature and pressure before the shock are

$$T_{01} = T_1 + \frac{V_1^2}{2c_p} = 230 + \frac{(815\ \text{m/s})^2}{2(1.005\ \text{kJ/kg}\cdot\text{K})}\left(\frac{1\ \text{kJ/kg}}{1000\ \text{m}^2/\text{s}^2}\right) = 560.5\ \text{K}$$

$$P_{01} = P_1\left(\frac{T_{01}}{T_1}\right)^{k/(k-1)} = (26\ \text{kPa})\left(\frac{560.5\ \text{K}}{230\ \text{K}}\right)^{1.4/(1.4-1)} = 587.3\ \text{kPa}$$

The velocity and the Mach number before the shock are determined from

$$c_1 = \sqrt{kRT_1} = \sqrt{(1.4)(0.287\ \text{kJ/kg}\cdot\text{K})(230\ \text{K})\left(\frac{1000\ \text{m}^2/\text{s}^2}{1\ \text{kJ/kg}}\right)} = \mathbf{304.0 m/s}$$

and

$$\text{Ma}_1 = \frac{V_1}{c_1} = \frac{815\ \text{m/s}}{304.0\ \text{m/s}} = \mathbf{2.681}$$

The fluid properties after the shock (denoted by subscript 2) are related to those before the shock through the functions listed in Table A-14. For $\text{Ma}_1 = 2.681$ we read

$$\text{Ma}_2 = \mathbf{0.4972},\quad \frac{P_{02}}{P_1} = 9.7330,\quad \frac{P_2}{P_1} = 8.2208,\quad \text{and}\quad \frac{T_2}{T_1} = 2.3230$$

Then the stagnation pressure P_{02}, static pressure P_2, and static temperature T_2, are determined to be

$$P_{02} = 9.7330P_1 = (9.7330)(26\ \text{kPa}) = \mathbf{253.1\ kPa}$$
$$P_2 = 8.2208P_1 = (8.2208)(26\ \text{kPa}) = \mathbf{213.7\ kPa}$$
$$T_2 = 2.3230T_1 = (2.3230)(230\ \text{K}) = \mathbf{534.3\ K}$$

The air velocity after the shock can be determined from $V_2 = \text{Ma}_2 c_2$, where c_2 is the speed of sound at the exit conditions after the shock,

$$V_2 = \text{Ma}_2 c_2 = \text{Ma}_2\sqrt{kRT_2} = (0.4972)\sqrt{(1.4)(0.287\ \text{kJ/kg.K})(534.3\ \text{K})\left(\frac{1000\ \text{m}^2/\text{s}^2}{1\ \text{kJ/kg}}\right)} = \mathbf{230.4 m/s}$$

This problem could also be solved using the relations for compressible flow and normal shock functions. The results would be identical.

4. For an ideal gas flowing through a normal shock, develop a relation for V_2/V_1 in terms of k, Ma_1, and Ma_2.

Solution:

The conservation of mass relation across the shock is

$$\rho_1 V_1 = \rho_2 V_2$$

and it can be expressed as

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = \frac{P_1/RT_1}{P_2/RT_2} = \left(\frac{P_1}{P_2}\right)\left(\frac{T_2}{T_1}\right)$$

From the following equations,

$$\frac{T_2}{T_1} = \frac{1 + \mathrm{Ma}_1^2(k - 1)/2}{1 + \mathrm{Ma}_2^2(k - 1)/2}$$

$$\frac{P_2}{P_1} = \frac{1 + k\mathrm{Ma}_1^2}{1 + k\mathrm{Ma}_2^2}$$

We can get

$$\frac{V_2}{V_1} = \left(\frac{1 + k\mathrm{Ma}_2^2}{1 + k\mathrm{Ma}_1^2}\right)\left(\frac{1 + \mathrm{Ma}_1^2(k - 1)/2}{1 + \mathrm{Ma}_2^2(k - 1)/2}\right)$$

This is an important relation as it enables us to determine the velocity ratio across a normal shock when the Mach numbers before and after the shock are known.

5. Air enters a converging–diverging nozzle with low velocity at 2.0 MPa and 100°C. If the exit area of the nozzle is 3.5 times the throat area, what must the back pressure be to produce a normal shock at the exit plane of the nozzle?

Solution:

Assumptions: 1. Air is an ideal gas.
2. Flow through the nozzle is steady, one-dimensional, and isentropic before the shock occurs.
3. The shock wave occurs at the exit plane.

The inlet stagnation pressure in this case is identical to the inlet pressure since the inlet velocity is negligible. Since the flow before the shock to be isentropic,

$$P_{01} = P_i = 2 \text{ MPa}$$

It is specified that A/A* =3.5. From Table A-13, Mach number and the pressure ratio which corresponds to this area ratio are the Ma_1 =2.80 and P_1/P_{01} = 0.0368. The pressure ratio across the shock for this Ma_1 value is, from Table A-14, P_2/P_1 = 8.98.

Thus, the back pressure, which is equal to the static pressure at the nozzle exit, must be

$$P_2 = 8.98P_1 = 8.98\times0.0368P_{01} = 8.98\times0.0368\times(2\text{ MPa}) = \mathbf{0.661\ MPa}$$

We can also solve this problem using the relations for compressible flow and normal shock functions. The results would be identical.

6. What must the back pressure be in Prob. 5 for a normal shock to occur at a location where the cross-sectional area is twice the throat area?

Solution:

Assumptions: 1. Air is an ideal gas.
2. Flow through the nozzle is steady, one-dimensional, and isentropic before the shock occurs.
3. The shock wave occurs at the exit plane.

The inlet stagnation pressure in this case is identical to the inlet pressure since the inlet velocity is negligible. Since the flow before the shock to be isentropic,

$$P_{01} = P_i = 2\text{ MPa}$$

It is specified that A/A* =2. From Table A-13, Mach number and the pressure ratio which corresponds to this area ratio are the Ma_1 =2.2 and P_1/P_{01} = 0.0935. The pressure ratio across the shock for this M_1 value is, from Table A-14, P_2/P_1 = 5.48.

Thus, the back pressure, which is equal to the static pressure at the nozzle exit, must be

$$P_2 = 5.48P_1 = 5.48\times0.0935P_{01} = 5.48\times0.0935\times(2\text{ MPa}) = \mathbf{1.02\ MPa}$$

We can also solve this problem using the relations for compressible flow and normal shock functions. The results would be identical.

7. Air enters a converging–diverging nozzle of a supersonic wind tunnel at 1 MPa and 300 K with a low velocity. If a normal shock wave occurs at the exit plane of the nozzle at Ma = 2.4, determine the pressure, temperature, Mach number, velocity, and stagnation pressure after the shock wave.

Solution:

Assumptions: 1. Air is an ideal gas with constant specific heats.
2. Flow through the nozzle is steady, one-dimensional, and isentropic before the shock occurs.
3. The shock wave occurs at the exit plane.

Properties: The properties of air are k = 1.4 and R = 0.287 kJ/kg·K. The inlet stagnation properties in this case are identical to the inlet properties since the inlet velocity is negligible. Then,

$$P_{01} = P_i = 1 \text{ MPa}$$

$$T_{01} = T_i = 300 \text{ K}$$

Then,

$$T_1 = T_{01}\left(\frac{2}{2+(k-1)\text{Ma}_1^2}\right) = (300 \text{ K})\left(\frac{2}{2+(1.4-1)2.4^2}\right) = 139.4 \text{ K}$$

and

$$P_1 = P_{01}\left(\frac{T_1}{T_0}\right)^{k/(k-1)} = (1 \text{ MPa})\left(\frac{139.4}{300}\right)^{1.4/0.4} = 0.06840 \text{ MPa}$$

The fluid properties after the shock (denoted by subscript 2) are related to those before the shock through the functions listed in Table A-14. For $\text{Ma}_1 = 2.4$ we read

$$\text{Ma}_2 = 0.5231 \cong \mathbf{0.523}, \quad \frac{P_{02}}{P_{01}} = 0.5401, \quad \frac{P_2}{P_1} = 6.5533, \quad \text{and} \quad \frac{T_2}{T_1} = 2.0403$$

Then the stagnation pressure P_{02}, static pressure P_2, and static temperature T_2, are determined to be

$$P_{02} = 0.5401 P_{01} = (0.5401)(1.0 \text{ MPa}) = \mathbf{0.540\ MPa = 540\ kPa}$$

$$P_2 = 6.5533 P_1 = (6.5533)(0.06840 \text{ MPa}) = \mathbf{0.448\ MPa = 448\ kPa}$$

$$T_2 = 2.0403 T_1 = (2.0403)(139.4 \text{ K}) = \mathbf{284\ K}$$

The air velocity after the shock can be determined from $V_2 = \text{Ma}_2\, c_2$, where c_2 is the speed of sound at the exit conditions after the shock,

$$V_2 = \text{Ma}_2 c_2 = \text{Ma}_2\sqrt{kRT_2} = (0.5231)\sqrt{(1.4)(0.287 \text{ kJ/kg}\cdot\text{K})(284 \text{ K})\left(\frac{1000 \text{ m}^2/\text{s}^2}{1 \text{ kJ/kg}}\right)} = \mathbf{177\ m/s}$$

We can also solve this problem using the relations for normal shock functions. The results would be
identical.