Continuous Systems

INTRODUCTION

We have so far dealt with discrete systems where mass, damping, and elasticity were assumed to be present only at certain discrete points in the system. There are many cases, known as *distributed or continuous systems*, in which it is not possible to identify discrete masses, dampers, or springs. We must then consider the continuous distribution of the mass, damping, and elasticity and assume that each of the infinite number of points of the system can vibrate. This is why a continuous system is also called *a system of infinite degrees of freedom*.

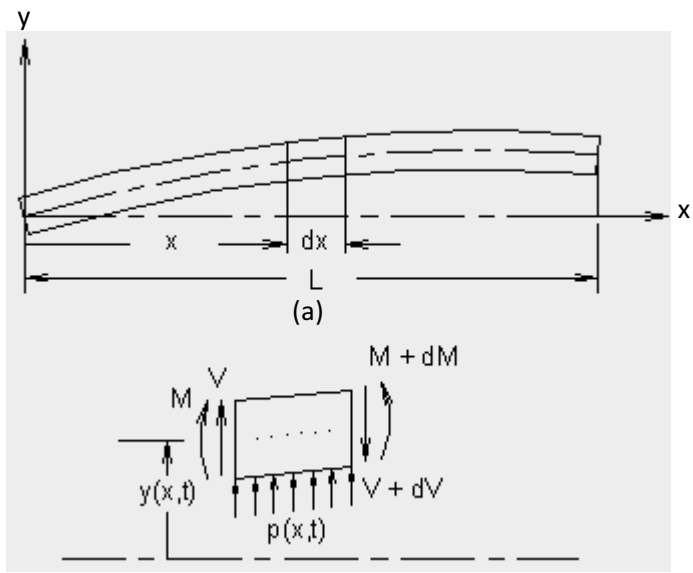
If a system is modeled as a discrete one, the governing equations are ordinary differential equations, which are relatively easy to solve. On the other hand, if the system is modeled as a continuous one, the governing equations are partial differential equations, which are more difficult. However, the information obtained from a discrete model of a system may not be as accurate as that obtained from a continuous model. The choice between the two models must be made carefully, with due consideration of factors such as the purpose of the analysis, the influence of the analysis on design, and the computational time available.

In this chapter, we shall consider the vibration of simple continuous systems-strings, bars, shafts, beams, and membranes. In general the frequency equation of a continuous system is a transcendental equation that yields an infinite number of natural frequencies and normal modes. This is in contrast to the behavior of discrete systems, which yield a finite number of such frequencies and modes. We need to apply boundary conditions to find the natural frequencies of a continuous system. The question of boundary conditions does not arise in the case of discrete systems except in an indirect way, because the influence coefficients depend on the manner in which the system is supported.

1- Lateral Vibration of Beams

Equation of Motion

The beam shown in Figure-a has a length "L", area moment of inertia "I", and a mass per unit length "ρ". To study the lateral vibration of the beam, consider an element of length "dx" at a distance "x" from the left end, Figure-b.



The beam is subjected to a uniform load "p(x, t)". The positive directions of the moments and the forces are indicated on the element.

The equilibrium of moments on the element gives

$$M + d\,M + p\,(dx)^2 / 2 - [V + d\,V]\,d\,x - M = 0$$

Neglecting second order infinitesimals. That is

$$(d\,x)^2 \approx 0$$

$$dV\,dx \approx 0$$

Then,

$$V = \frac{\partial M}{\partial x} \qquad (1)$$

The equilibrium of forces gives

$$V + p(x,t)\, dx - [V + d\, V] = 0$$

Or

$$p(x,t) = \frac{\partial V}{\partial x} \qquad (2)$$

From the fundamental theory of beams

$$M = E\, I\, \frac{\partial^2 y}{\partial x^2}$$

Substitute in Eqs. (1) and (2), then

$$V(x,t) = \frac{\partial}{\partial x}\left[EI\frac{\partial^2 y}{\partial x^2}\right] \qquad (3)$$

$$p(x,t) = \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 y}{\partial x^2}\right] \qquad (4)$$

In the case of the vibration of the beam, "p(x,t)" is divided into two parts. The first is the inertia effect and is equal to "- $\rho\frac{\partial^2 y}{\partial t^2}$". The second is the external effect and is equal to "f(x,t)". If the beam is homogenous and has a uniform section, the equation of motion is given by

$$E\, I\, \frac{\partial^4 y}{\partial x^4} + \rho\frac{\partial^2 y}{\partial t^2} = f(x,t) \qquad (5)$$

Analysis of Free Vibrations

To determine the natural frequencies of a beam, the external effect is considered zero. The equation of motion in this case is given by

$$E\,I\,\frac{\partial^4 y}{\partial x^4} + \rho\frac{\partial^2 y}{\partial t^2} = 0 \qquad (6)$$

Let

$$y(x,t) = Y(x)\,(A\cos\omega t + B\sin\omega t)$$

Where, "A" and "B" are arbitrary constants which are determined from the initial conditions. Substituting in Eq. (35)

$$\left[EI\,\frac{d^4\,Y(x)}{dx^4} - \rho\omega^2\,Y(x)\right]\,(A\cos\omega t + B\sin\omega t) = 0 \qquad (7)$$

Let

$$\frac{\rho\omega^2}{EI} = \beta^4, \text{ then}$$

$$\frac{d^4\,Y(x)}{dx^4} - \beta^4\,Y(x) = 0 \qquad (8)$$

The solution of Eq. (8) is determined by assuming that

$$Y(x) = K\,e^{sx}$$

Substitute in Eq. (8), then

$$s^4 - \beta^4 = 0$$

The roots of this equation are "β, $-\beta$, $i\,\beta$, $-i\,\beta$"

The general solution of Eq. (8) is

$$Y(x) = K_1\,e^{\beta x} + K_2\,e^{-\beta x} + K_3\,e^{i\beta x} + K_4\,e^{-i\beta x}$$

Expanding the exponentials and collecting the constants, hence

$$Y(x) = C_1\cos\beta x + C_2\sin\beta x + C_3\cosh\beta x + C_4\sinh\beta x \qquad (9)$$

The general form of the solution of the differential equation is

$$y(x,t) = (C_1\cos\beta x + C_2\sin\beta x + C_3\cosh\beta x + C_4\sinh\beta x)\,(A\cos\omega t + B\sin\omega t) \qquad (10)$$

The natural frequencies are determined by applying the boundary conditions.

Boundary Conditions

The common boundary conditions are

1. Free end, the moment and the shearing force are zero.

   Moment =0 $\Rightarrow$ $\frac{\partial^2 y}{\partial x^2} = 0$

   Shear =0 $\Rightarrow$ $\frac{\partial^3 y}{\partial x^3} = 0$

2. Simply supported, the deflection and the moment are zero.

   Deflection =0 $\Rightarrow$ $y(x,t) = 0$

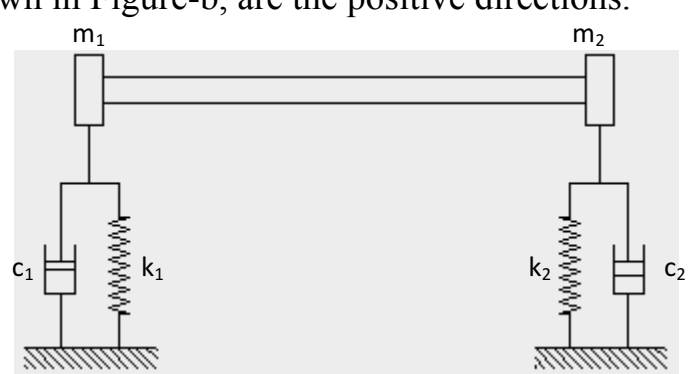
   Moment =0 $\Rightarrow$ $\frac{\partial^2 y}{\partial x^2} = 0$

3. Fixed end, the deflection and the slope of the deflection are zero.

   Deflection =0 $\Rightarrow$ $y(x,t) = 0$

   Slope =0 $\Rightarrow$ $\frac{\partial y}{\partial x} = 0$

4. The case when the ends are attached to a spring-mass-damper systems is shown in Figure. The positive directions of the forces applied on the element, as shown in Figure-b, are the positive directions.



$$E\,I\,\frac{\partial^3 y}{\partial x^3} = m\,\frac{\partial^2 y}{\partial t^2} + c\,\frac{\partial y}{\partial t} + k\,y \qquad \text{for the right end}$$

$$E\,I\,\frac{\partial^3 y}{\partial x^3} = -\,m\,\frac{\partial^2 y}{\partial t^2} - c\,\frac{\partial y}{\partial t} - k\,y \qquad \text{for the left end}$$

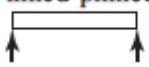
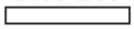
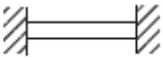
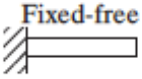
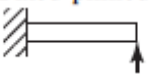
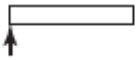
End Conditions of Beam	Frequency Equation	Mode Shape (Normal Function)	Value of $\beta_n l$
Pinned-pinned 	$\sin \beta_n l = 0$	$W_n(x) = C_n[\sin \beta_n x]$	$\beta_1 l = \pi$ $\beta_2 l = 2\pi$ $\beta_3 l = 3\pi$ $\beta_4 l = 4\pi$
Free-free 	$\cos \beta_n l \cdot \cosh \beta_n l = 1$	$W_n(x) = C_n[\sin \beta_n x + \sinh \beta_n x + \alpha_n(\cos \beta_n x + \cosh \beta_n x)]$ where $\alpha_n = \left(\frac{\sin \beta_n l - \sinh \beta_n l}{\cosh \beta_n l - \cos \beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$ ($\beta l = 0$ for rigid-body mode)
Fixed-fixed 	$\cos \beta_n l \cdot \cosh \beta_n l = 1$	$W_n(x) = C_n[\sinh \beta_n x_n x - \sin \beta_n x + \alpha_n(\cosh \beta_n x - \cos \beta_n x)]$ where $\alpha_n = \left(\frac{\sinh \beta_n l - \sin \beta_n l}{\cos \beta_n l - \cosh \beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$
Fixed-free 	$\cos \beta_n l \cdot \cosh \beta_n l = -1$	$W_n(x) = C_n[\sin \beta_n x - \sinh \beta_n x - \alpha_n(\cos \beta_n x - \cosh \beta_n x)]$ where $\alpha_n = \left(\frac{\sin \beta_n l + \sinh \beta_n l}{\cos \beta_n l + \cosh \beta_n l}\right)$	$\beta_1 l = 1.875104$ $\beta_2 l = 4.694091$ $\beta_3 l = 7.854757$ $\beta_4 l = 10.995541$
Fixed-pinned 	$\tan \beta_n l - \tanh \beta_n l = 0$	$W_n(x) = C_n[\sin \beta_n x - \sinh \beta_n x + \alpha_n(\cosh \beta_n x - \cos \beta_n x)]$ where $\alpha_n = \left(\frac{\sin \beta_n l - \sinh \beta_n l}{\cos \beta_n l - \cosh \beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$
Pinned-free 	$\tan \beta_n l - \tanh \beta_n l = 0$	$W_n(x) = C_n[\sin \beta_n x + \alpha_n \sinh \beta_n x]$ where $\alpha_n = \left(\frac{\sin \beta_n l}{\sinh \beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$ ($\beta l = 0$ for rigid-body mode)

Figure. Common boundary conditions for the transverse vibration of a beam.

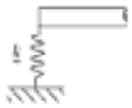
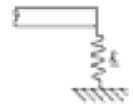
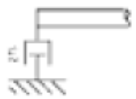
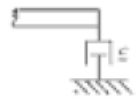
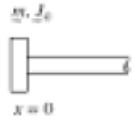
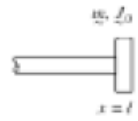
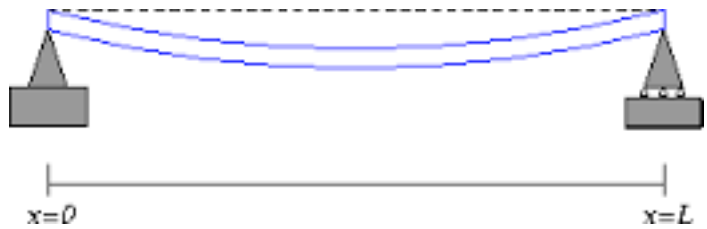
Boundary condition	At left end ($x = 0$)		At right end ($x = l$)	
1. Free end (bending moment = 0, shear force = 0)	 $x = 0$	$EI\frac{\partial^2 w}{\partial x^2}(0,t) = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(0,t)} = 0$	 $x = l$	$EI\frac{\partial^2 w}{\partial x^2}(l,t) = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(l,t)} = 0$
2. Fixed end (deflection = 0, slope = 0)	 $x = 0$	$w(0,t) = 0$ $\frac{\partial w}{\partial x}(0,t) = 0$	 $x = l$	$w(l,t) = 0$ $\frac{\partial w}{\partial x}(l,t) = 0$
3. Simply supported end (deflection = 0, bending moment = 0)	 $x = 0$	$w(0,t) = 0$ $EI\frac{\partial^2 w}{\partial x^2}(0,t) = 0$	 $x = l$	$w(l,t) = 0$ $EI\frac{\partial^2 w}{\partial x^2}(l,t) = 0$
4. Sliding end (slope = 0, shear force = 0)	 $x = 0$	$\frac{\partial w}{\partial x}(0,t) = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(0,t)} = 0$	 $x = l$	$\frac{\partial w}{\partial x}(l,t) = 0$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(l,t)} = 0$
5. End spring (spring constant = k)	 $x = 0$	$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(0,t)} =$ $-kw(0,t)$ $EI\frac{\partial^2 w}{\partial x^2}(0,t) = 0$	 $x = l$	$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|(l,t) =$ $kw(l,t)$ $EI\frac{\partial^2 w}{\partial x^2}(l,t) = 0$
6. End damper (damping constant = c)	 $x = 0$	$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(0,t)} =$ $-c\frac{\partial w}{\partial t}(0,t)$ $EI\frac{\partial^2 w}{\partial x^2}(0,t) = 0$	 $x = l$	$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(l,t)} =$ $c\frac{\partial w}{\partial t}(l,t)$ $EI\frac{\partial^2 w}{\partial x^2}(l,t) = 0$
7. End mass (mass = m with negligible moment of inertia)	 $x = 0$	$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(0,t)} =$ $-m\frac{\partial^2 w}{\partial t^2}(0,t)$ $EI\frac{\partial^2 w}{\partial x^2}(0,t) = 0$	 $x = l$	$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(l,t)} =$ $m\frac{\partial^2 w}{\partial t^2}(l,t)$ $EI\frac{\partial^2 w}{\partial x^2}(l,t) = 0$
8. End mass with moment of inertia (mass = m, moment of inertia = J_0)	 $x = 0$	$EI\frac{\partial^2 w}{\partial x^2}(0,t) =$ $-J_0\frac{\partial^3 w}{\partial x \partial t^2}(0,t)$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(0,t)} =$ $-m\frac{\partial^2 w}{\partial t^2}(0,t)$	 $x = l$	$EI\frac{\partial^2 w}{\partial x^2}(l,t) =$ $J_0\frac{\partial^3 w}{\partial x \partial t^2}(l,t)$ $\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big\|_{(l,t)} =$ $m\frac{\partial^2 w}{\partial t^2}(l,t)$

Figure Boundary Conditions for Beams

Example: 1

Find the natural frequencies of a beam simply supported at both ends.



Solution

$y(x,t) = (C_1\cos \beta x + C_2\sin \beta x + C_3\cosh \beta x + C_4\sinh \beta x)\,(A\cos \omega t + B\sin \omega t)$

For a simply supported beam we apply boundary conditions "2" at both ends.

- $y(0,t)$ gives

$$0 = C_1 \times \cos 0 + C_2 \times \sin 0 + C_3 \times \cosh 0 + C_4 \times \sinh 0$$

$$0 = C_1 \times 1 + C_2 \times 0 + C_3 \times 1 + C_4 \times 0$$

$$0 = C_1 + C_3$$

- $y_{xx}(x,t)|_{x=0}$ gives

$$0 = -C_1 + C_3$$

Thus

$$C_1 = C_3 = 0$$

- $y(x,t)|_{x=L}$ gives

$$C_2 \sin \beta L + C_4 \sinh \beta L = 0$$

- $Y_{xx}(x,t)|_{x=L}$ gives

$$- C_2 \sin \beta L + C_4 \sinh \beta L = 0$$

From the above two equations

$$C_4 \sinh \beta L = 0$$

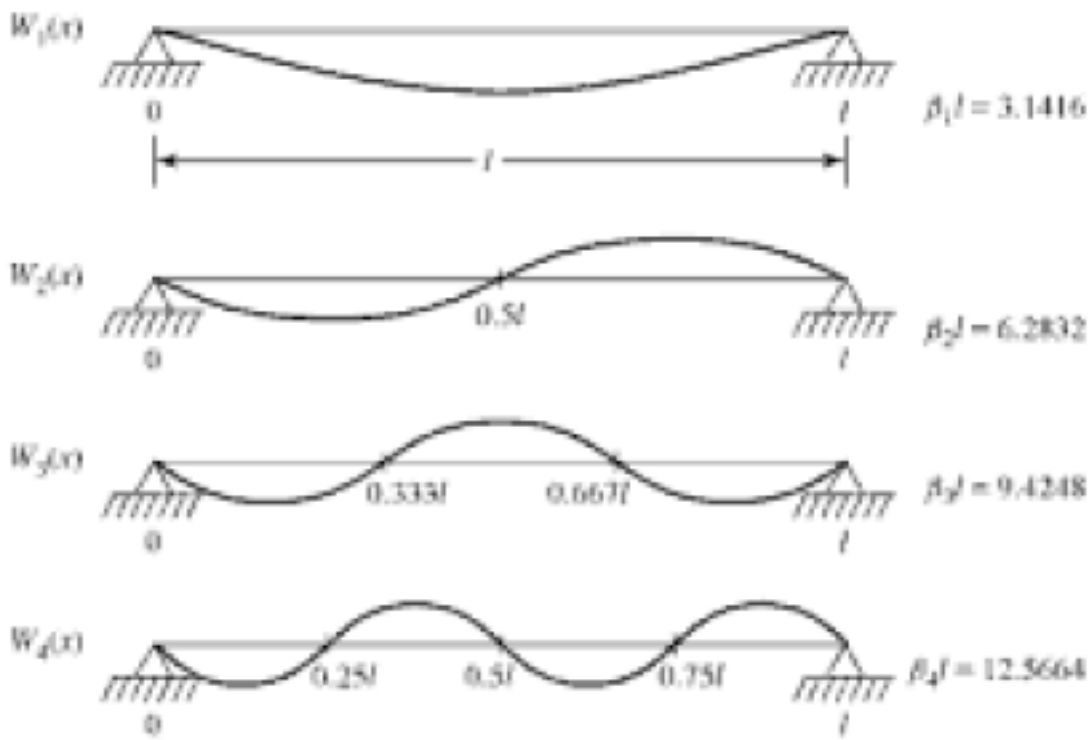
"sinh βL" is only zero when "β = 0". This means that "ω = 0", and, consequently, there is no motion. Then, we conclude that "$C_4 = 0$".

Therefore, the characteristic equation is

$$\sin \beta L = 0$$

$$\beta L = n\pi \qquad n = 1, 2, 3, ...$$

The natural frequency of mode number "n" is equal to "$\left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho}}$".



Example: 2

Find the natural frequencies of a beam fixed at one end and free at the other end.

Solution

For the left end, we apply conditions "3",

- $y(x,t)|_{x=0}$ gives $\qquad C_1 + C_3 = 0$

- $y_x(x,t)|_{x=0}$ gives $C_2 + C_4 = 0$

For the right end, we apply conditions "1".

- $Y_{xx}(x,t)|_{x=L}$ gives $-C_1 \cos \beta L - C_2 \sin \beta L + C_3 \cosh \beta L + C_4 \sinh \beta L = 0$
- $Y_{xxx}(x,t)|_{x=L}$ gives $C_1 \sin \beta L - C_2 \cos \beta L + C_3 \sinh \beta L + C_4 \cosh \beta L = 0$

Substituting the values of "C_3" and "C_4" in the last two equations, then

$$- C_1 (\cos \beta L + \cosh \beta L) - C_2 (\sin \beta L + \sin h\beta L) = 0$$

$$C_1 (\sin \beta L - \sinh \beta L) - C_2 (\cos \beta L + \cosh \beta L) = 0$$

For non trivial solution, the determinant of the coefficients is zero. Thus

$$(\cos \beta L + \cosh \beta L)^2 + (\sin \beta L - \sinh \beta L)(\sin \beta L + \sinh \beta L) = 0$$

This is the characteristic equation and can be reduced to the form

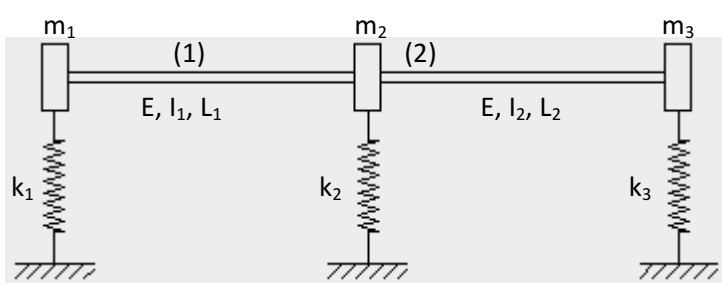
$$1 + \cos \beta L \ \cosh \beta L = 0$$

The first four values for "βL" which satisfy this equation are "1.88, 4.7, 7.9 and 11.0". The corresponding values of the natural frequencies are

"$3.15\sqrt{\frac{EI}{\rho}}$, $22.1\sqrt{\frac{EI}{\rho}}$, $61.78\sqrt{\frac{EI}{\rho}}$, and $121\sqrt{\frac{EI}{\rho}}$ rad/s".

Example: 3

Find the natural frequencies of a beam supported as shown in Figure.



Solution

Any interruption in the beam either by changing the cross-section, the property, or the loading makes it necessary to divide the beam to separate regions, each region is represented by an equation of motion. In this example, the beam is divided to regions (1) and (2).

The equations of motion are

$$E\,I_1 \frac{\partial^4 y_1}{\partial x^4} + \rho \frac{\partial^2 y_1}{\partial t^2} = 0$$

$$E\,I_2 \frac{\partial^4 y_2}{\partial x^4} + \rho \frac{\partial^2 y_2}{\partial t^2} = 0$$

The solutions are given by

$y_1(x,t)= (C_1\cos \beta_1 x+C_2\sin \beta_1 x+C_3\cosh \beta_1 x+C_4\sinh \beta_1 x)\ (A\cos \omega t+B\sin \omega t)$

$y_2(\xi,t)=(D_1\cos \beta_2\xi+D_2\sin \beta_2\xi+D_3\cosh \beta_2\xi+D_4\sinh \beta_2\xi)\ (A\cos \omega t+B\sin \omega t)$

"x" represents the axial position for the first region and "ξ" represents the axial position for the second region; "$\xi= 0$" at the right side of "m_2". The application the boundary conditions is performed as follows:

For the left side of region (1), we apply the moment condition in "1" and shear condition "4".

- The moment condition

$$EI_1 \left.\frac{\partial^2 y_1}{\partial x^2}\right|_{x=0} = 0$$

$$- C_1 + C_3 = 0 \qquad \text{(i)}$$

- Condition 4

$$EI_1 \left.\frac{\partial^3 y_1}{\partial x^3}\right|_{x=0} = - m_1 \left.\frac{\partial^2 y_1}{\partial t^2}\right|_{x=0} - k_1 y_1$$

$$E\,I_1\,\beta_1^3 (- C_2 + C_4) = (C_1 + C_3)\,(m_1\,\omega^2 - k_1) \qquad \text{(ii)}$$

For the right side of region (1) and the left side of region (2), we apply:

- The deflection is the same

$y_1(L_1,t) = y_2(0,t)$

$$C_1 \cos \beta_1 L_1 + C_2 \sin \beta_1 L_1 + C_3 \cosh \beta_1 L_1 + C_4 \sinh \beta_1 L_1 = D_1 + D_3 \qquad \text{(iii)}$$

- The slope is the same

$$EI_1 \left.\frac{\partial y_1}{\partial x}\right|_{x=L_1} = EI_2 \left.\frac{\partial y_2}{\partial \xi}\right|_{\xi=0}$$

$$E\, I_1\, \beta_1 (- C_1 \sin \beta_1 L_1 + C_2 \cos \beta_1 L_1 + C_3 \sinh \beta_1 L_1 + C_4 \cosh \beta_1 L_1)$$

$$= E\, I_2\, \beta_2 (D_2 + D_4) \qquad \text{(iv)}$$

- The moment is the same

$$EI_1 \left.\frac{\partial^2 y_1}{\partial x^2}\right|_{x=L_1} = EI_2 \left.\frac{\partial^2 y_2}{\partial \xi^2}\right|_{\xi=0}$$

$$E\, I_1\, \beta_1^2 (- C_1 \cos \beta_1 L_1 - C_2 \sin \beta_1 L_1 + C_3 \cosh \beta_1 L_1 + C_4 \sinh \beta_1 L_1)$$

$$= E\, I_2\, \beta_2^2 (- D_1 + D_3) \qquad \text{(v)}$$

- The sum of the forces on "m_2" is zero

$$-EI_1 \left.\frac{\partial^3 y_1}{\partial x^3}\right|_{x=L_1} + EI_2 \left.\frac{\partial^3 y_2}{\partial \xi^3}\right|_{\xi=0} + (k_2 - m_2 \omega^2) y_2\big|_{\xi=0} = 0$$

$$- E\, I_1\, \beta_1^3 (C_1 \sin \beta_1 L_1 - C_2 \cos \beta_1 L_1 + C_3 \sinh \beta_1 L_1 + C_4 \cosh \beta_1 L_1)$$

$$+ E\, I_2\, \beta_2^3 (- D_2 + D_4) + (D_1 + D_3)(m_2\, \omega^2 - k_2) = 0 \qquad \text{(vi)}$$

For the right side of region (2), we apply the moment condition in "1" and condition "4".

- The moment condition

$$EI_1 \left.\frac{\partial^2 y_2}{\partial \xi^2}\right|_{\xi=L_2} = 0$$

$$(- D_1 \cos \beta_2 L_2 - D_2 \sin \beta_2 L_2 + D_3 \cosh \beta_2 L_2 + D_4 \sinh \beta_2 L_2) = 0 \qquad \text{(vii)}$$

- Condition 4

$$EI_2 \frac{\partial^3 y_2}{\partial \xi^3}\bigg|_{\xi=L_2} = -m_3 \frac{\partial^2 y_2}{\partial t^2}\bigg|_{\xi=L_2} - k_3\, y_3\big|_{\xi=L_2}$$

$$E\, I_2\, \beta_2^3 (D_1 \sin \beta_2 L_2 - D_2 \cos \beta_2 L_2 + D_3 \sinh \beta_2 L_2 + D_4 \cosh \beta_2 L_2) =$$

$$(D_1 \cos \beta_2 L_2 + D_2 \sin \beta_2 L_2 + D_3 \cosh \beta_2 L_2 + D_4 \sinh \beta_2 L_2)\,(k_3 - m_3\, \omega^2) \quad \text{(viii)}$$

The determinant of the coefficients of the above eight equations is zero and represents the characteristic equation for obtaining the natural frequencies.