Multiple Degree of Freedom Systems

Dr./ Ahmed Nagib

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Multiple Degree of Freedom Systems

Real systems can not be modelled as one degree of freedom system, and are modelled by using multiple degree of freedom systems.
We will extend the previous chapters for two degree of freedom system.

Multiple Degree of Freedom Systems

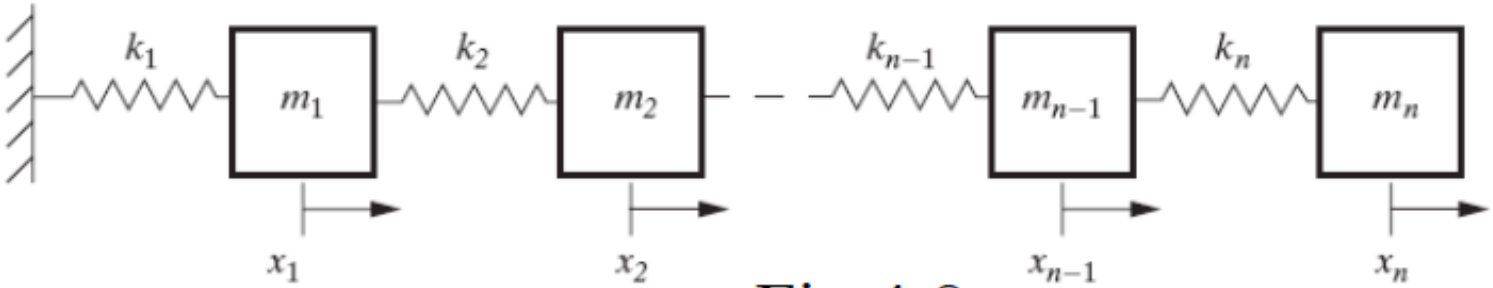


Fig 4.8

Extending previous section to any number of degrees of freedom

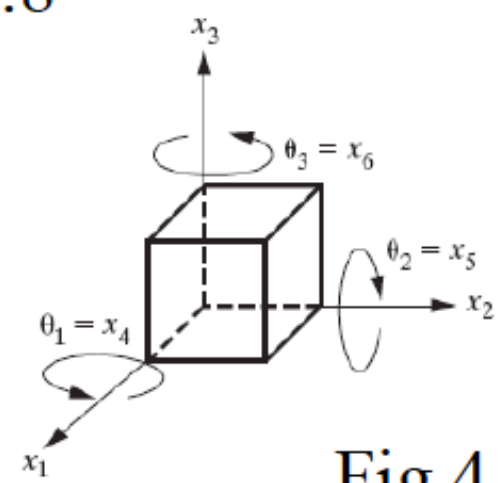


Fig 4.7

Multiple Degree of Freedom Systems

A FBD of the system of shown in the figure yields the n equations of motion o the form:

$$m_i + k_i\left(x_i - x_{i-1}\right) - k_{i+1}\left(x_{i-1} - x_i\right) = 0, \quad i = 1,2,3\cdots n \qquad (4.83)$$

Writing all n of these equations and casting them in matrix form yields:

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}, \qquad (4.80)$$

where:

Multiple Degree of Freedom Systems

The relevant matrices and vectors are:

$$M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & & 0 \\ 0 & -k_3 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & k_{n-1} + k_n & -k_n \\ 0 & 0 & \cdots & -k_n & k_n \end{bmatrix} \quad (4.83)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \ddot{\mathbf{x}}(t) = \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \vdots \\ \ddot{x}_n(t) \end{bmatrix}$$

Multiple Degree of Freedom Systems

For such Free and undamped systems the solving process of the response stays the same…just more modal equations result:

The solving process stays the same as Two Degree of Freedom System, Just get more modal equations, one for each degree of freedom (n is the number of dof)

$$\ddot{r}_1(t) + \omega_1^2 r_1(t) = 0$$

$$\ddot{r}_2(t) + \omega_2^2 r_2(t) = 0$$

$$\ddot{r}_3(t) + \omega_3^2 r_3(t) = 0$$

$$\vdots$$

$$\ddot{r}_n(t) + \omega_n^2 r_n(t) = 0$$

Multiple Degree of Freedom Systems Example 4.4.2

Consider a simple model pf horizontal vibration of a four story building, subject to a wind that gives the building an initial displacement of $x(0) = [0.025 \quad 0.02 \quad 0.01 \quad 0.001]^T$. Find the response of each floor.

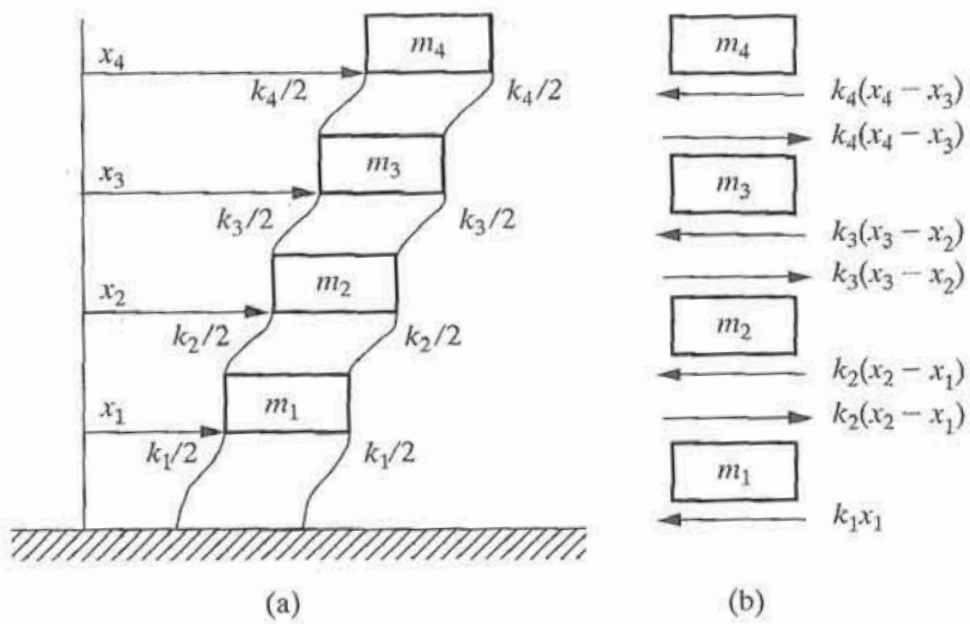


Figure 4.9 (a) Simple model of the horizontal vibration of a four-story building. Here each floor is modeled as a lumped mass and the walls are modeled as providing horizontal stiffness. (b) Restoring forces acting on each mass (floor).

Multiple Degree of Freedom Systems Example 4.4.2

The equations of motion of each floor are

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0$$

$$m_3\ddot{x}_3 - k_3x_2 + (k_3 + k_4)x_3 - k_4x_4 = 0$$

$$m_4\ddot{x}_4 - k_4x_3 + k_4x_4 = 0$$

Multiple Degree of Freedom Systems Example 4.4.2

In Matrix form, these four equations can be written as

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Some reasonable values for a building are

$m_1 = m_2 = m_3 = m_4 = 4000$ kg

and

$k_1 = k_2 = k_3 = k_4 = 5000$ N/m

Multiple Degree of Freedom Systems Example 4.4.2

In this case the numerical values of M and K becomes

$$M = 4000I \qquad K = \begin{bmatrix} 10{,}000 & -5000 & 0 & 0 \\ -5000 & 10{,}000 & -5000 & 0 \\ 0 & -5000 & 10{,}000 & -5000 \\ 0 & 0 & -5000 & 5000 \end{bmatrix}$$

To simplify the calculations, each Matrix is divided by 1000. Since the equation of motion is homogeneous, this corresponds to dividing both sides of the matrix equation by 1000 so the equality is preserved.

Multiple Degree of Freedom Systems Example 4.4.2

The initial conditions are

$$\mathbf{x}(0) = \begin{bmatrix} 0.025 \\ 0.020 \\ 0.010 \\ 0.001 \end{bmatrix} \qquad \dot{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The Matrix $M^{-1/2}$ and $\widetilde{K}$

$$M^{-1/2} = \frac{1}{2} I \qquad \widetilde{K} = \begin{bmatrix} 2.5 & -1.25 & 0 & 0 \\ -1.25 & 2.5 & -1.25 & 0 \\ 0 & -1.25 & 2.5 & -1.25 \\ 0 & 0 & -1.25 & 1.25 \end{bmatrix}$$

Multiple Degree of Freedom Systems Example 4.4.2

The eigen values and normalized eigen vectors for $\widetilde{K}$ yields

$$\lambda_1 = 0.1508 \qquad \lambda_2 = 1.2500 \qquad \lambda_3 = 2.9341 \qquad \lambda_4 = 4.4151$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.2280 \\ 0.4285 \\ 0.5774 \\ 0.6565 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0.5774 \\ 0.5774 \\ 0.0 \\ -0.5774 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0.6565 \\ -0.2280 \\ -0.5774 \\ 0.4285 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} -0.4285 \\ 0.6565 \\ -0.5774 \\ 0.2280 \end{bmatrix}$$

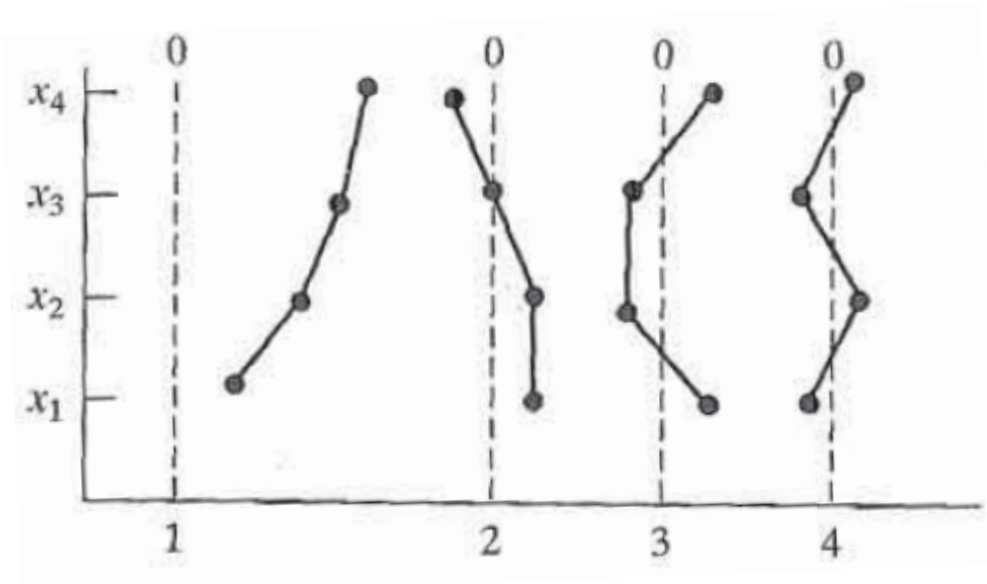
Converting this into naturasl frequencies and mode shapes ($\omega_i = \sqrt{\lambda_i}$ and $u_i = M^{-1/2}\ \mathrm{v}^i$), yields

Multiple Degree of Freedom Systems Example 4.4.2

$\omega_1 = 0.3883, \omega_2 = 0.1118, \omega_3 = 1.7129, \omega_{14} = 2.1012$ and

$$\mathbf{u}_1 = \begin{bmatrix} 0.1140 \\ 0.2143 \\ 0.2887 \\ 0.3283 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.2887 \\ 0.2887 \\ 0.0 \\ -0.2887 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.3283 \\ -0.1140 \\ -0.2887 \\ 0.2143 \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} -0.2143 \\ 0.3283 \\ -0.2887 \\ 0.1140 \end{bmatrix}$$

The four mode shapes can be plotted as



Multiple Degree of Freedom Systems
Example 4.4.2

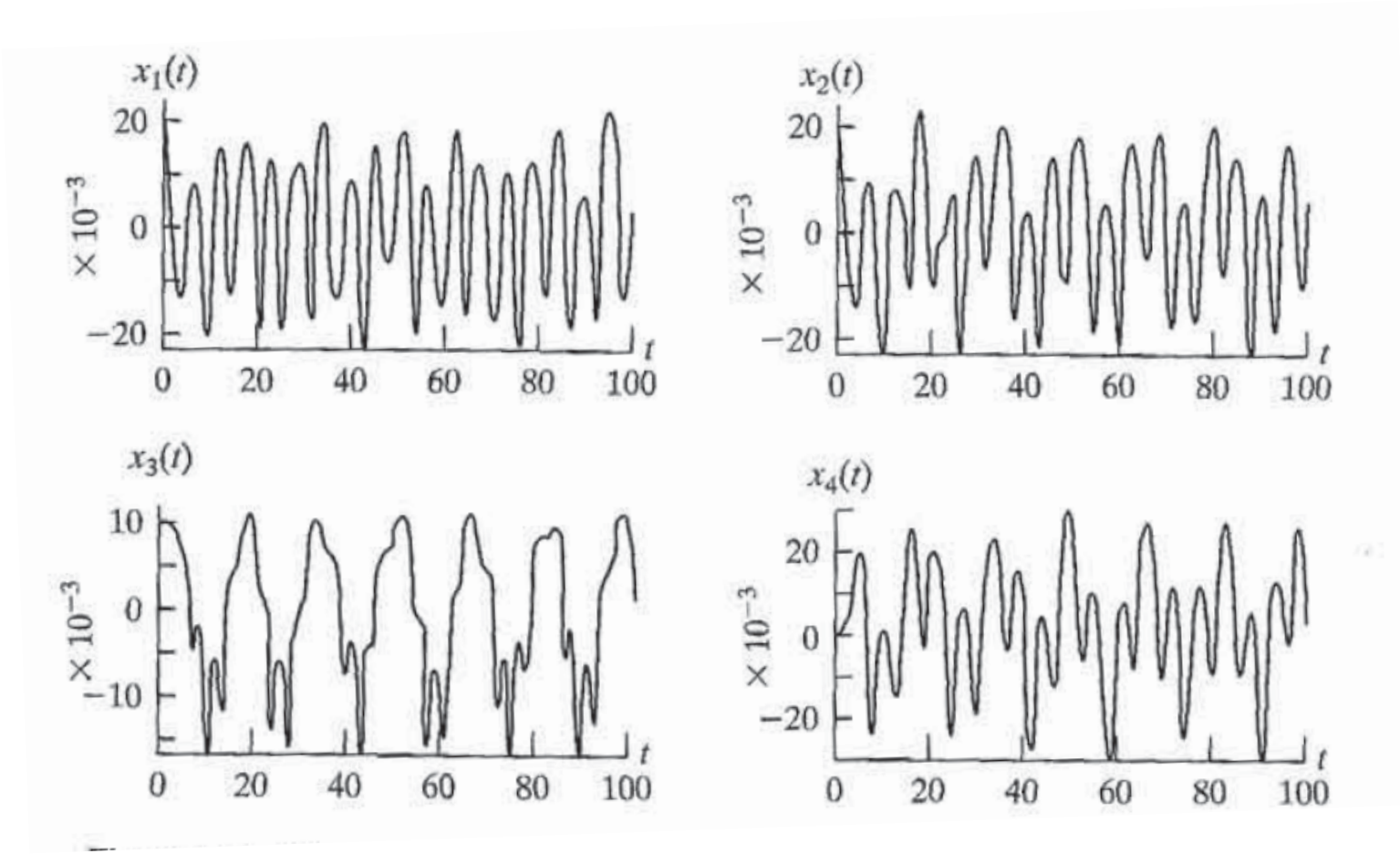
After Calculating the S matrix, the response can be written in the form of

$$\mathbf{x}(t) = \begin{bmatrix} 0.0047 \\ 0.0089 \\ 0.0120 \\ 0.0136 \end{bmatrix} \cos(0.3883t) + \begin{bmatrix} 0.0147 \\ 0.0147 \\ 0 \\ -0.0147 \end{bmatrix} \cos(1.1180t)$$

$$+ \begin{bmatrix} 0.0043 \\ -0.0015 \\ -0.0038 \\ 0.0028 \end{bmatrix} \cos(1.7129t) + \begin{bmatrix} 0.0013 \\ -0.0021 \\ 0.0018 \\ -0.0007 \end{bmatrix} \cos(2.1012t)$$

Multiple Degree of Freedom Systems Example 4.4.2

The response of each mass can be plotted as



Multiple Degree of Freedom Systems
Nodes of a mode

A node of a mode shape is simply the coordinates of a zero entry in the mode shape. For instance, the second mode shape shown in the previous slide, shows that the third entry of the second mode shape is zero!

- Zero elements in a mode shape are called nodes.
- A node of a mode means there is no motion of the mass or (coordinate) corresponding to that entry at the frequency associated with that mode.

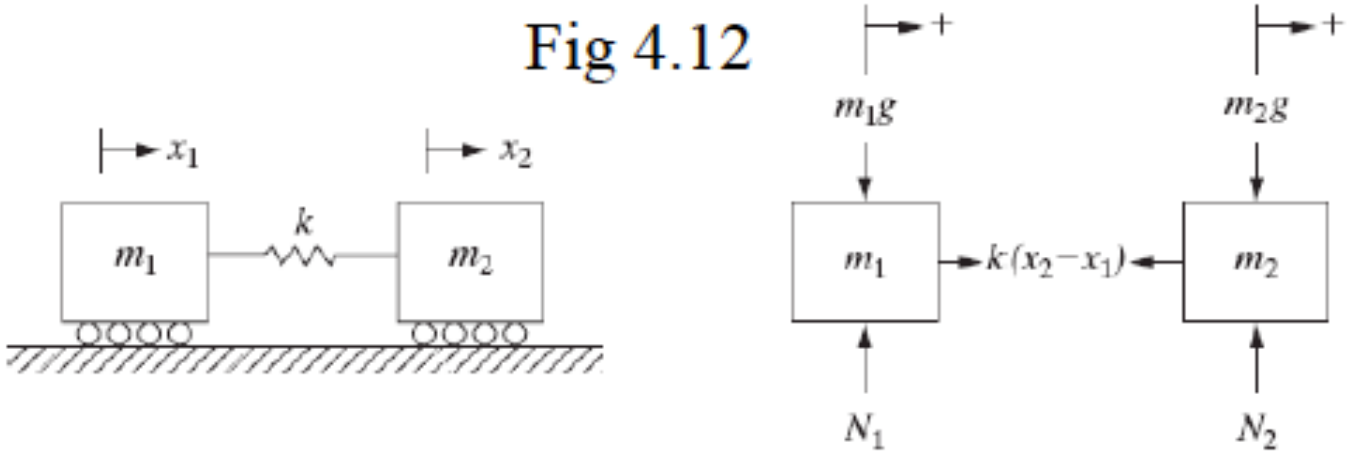
$$\mathbf{u}_2 = \begin{bmatrix} 0.2887 \\ 0.2887 \\ 0 \\ -0.2887 \end{bmatrix}$$

node

They make great mounting points in machines

Multiple Degree of Freedom Systems
Rigid Body mode

Fig 4.12



- Note that the system in Fig 4.12 is not constrained and can move as a rigid body
- Physically if this system is displaced we would expect it to move off the page whilst the two masses oscillate back and forth

Multiple Degree of Freedom Systems
Rigid Body mode

The free body diagram of figure 4.12 yields

$$m_1\ddot{x}_1 = k(x_2 - x_1) \quad \text{and} \quad m_2\ddot{x}_2 = -k(x_2 - x_1)$$

$$\Rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve for the free response given:

$$m_1 = 1 \text{ kg}, \; m_2 = 4 \text{ kg}, \; k = 400 \text{ N} \;\; \text{subject to}$$

$$\mathbf{x}_0 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \text{ m} \;\; \text{and } \mathbf{v}_0 = 0$$

Multiple Degree of Freedom Systems
Rigid Body mode

1. $M^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

2. $\tilde{K} = M^{-1/2} K M^{-1/2} = 400 \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 400 & -200 \\ -200 & 100 \end{bmatrix}$

3. $\det\left(\tilde{K} - \lambda I\right) = 100 \det\left(\begin{bmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{bmatrix}\right) = 100\left(\lambda^2 - 5\lambda\right) = 0$

$\Rightarrow \lambda_1 = 0$ and $\lambda_2 = 5 \Rightarrow \quad \omega_1 = 0, \quad \omega_2 = 2.236$ rad/s

Indicates a rigid body motion

Multiple Degree of Freedom Systems
Rigid Body mode

Now calculate the eigenvectors and note in particular that they cannot be zero even if the eigenvalue is zero

$$\lambda = 0 \Rightarrow 100\begin{bmatrix} 4-0 & -2 \\ -2 & 1-0 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4v_{11} - 2v_{21} = 0$$

$$\Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ or after normalizing } \mathbf{v}_1 = \begin{bmatrix} 0.4472 \\ 0.8944 \end{bmatrix}$$

$$\text{Likewise: } \mathbf{v}_2 = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix}$$

As a check note that

$$P^T P = I \text{ and } P^T \tilde{K} P = \text{diag}\begin{bmatrix} 0 & 5 \end{bmatrix}$$

Multiple Degree of Freedom Systems
Rigid Body mode

$$S = M^{-1/2}P = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0.4472 & -0.8944 \\ 0.8944 & 0.4472 \end{bmatrix} = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.4472 & 0.2236 \end{bmatrix}$$

$$\Rightarrow S^{-1} = \begin{bmatrix} 0.4472 & 1.7889 \\ -0.8944 & 0.8944 \end{bmatrix}$$

7. Calculate the modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}_0 = \begin{bmatrix} 0.4472 & 1.7889 \\ -0.8944 & 0.8944 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.004472 \\ -0.008944 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}_0 = 0$$

Multiple Degree of Freedom Systems
Rigid Body mode

Now compute the solution in modal coordinates and note what happens to the first mode.

Since $\omega_1 = 0$ the first modal equation is

$$\ddot{r}_1 + (0)r_1 = 0$$

$$\Rightarrow r_1(t) = a + bt \qquad \text{Rigid body translation}$$

And the second modal equation is

$$\ddot{r}_2(t) + 5r_2(t) = 0$$

$$\Rightarrow r_2(t) = a_2 \cos\sqrt{5}t \qquad \text{Oscillation}$$

Multiple Degree of Freedom Systems
Rigid Body mode

Applying the modal initial conditions to these two solution forms yields:

$$r_1(0) = a = 0.004472$$

$$\dot{r}_1(0) = b = 0.0$$

$$\Rightarrow r_1(t) = 0.0042$$

as in the past problems the initial conditions for r_2 yield

$$r_2(t) = -0.0089\cos\sqrt{5}t$$

$$\Rightarrow \mathbf{r}(t) = \begin{bmatrix} 0.0042 \\ -0.0089\cos\sqrt{5}t \end{bmatrix}$$

Multiple Degree of Freedom Systems
Rigid Body mode

Transform the modal solution to the physical coordinate system

$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.4472 & -0.8944 \\ 0.4472 & 0.2236 \end{bmatrix} \begin{bmatrix} 0.0045 \\ -0.0089\cos\sqrt{5}t \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2.012 + 7.60\cos\sqrt{5}t \\ 2.012 - 1.990\cos\sqrt{5}t \end{bmatrix} \times 10^{-3} \text{ m}$$

Each mass is moved a constant distance and then oscillates at a single frequency.

Order the frequencies

- It is convention to call the lowest frequency ω_1 so that $\omega_1 \leq \omega_2 \leq \omega_3 < \ldots$
- Order the modes (or eigenvectors) accordingly
- It really does not make a difference in computing the time response
- However:
  - When we measuring frequencies, they appear lowest to highest
  - Physically the frequencies respond with the highest energy in the lowest mode (important in flutter calculations, run up in rotating machines, etc.)

Multiple Degree of Freedom Systems

Till now we studied the response of undamped, free two and multiple degree of freedom systems. The modal analysis performed in the previous chaptered can be extended for damped and forced systems but with limitations.
The response of the system in commercial softwares is calculated by using numerical integration as we discussed before in the solving of vibration equation lecture.