OPTIMUM DESIGN

Dr. / Ahmed Nagib Elmekawy

Lecture 4

Fall 2017

Infinite/multiple solutions

When $f(\mathbf{x})$ is parallel to a binding constraint

$$f(\mathbf{x}) = -x_1 - 0.5x_2$$

$$g_1: \quad 2x_1 + 3x_2 \le 12$$

$$g_2: \quad 2x_1 + x_2 \le 8$$

$$g_3: \; -x_1 \le 0$$

$$g_4: \; -x_2 \le 0$$

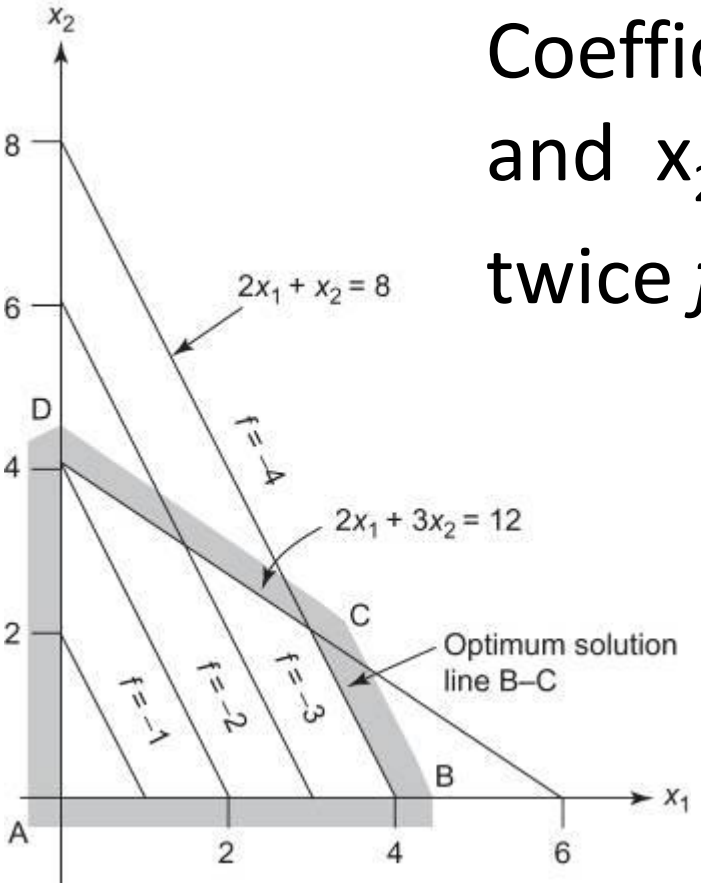
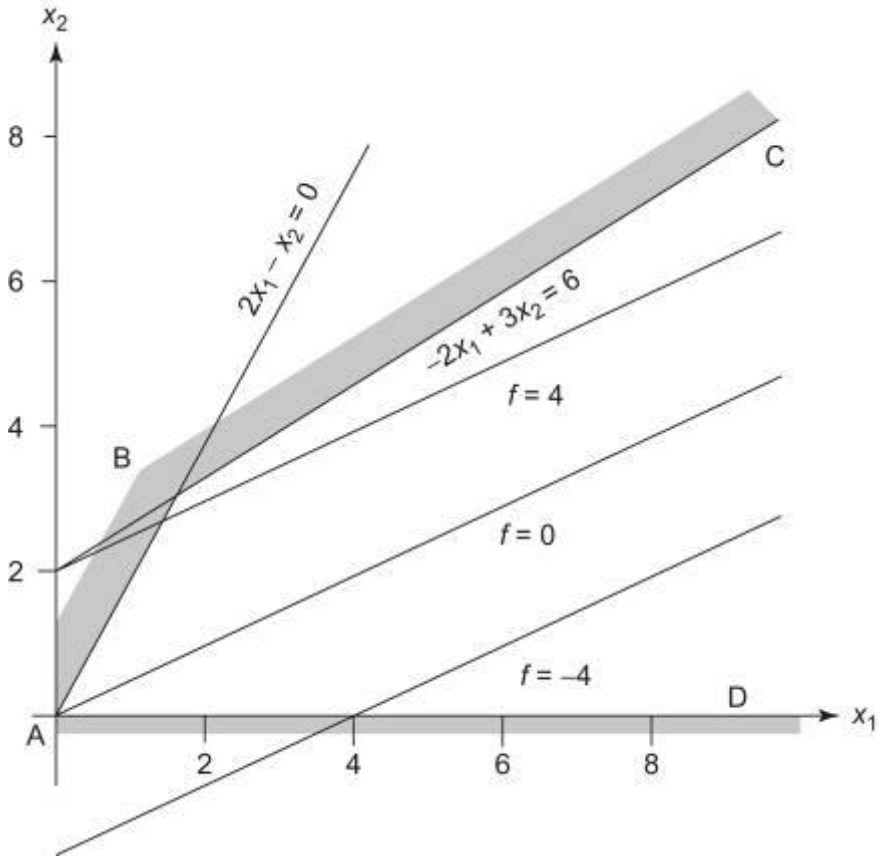


Figure 3.7 Example problem with multiple solutions.

Coefficient of x_1 and x_2 in g_2 are twice $f(x)$

Unbound Solution

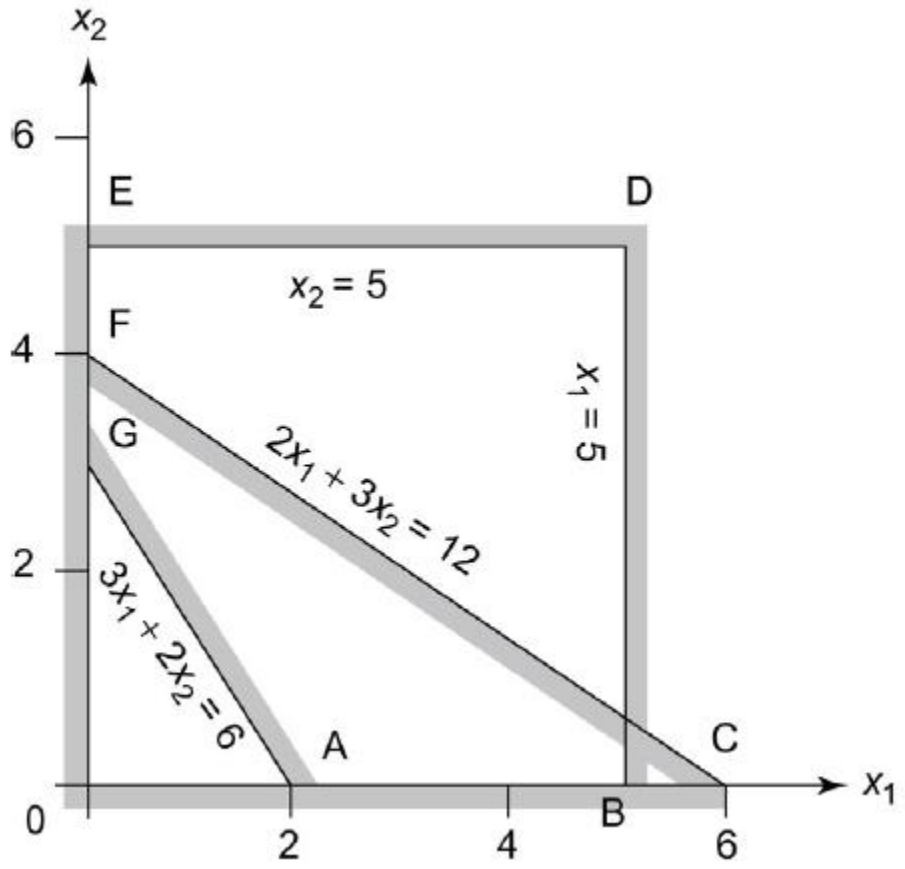
Figure 3.8 Example problem with an unbounded solution.



Open region

On R.H.S.

Infeasible design



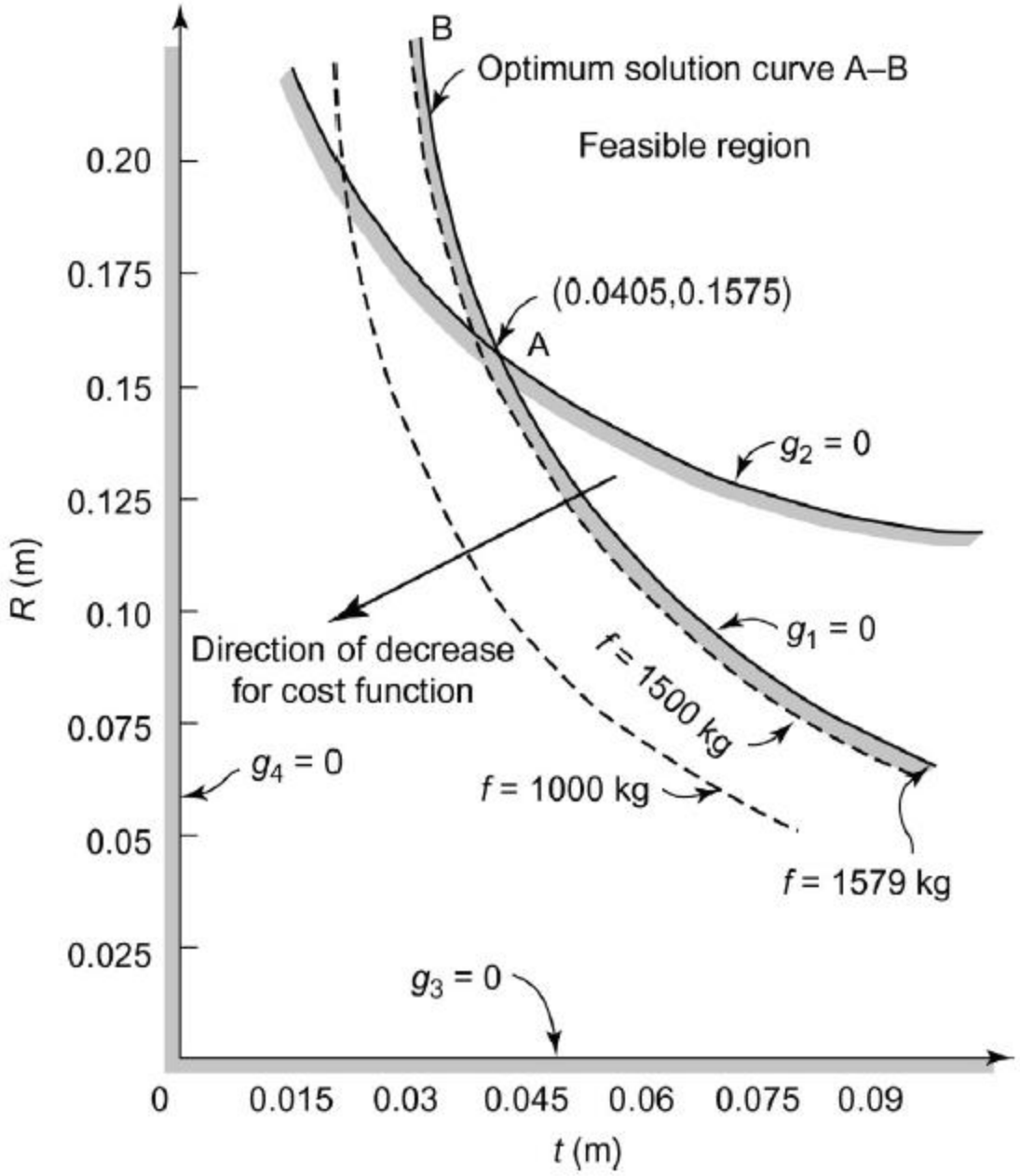
The design problem formulated in Section 2.7 will now be solved by the graphical method using the following data: $P = 10\,\text{MN}, E = 207\,\text{GPa}, \rho = 7833\,\text{kg/m}^3, l = 5.0\,\text{m}$, and $\sigma_a = 248\,\text{MPa}$. Using these data, formulation 1 for the problem is defined as, "find thickness t (m) and mean radius R (m) to minimize the mass function."

$$f(R,t) = 2\rho l\pi Rt = 2(7833)(5)\pi Rt = 2.4608\times 10^5 Rt,\ \text{kg} \tag{a}$$

subject to the four inequality constraints

$$g_1(R,t) = \frac{P}{2\pi Rt} - \sigma_a = \frac{10\times 10^6}{2\pi Rt} - 248\times 10^6 \le 0\ \text{(stress constraint)} \tag{b}$$

$$g_2(R,t) = P - \frac{\pi^3 ER^3 t}{4l^2} = 10\times 10^6 - \frac{\pi^3 (207\times 10^9) R^3 t}{4(5)(5)} \le 0\ \text{(buckling load constraint)} \tag{c}$$


B
Optimum solution curve A–B
Feasible region
0.20
0.175
0.15
0.125
0.10
0.075
0.05
0.025
R (m)
(0.0405,0.1575)
A
$g_2 = 0$
$g_1 = 0$
Direction of decrease
for cost function
f = 1500 kg
$g_4 = 0$
f = 1000 kg
f = 1579 kg
$g_3 = 0$
0
0.015
0.03
0.045
0.06
0.075
0.09
t (m)

Graphical Solution

1. Sketch coordinate system
2. Plot constraints
3. Determine feasible region
4. Plot f(**x**) contours (2 or 3)
5. Find opt solution **x*** & opt value f(**x***)

“Unique” Solution

Recall a typical system of linear eqns

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & -3 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2 + x_3) \\ (5x_1 - 3x_2 - x_3) \\ (2x_1 + x_2 + x_3) \end{bmatrix}$$

$$\mathbf{A}_{3x3}\mathbf{x}_{3x1} = \mathbf{y}_{3x1}$$

$$\mathrm{h_j}(\mathbf{x}) = 0 \qquad j = 1 \ldots \mathrm{p}$$

The number of independent h_j must be less than or equal to n

i.e. $p \leq n$

Infeasible Problem

Constraints are:

inconsistent

conflicting

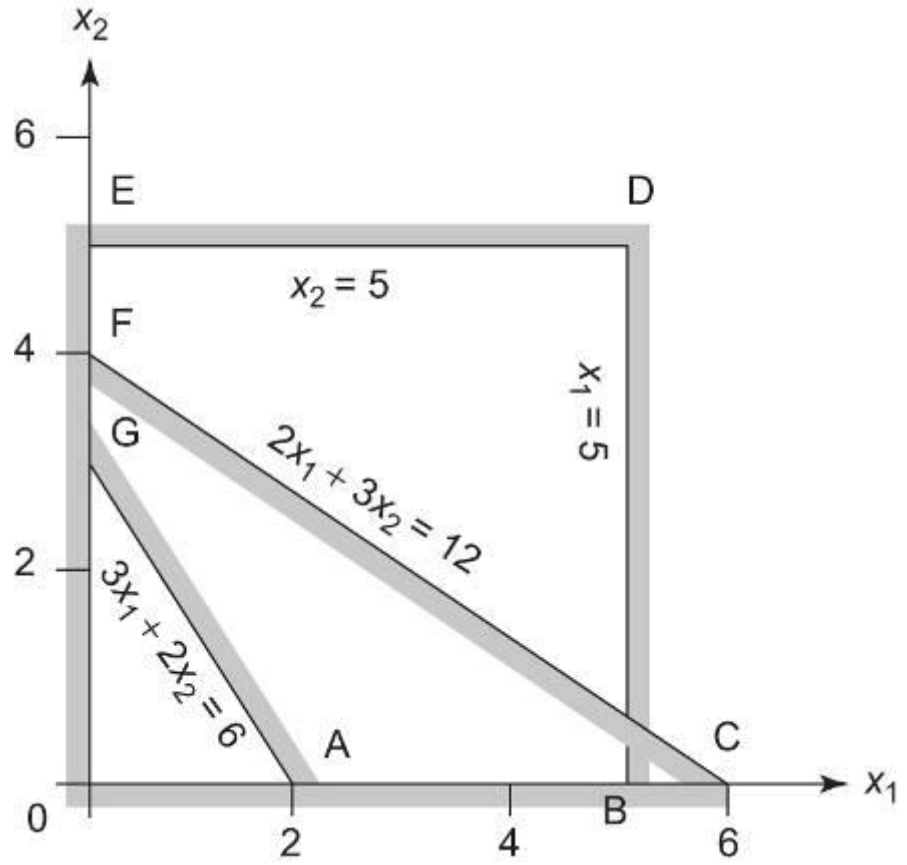


Figure 3.9 Infeasible design optimization problem.

How many inequality constraints can we have?

How many active inequality constraints?

Optimization Methods

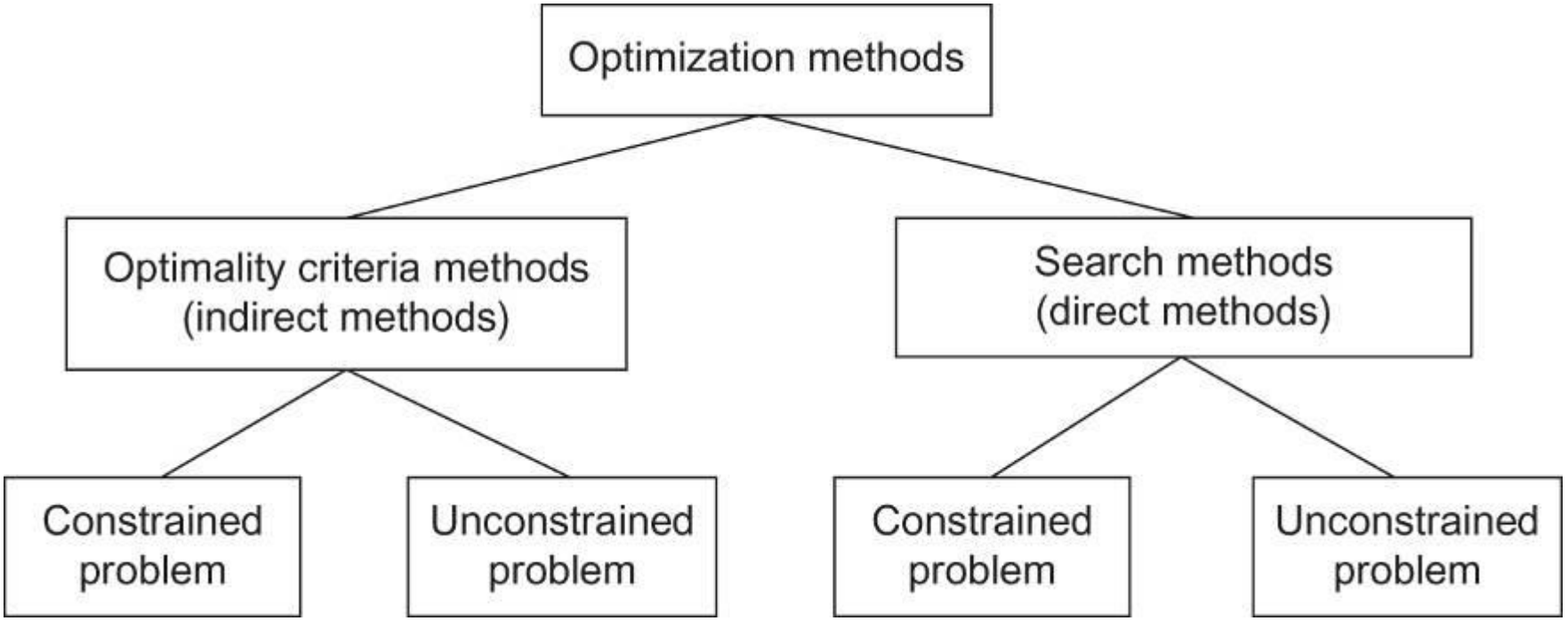


Figure 4.1 Classification of optimization methods.

Global/local optima

Global Maximum?
Local Maximum?
Global Minimum?
Local Minimum?

Global = absolute
Local = relative

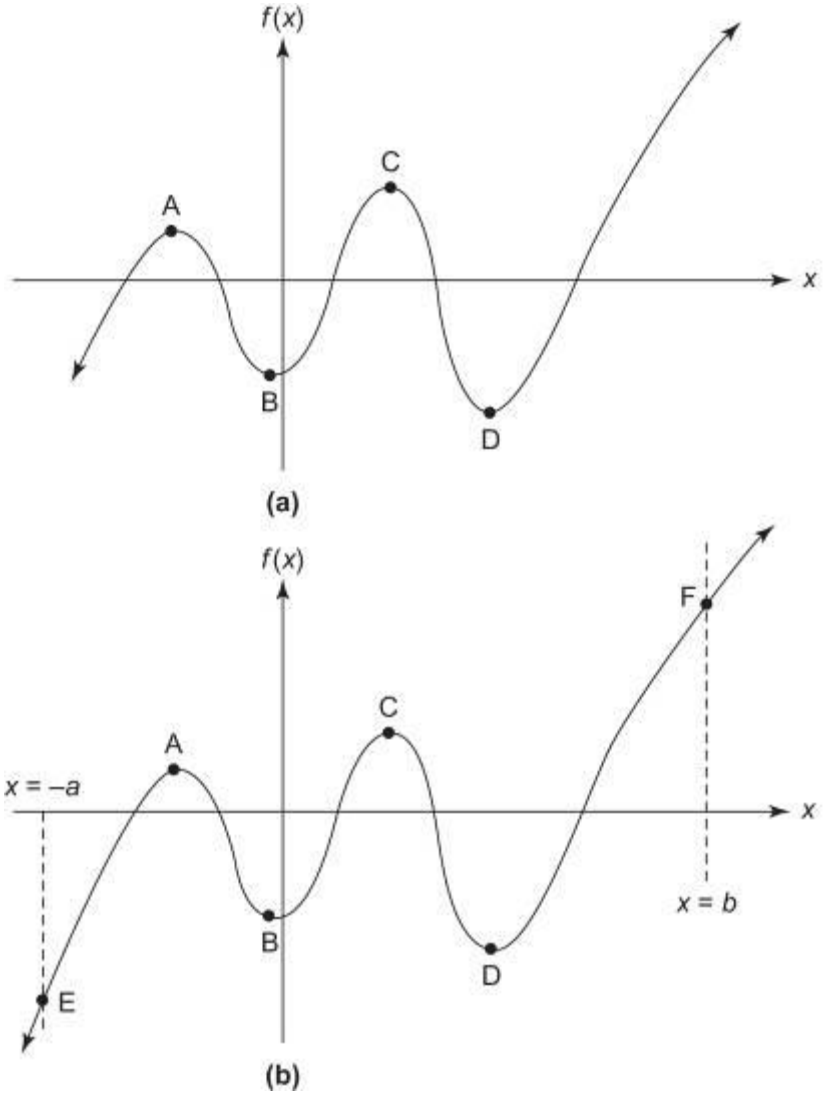


Figure 4.2 Representation of optimum points. (a) The unbounded domain and function (no global optimum). (b) The bounded domain and function (global minimum and maximum exist).

Closed & Bounded

Global (Absolute) Min

$\mathbf{x}^*$ is a global minimum of $f(\mathbf{x})$ IFF

$$f(\mathbf{x}^*) \leq f(\mathbf{x})$$

for all $\mathbf{x}$ in the feasible set $S = \{\mathbf{x} \mid \mathbf{h}(\mathbf{x}) = 0 \;\; \mathbf{g}(\mathbf{x}) \leq 0\}$

Note that $S = \{\mathbf{x} \mid h_j(\mathbf{x}) = 0, \;\; j = 1 \text{ to } p; \;\; g_i(\mathbf{x}) \leq 0; i = 1 \text{ to } m\}$

i.e.

$S = \{\mathbf{x} \mid \mathbf{h}(\mathbf{x}) = 0 \;\; \mathbf{g}(\mathbf{x}) \leq 0\}$

"Strong (strict)" versus "weak" minima

$f(\mathbf{x}^*) < f(\mathbf{x})$ $\qquad$ $f(\mathbf{x}^*) \leq f(\mathbf{x})$

Local (Relative) Min

$\mathbf{x}^*$ is a global minimum of $f(\mathbf{x})$ IFF

$$f(\mathbf{x}^*) \leq f(\mathbf{x})$$

For all $\mathbf{x}$ in a small neighborhood N of $\mathbf{x}^*$ in the feasible set $S = \{\mathbf{x} \mid \mathbf{h}(\mathbf{x}) = 0 \;\; \mathbf{g}(\mathbf{x}) \leq 0\}$

Where N is defined as:

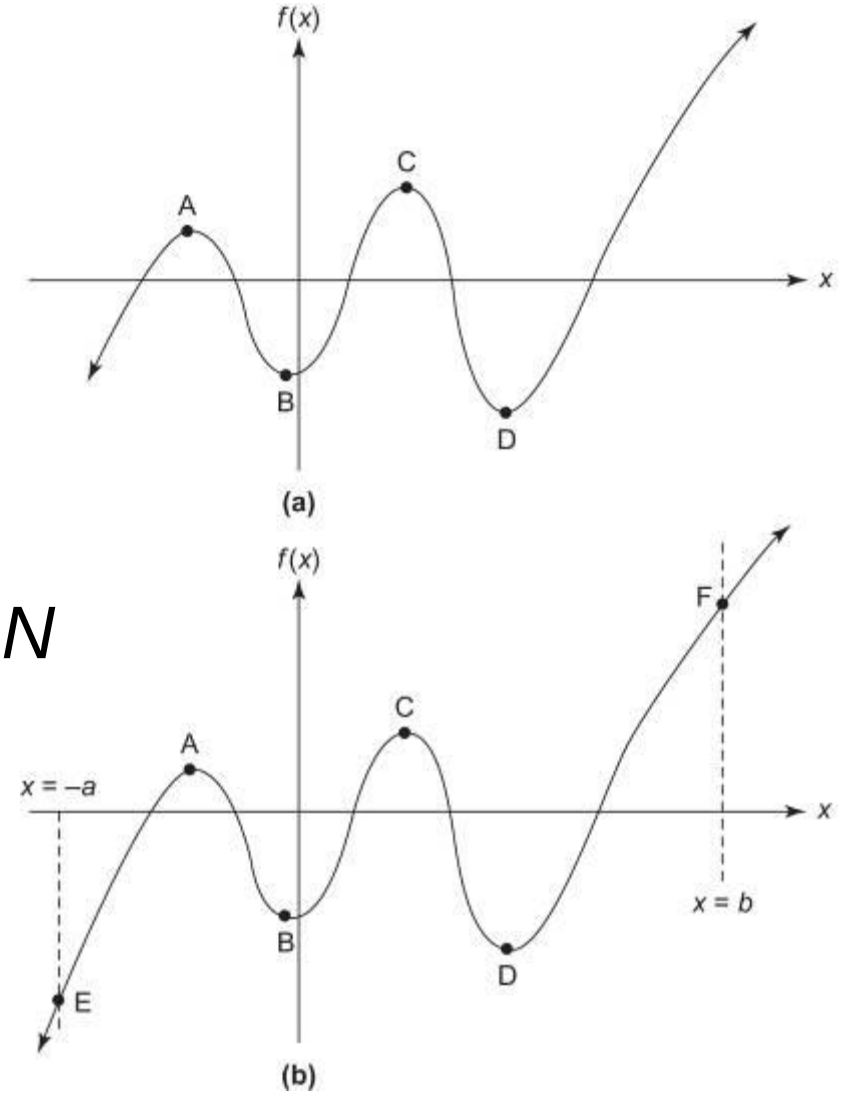
$$N = \{\mathbf{x} \mid \mathbf{x} \in S \text{ with } \|\mathbf{x} - \mathbf{x}^*\| < \delta$$

$$\text{for some small } \delta$$

Global/local optima

Global Maximum?
$f(\mathbf{x}^*) \leq f(\mathbf{x})$
Anywhere in S

Local Maximum?
$f(\mathbf{x}^*) \leq f(\mathbf{x})$
In small neighborhood N



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Closed & Bounded

Unconstrained Minimum

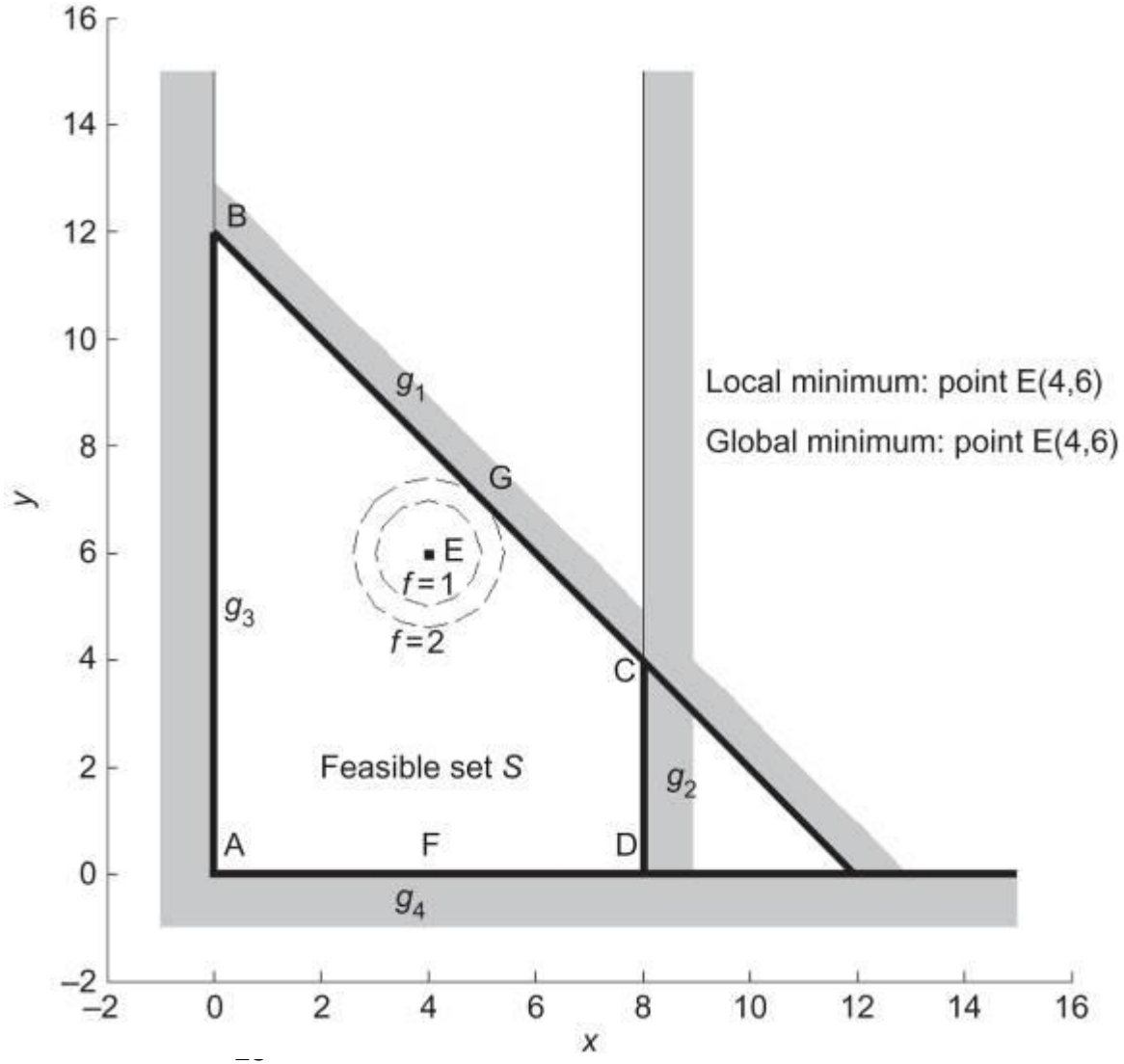
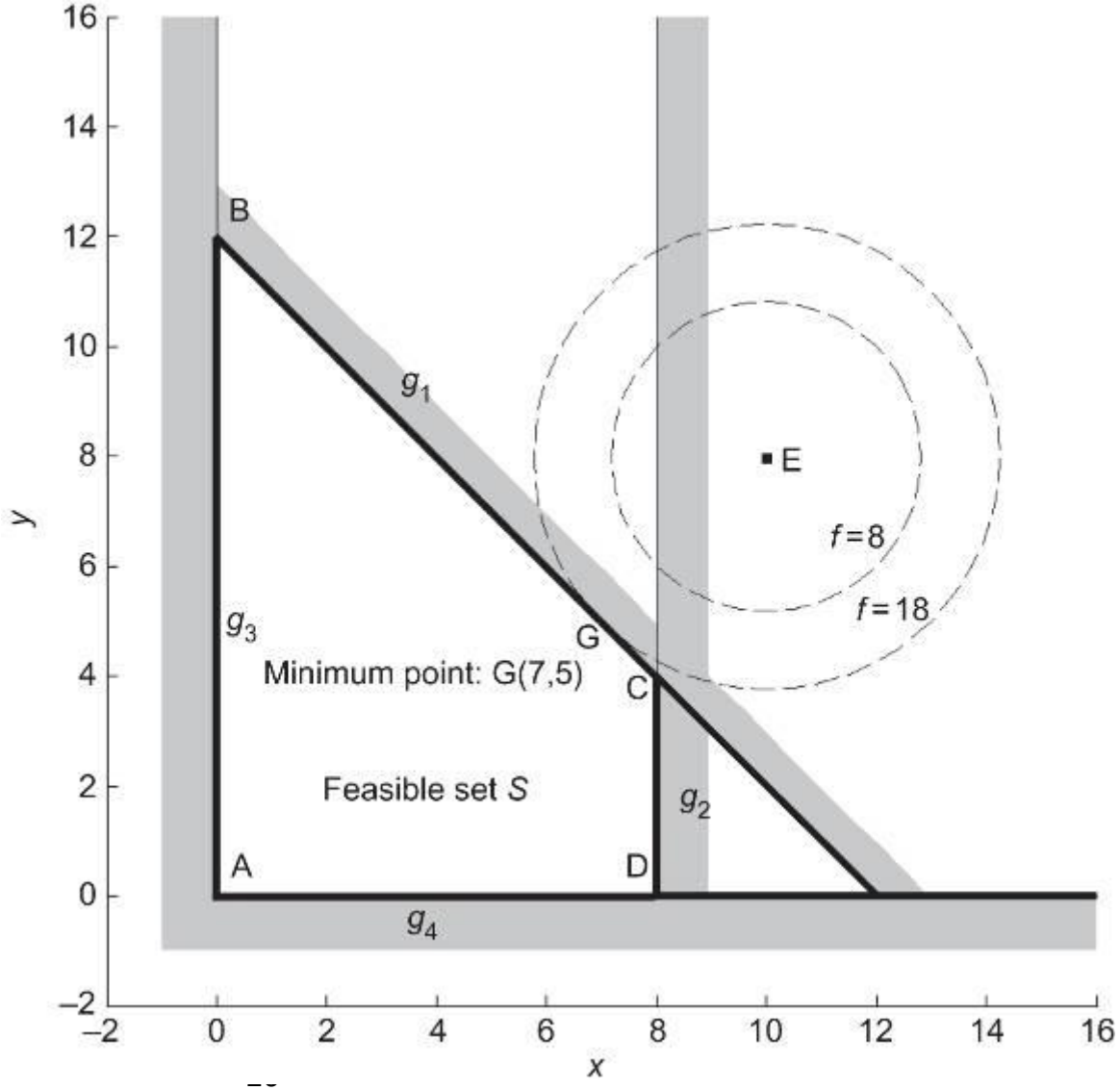


Figure 4.3 Representation of unconstrained minimum for Example 4.1.

Constrained Minimum

Figure 4.4 Representation of constrained minimum for Example 4.2.



Existence of a Global Minimum

Weierstrass Theorem:

1. If $f(\mathbf{x})$ is continuous
2. On non-empty set S *and*
3. S is closed and bounded

Then $f(\mathbf{x})$ has a global minimum in S

NOTE: If Weierstrass conditions are not satisfied

A global minimum may still exist!

The power of the theorem is that it guarantees a global min if conditions are satisfied.

e.g. $f(\mathbf{x}) = x^2$ *on open interval*

Gradient is ⊥ to tangent plane

$$\nabla f(\mathbf{x}^*) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{bmatrix}_{\mathbf{x}^*}$$

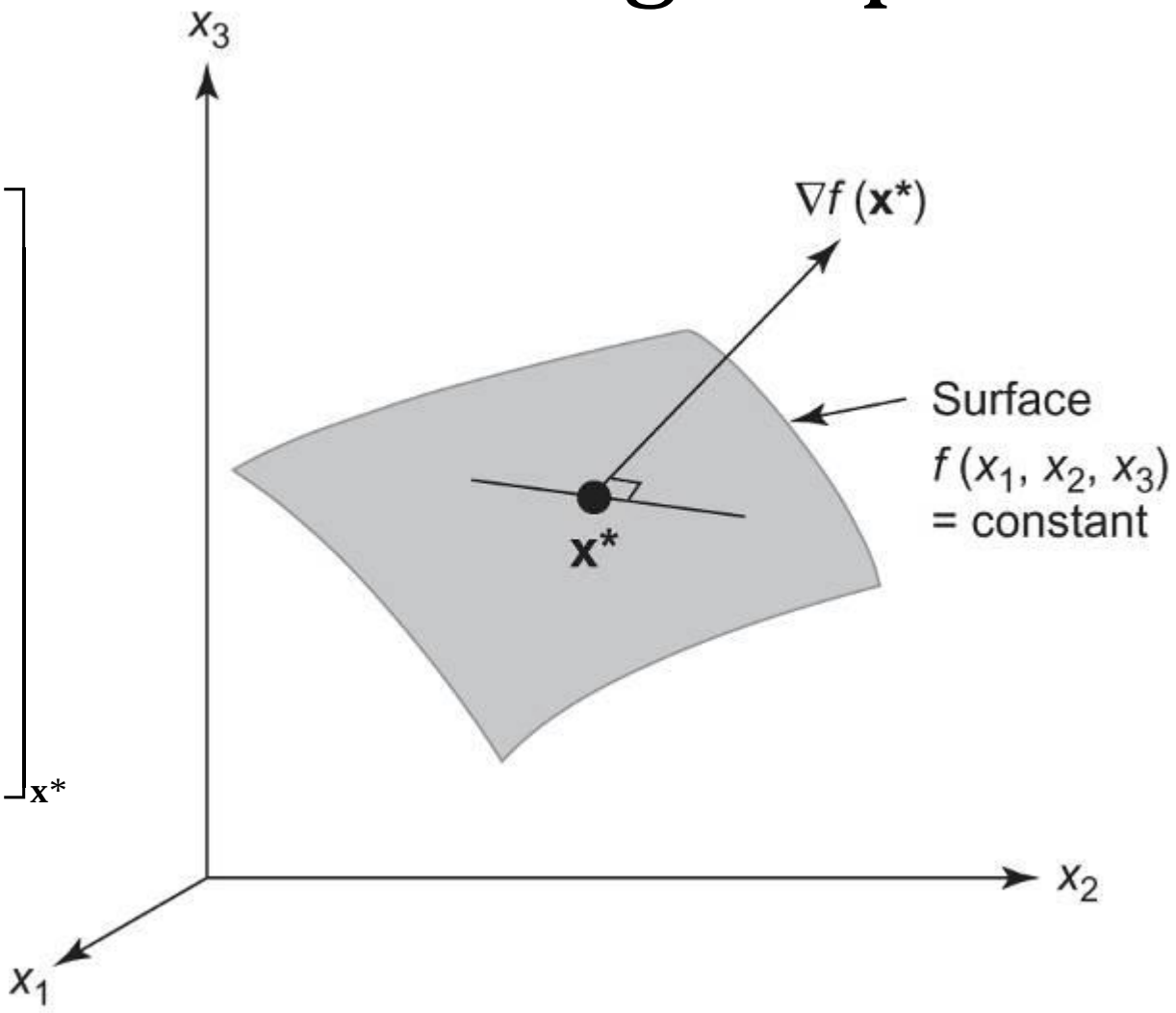


Figure 4.5 Gradient vector for $f(x_1, x_2, x_3)$ at the point x^*.

Gradient example

$$\nabla f(\mathbf{x}^*) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix}_{\mathbf{x}^*}$$

$$\mathbf{c} = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 - 1) \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} 2(1.8 - 1) \\ 2(1.6 - 1) \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} 1.6 \\ 1.2 \end{bmatrix}$$

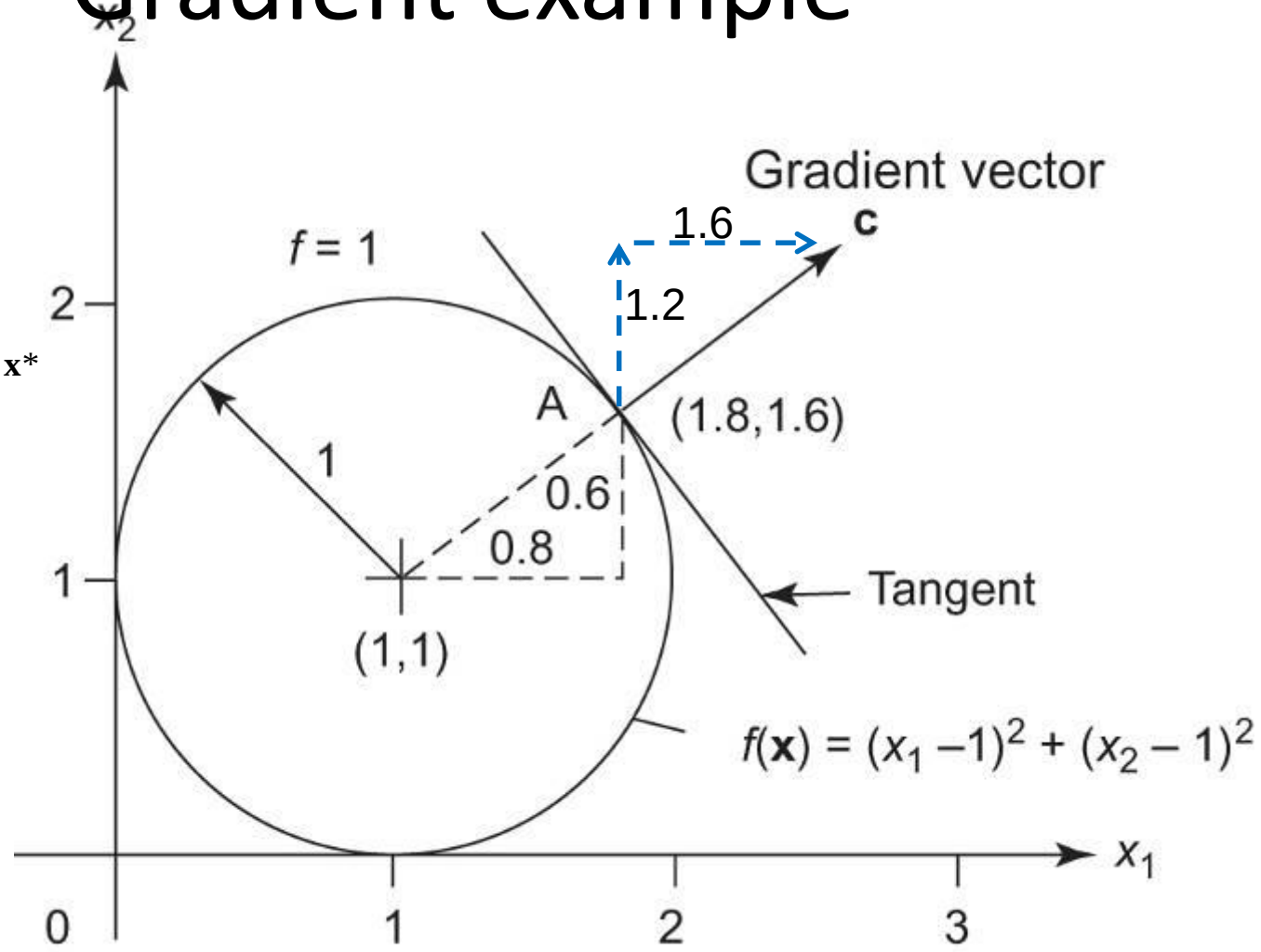


Figure 4.6 Gradient vector (that is not to scale) for the function $f(x)$ of Example 4.5 at the point (1.8, 1.6).

Second Partial Deriivatives of a function $f(\mathbf{x})$

Hessian Matrix

$$\nabla^2 f = \mathbf{H}(\mathbf{x}^*) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_n^2} \\ \frac{\partial^2 f}{\partial x_2^2}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2^2}\frac{\partial^2 f}{\partial x_n^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n^2}\frac{\partial^2 f}{\partial x_1^2} & & & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{\mathbf{x}^*}$$

What does the **x*** mean?

EXAMPLE 4.6 EVALUATION OF THE GRADIENT AND HESSIAN OF A FUNCTION

For the following function, calculate the gradient vector and the Hessian matrix at the point (1, 2):

$$f(\mathbf{x}) = x_1^3 + x_2^3 + 2x_1^2 + 3x_2^2 - x_1x_2 + 2x_1 + 4x_2 \tag{a}$$

Solution

The first partial derivatives of the function are given as

$$\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1 - x_2 + 2; \quad \frac{\partial f}{\partial x_2} = 3x_2^2 + 6x_2 - x_1 + 4 \tag{b}$$

Substituting the point $x_1 = 1$, $x_2 = 2$, the gradient vector is given as $\mathbf{c} = (7, 27)$.

The second partial derivatives of the function are calculated as

$$\frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4; \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = -1; \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = -1; \quad \frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 6. \tag{c}$$

Therefore, the Hessian matrix is given as

$$\mathbf{H}(\mathbf{x}) = \begin{bmatrix} 6x_1 + 4 & -1 \\ -1 & 6x_2 + 6 \end{bmatrix} \tag{d}$$

The Hessian matrix at the point (1, 2) is given as

$$\mathbf{H}(1, 2) = \begin{bmatrix} 10 & -1 \\ -1 & 18 \end{bmatrix} \tag{e}$$