OPTIMUM DESIGN

Dr. / Ahmed Nagib Elmekawy

Lecture 7

Fall 2017

Positive definiteness?

Eigenvalue Check for the Form of a Matrix

Let λ_i, $i = 1$ to n be the eigenvalues of a symmetric $n \times n$ matrix $\mathbf{A}$ associated with the quadratic form $F(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ (since $\mathbf{A}$ is symmetric, all eigenvalues are real). The following results can be stated regarding the quadratic form $F(\mathbf{x})$ or the matrix $\mathbf{A}$:

1. $F(\mathbf{x})$ is *positive definite if and only if* all eigenvalues of $\mathbf{A}$ are strictly positive that is, $\lambda_i > 0$, $i = 1$ to n.
2. $F(\mathbf{x})$ is *positive semidefinite if and only if* all eigenvalues of $\mathbf{A}$ are nonnegative that is, $\lambda_i \geq 0$, $i = 1$ to n (note that at least one eigenvalue must be zero for it to be called positive semidefinite).
3. $F(\mathbf{x})$ is *negative definite if and only if* all eigenvalues of $\mathbf{A}$ are strictly negative that is, $\lambda_i < 0$, $i = 1$ to n.
4. $F(\mathbf{x})$ is *negative semidefinite if and only if* all eigenvalues of $\mathbf{A}$ are nonpositive that is, $\lambda_i \leq 0$, $i = 1$ to n (note that at least one eigenvalue must be zero for it to be called negative semidefinite).
5. $F(\mathbf{x})$ is *indefinite if* some $\lambda_i < 0$ and some other $\lambda_j > 0$.

Positive definiteness?

THEOREM 4.3

Check for the Form of a Matrix Using Principal Minors

Let M_k be the kth leading principal minor of the $n \times n$ symmetric matrix $\mathbf{A}$ defined as the determinant of a $k \times k$ submatrix obtained by deleting the last $(n - k)$ rows and columns of $\mathbf{A}$ (Section A.3). Assume that *no two consecutive principal minors* are zero. Then

1. $\mathbf{A}$ is positive definite if and only if all $M_k > 0$, $k = 1$ to n.
2. $\mathbf{A}$ is positive semidefinite if and only if $M_k > 0$, $k = 1$ to r, where $r < n$ is the rank of $\mathbf{A}$ (refer Section A.4 for a definition of the rank of a matrix).
3. $\mathbf{A}$ is negative definite if and only if $M_k < 0$ for k odd and $M_k > 0$ for k even, $k = 1$ to n.
4. $\mathbf{A}$ is negative semidefinite if and only if $M_k \leq 0$ for k odd and $M_k \geq 0$ for k even, $k = 1$ to $r < n$.
5. $\mathbf{A}$ is indefinite if it does not satisfy any of the preceding criteria.

Positive definiteness?

- By inspection
- Leading principal minors
- Eigenvalues

Given $F(\mathbf{x}) = \frac{1}{2}\,\mathbf{x}^T\mathbf{A}\,\mathbf{x}$ where $\mathbf{A}$ is symmetric

e.g. by inspection $F(\mathbf{x}) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$

Find leading principal minors to check PD of **A**(**x**)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$M_1 = |a_{11}|$$

$$M_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$M_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Principal Minors Test for PD

A matrix is positive definite if:

1.No two consecutive minors can be zero AND

2. All minors are positive, i.e. $M_k > 0$

If two consecutive minors are zero
The test cannot be used.

Principal Minors Test for ND

A matrix is negative definite if:

1.No two consecutive minors can be zero AND

2. $M_k<0$ for k=odd

3. $M_k>0$ for k=even

If two consecutive minors are zero
The test cannot be used.

Eigenvalues

$$\mathbf{Ax} = \lambda\mathbf{x}$$

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

Since **x** should not be zero… we should find values for lambda such that

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$$

Eigenvalue test

$$\left|\mathbf{A}-\lambda\mathbf{I}\right|=0$$

Form	Eigenvalue Test
Positive Definite (PD) $\mathbf{x}^T\mathbf{A}\,\mathbf{x}>0$ for all $\mathbf{x}$, other than $\mathbf{x}=\mathbf{0}$	$\lambda_i>0$
Positive Semi-def (PSD) $\mathbf{x}^T\mathbf{A}\,\mathbf{x}\geq 0$ for all $\mathbf{x}$, and $\mathbf{x}^T\mathbf{A}\,\mathbf{x}=0$ for at least one $\mathbf{x}\neq\mathbf{0}$	$\lambda_i\geq 0$
Indefinite $\mathbf{x}^T\mathbf{A}\,\mathbf{x}\leq 0$ for some $\mathbf{x}$ $\mathbf{x}^T\mathbf{A}\,\mathbf{x}\geq 0$ for other $\mathbf{x}$	$\lambda_i>0$ $\lambda_i<0$

Eigenvalue example

EXAMPLE 4.12 DETERMINATION OF THE FORM OF A MATRIX

Determine the form of the matrices given

$$(1)\ \mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad (2)\ \mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

For a given matrix $\mathbf{A}$, the eigenvalue problem is defined as $\mathbf{Ax} = \lambda\mathbf{x}$, where λ is an eigenvalue and $\mathbf{x}$ is the corresponding eigenvector (refer Section A.6 for details). To determine the eigenvalues, we set the so-called characteristic determinant to zero $|(\mathbf{A} - \lambda\mathbf{I})| = 0$. Since the matrix (1) is diagonal, its eigenvalues are the diagonal elements (ie, $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = 4$). Since all eigenvalues are strictly positive, the matrix is positive definite. The principal minor check of Theorem 4.3 gives the three principal minors as

$$M_1 = a_{11} = 2; \quad M_2 = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8; \quad M_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 24 \qquad \text{(a)}$$

Eigenvalue example

2) $\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = 0$

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = (-1-\lambda)[(-1-\lambda)^2 - 1] = 0$$

$$(-1-\lambda)^2 - 1 = 0$$
$$(1 - 2\lambda + \lambda^2) - 1 = 0$$
$$-2\lambda + \lambda^2 = 0$$
$$\lambda(-2 + \lambda) = 0$$

$$\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = -2$$

Therefore **A** is NSD

Eigenvalue example

2) $\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

To use Theorem 4.3, we calculate the three leading principal minors as

$$M_1 = -1, \quad M_2 = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0, \quad M_3 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = 0 \qquad \text{(d)}$$

Since there are two consecutive zero leading principal minors, we cannot use Theorem 4.3.

EXAMPLE 4.13 CALCULATIONS FOR THE GRADIENT AND HESSIAN OF THE QUADRATIC FORM

Calculate the gradient and Hessian of the following quadratic form:

$$F(\mathbf{x}) = 2x_1^2 + 2x_1x_2 + 4x_1x_3 - 6x_2^2 - 4x_2x_3 + 5x_3^2 \tag{a}$$

Solution

Differentiating $F(\mathbf{x})$ with respect to x_1, x_2, and x_3, we get gradient components as

$$\frac{\partial F}{\partial x_1} = (4x_1 + 2x_2 + 4x_3); \quad \frac{\partial F}{\partial x_2} = (2x_1 - 12x_2 - 4x_3); \quad \frac{\partial F}{\partial x_3} = (4x_1 - 4x_2 + 10x_3) \tag{b}$$

Writing this in the vector form, we get gradient of $F(\mathbf{x})$ as

$$\nabla F(\mathbf{x}) = \begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \frac{\partial F}{\partial x_2} \\ \frac{\partial F}{\partial x_3} \end{bmatrix} = \begin{bmatrix} (4x_1 + 2x_2 + 4x_3) \\ (2x_1 - 12x_2 - 4x_3) \\ (4x_1 - 4x_2 + 10x_3) \end{bmatrix} \tag{c}$$

EXAMPLE 4.13

Differentiating the gradient components once again, we get the Hessian components as

$$\frac{\partial^2 F}{\partial x_1^2} = 4, \frac{\partial^2 F}{\partial x_1 \partial x_2} = 2, \frac{\partial^2 F}{\partial x_1 \partial x_3} = 4$$

$$\frac{\partial^2 F}{\partial x_2 \partial x_1} = 2, \frac{\partial^2 F}{\partial x_2^2} = -12, \frac{\partial^2 F}{\partial x_2 \partial x_3} = -4 \qquad \text{(d)}$$

$$\frac{\partial^2 F}{\partial x_3 \partial x_1} = 4, \frac{\partial^2 F}{\partial x_3 \partial x_2} = -4, \frac{\partial^2 F}{\partial x_3^2} = 10$$

Writing the Hessian in the matrix form, we get

$$\mathbf{H} = \begin{bmatrix} 4 & 2 & 4 \\ 2 & -12 & -4 \\ 4 & -4 & 10 \end{bmatrix} \qquad \text{(e)}$$

Writing the given quadratic form in the matrix notation, we identify the symmetric matrix **A** as

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & -6 & -2 \\ 2 & -2 & 5 \end{bmatrix} \qquad \text{(f)}$$

EXAMPLE 4.13

Comparing elements of the matrix **A** with Hessian **H** of F in Eq. (e), we observe that $\mathbf{H} = 2\mathbf{A}$. Using Eq. (4.21), the gradient of the quadratic form is also given as $\nabla F(\mathbf{x}) = 2\mathbf{A}\mathbf{x}$:

$$\nabla F(\mathbf{x}) = 2\begin{bmatrix} 2 & 1 & 2 \\ 1 & -6 & -2 \\ 2 & -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (4x_1 + 2x_2 + 4x_3) \\ (2x_1 - 12x_2 - 4x_3) \\ (4x_1 - 4x_2 + 10x_3) \end{bmatrix} \tag{g}$$

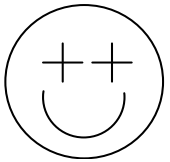
Single variable optimization

First-order necessary condition

$f'(x^*) = 0$ "stationary point(s)"

$f''(x^*) > 0$

Second-order sufficient condition for a minimum



$f''(x^*) < 0$

Second-order sufficient condition for a maximum

EXAMPLE 4.15 DETERMINATION OF LOCAL MINIMUM POINTS USING FIRST-ORDER NECESSARY CONDITIONS

Find the local minima for the function

$$f(x) = x^2 - 4x + 4 \tag{a}$$

Solution

Fig. 4.7 shows a graph for the function $f(x) = x^2 - 4x + 4$. It can be seen that the function always has a positive value except at $x = 2$, where it is zero. Therefore, this is a local as well as a global minimum point for the function. Let us see how this point will be determined using the necessary and sufficient conditions.

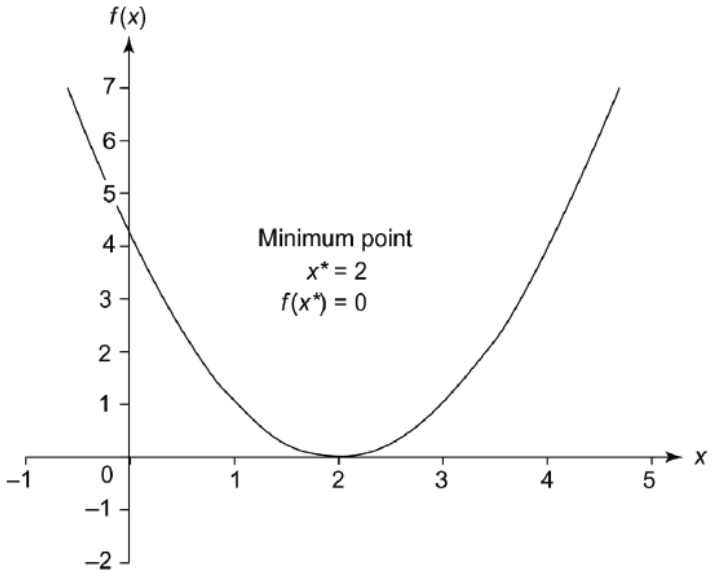
Differentiating the function twice,

$$f' = 2x - 4; \quad f'' = 2 \tag{b}$$

The necessary condition $f' = 0$ implies that $x^* = 2$ is a stationary point. Since $f'' > 0$ at $x^* = 2$ (actually for all x), the sufficiency condition of Eq. (4.28) is satisfied. Therefore $x^* = 2$ is a local minimum for $f(x)$. The minimum value of f is 0 at $x^* = 2$.

Note that at $x^* = 2$, the second-order necessary condition for a local maximum $f'' \leq 0$ is violated since $f''(2) = 2 > 0$. Therefore the point $x^* = 2$ cannot be a local maximum point. In fact the graph of the function shows that there is no local or global maximum point for the function.

Checking the optimality of a given point: If we are required to check optimality of a given point, say $x = 1$, we can substitute it into the necessary condition and see if it is satisfied. From Eq. (b), $f' = -2$ at $x = 1$. Therefore the necessary condition is violated and $x = 1$ is not a candidate minimum point.



$$f(x) = x^3 - x^2 - 4x + 4$$

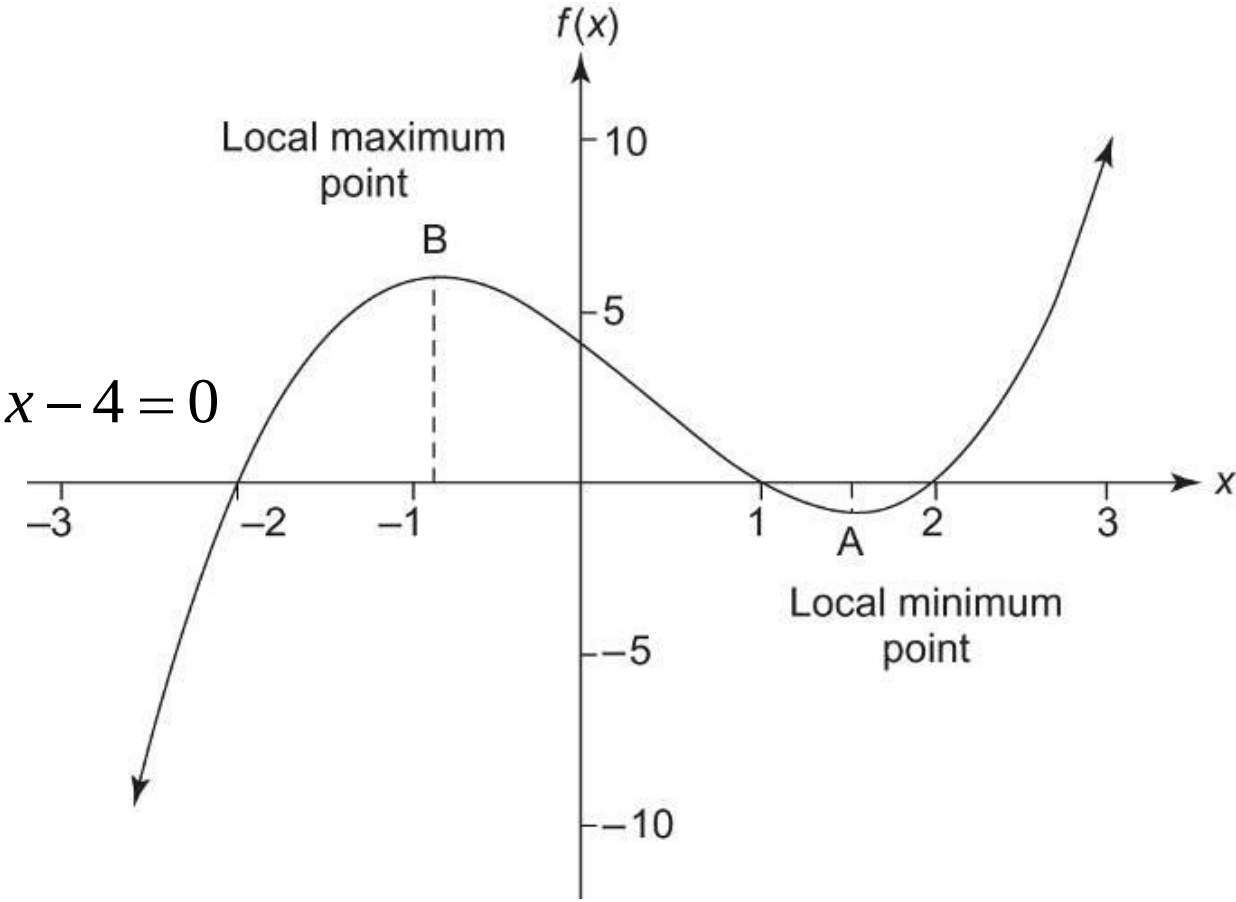
Necessary condition

$$f'(x) = 3x^2 - 2x - 4$$

$$f'(x) = 0 \quad i.e. \quad 3x^2 - 2x - 4 = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_1 = \quad 1.535 \text{ pt A}$$
$$x_2 = -0.8685 \text{ pt B}$$



Sufficient condition

$$f''(x) = 6x - 2$$
$$f''(1.535) = 7.211 > 0 \quad \therefore \quad Min$$
$$f''(-0/8685) = -7.211 < 0) \quad \therefore \quad Max$$

What happens when
$f''(x)=0$?
i.e. $x=2/6=1/3$

EXAMPLE 4.17 DETERMINATION OF LOCAL MINIMUM POINTS USING FIRST-ORDER NECESSARY CONDITIONS

Find the minimum value for the function

$$f(x) = x^4 \tag{a}$$

Solution

Differentiating the function twice,

$$f' = 4x^3; \quad f'' = 12x^2 \tag{b}$$

The first order necessary condition ($f' = 0$) gives $x^* = 0$ as a stationary point. Since $f''(x^*) = 0$, we cannot conclude from the sufficiency condition of Eq. (4.28) that x^* is a minimum point. However, the second-order necessary condition of Eq. (4.29) is satisfied, so we cannot rule out the possibility of x^* being a minimum point. In fact, a graph of $f(x)$ versus x will show that x^* is indeed the global minimum point. $f''' = 24x$, which is zero at $x^* = 0$. $f''''(x^*) = 24$, which is strictly greater than zero. Therefore, the fourth-order sufficiency condition is satisfied, and $x^* = 0$ is indeed a local minimum point. It is actually a global minimum point with $f(0) = 0$.

MV Optimization

For **x*** to be a local minimum: $\Delta f = f(\mathbf{x}) - f(\mathbf{x}^*) > 0$

$$\Delta f = \nabla f^T(\mathbf{x}^*)\mathbf{d} + \frac{1}{2}\mathbf{d}^T\mathbf{H}\,\mathbf{d}$$

$$\nabla f^T(\mathbf{x}^*) = \mathbf{0}$$

1st order
Necessary
Condition

$$\mathbf{d}^T\mathbf{H}\,\mathbf{d} > 0$$

2nd order
Sufficient
Condition

i.e. **H(x***) must be positive definite

Positive definiteness Tests?

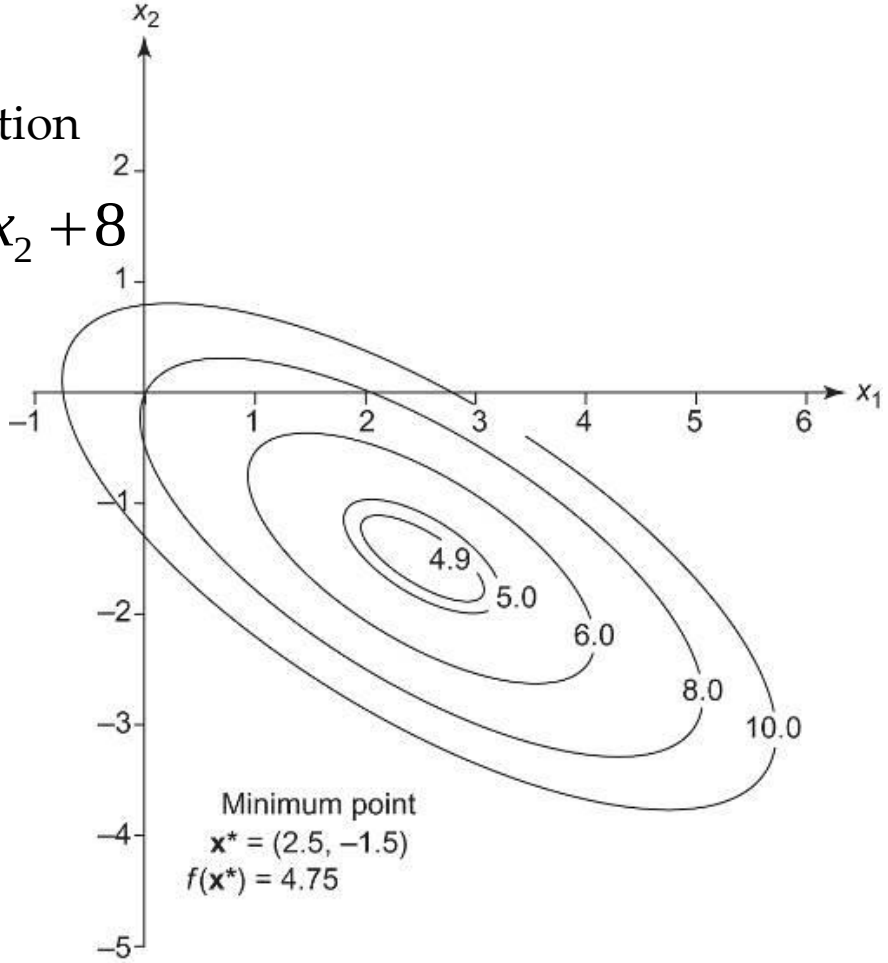
- By inspection
- Leading principal minors
- Eigenvalues

Given $F(\mathbf{x}) = \frac{1}{2}\,\mathbf{x}^T\mathbf{A}\,\mathbf{x}$ where $\mathbf{A}$ is symmetric

e.g. by inspection $F(\mathbf{x}) = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$

Find local minimum points for the function

$$f(\mathbf{x}) = x_1^2 + 2x_1x_2 + 2x_2^2 - 2x_1 + x_2 + 8$$



Necessary condition

$$\nabla f(\mathbf{x}) = \begin{bmatrix} (2x_1 + 2x_2 - 2) \\ (2x_1 + 4x_2 + 1) \end{bmatrix}_{\mathbf{x}^*} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x}^* = \begin{bmatrix} 2.5 & -1.5 \end{bmatrix}$$

Sufficient condition

$$\nabla^2 f = \mathbf{H}(\mathbf{x}^*) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix}_{\mathbf{x}^*} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

H(x) is Pos Def
x* is local min!

Effects of scaling $f(\mathbf{x})$ or adding a constant

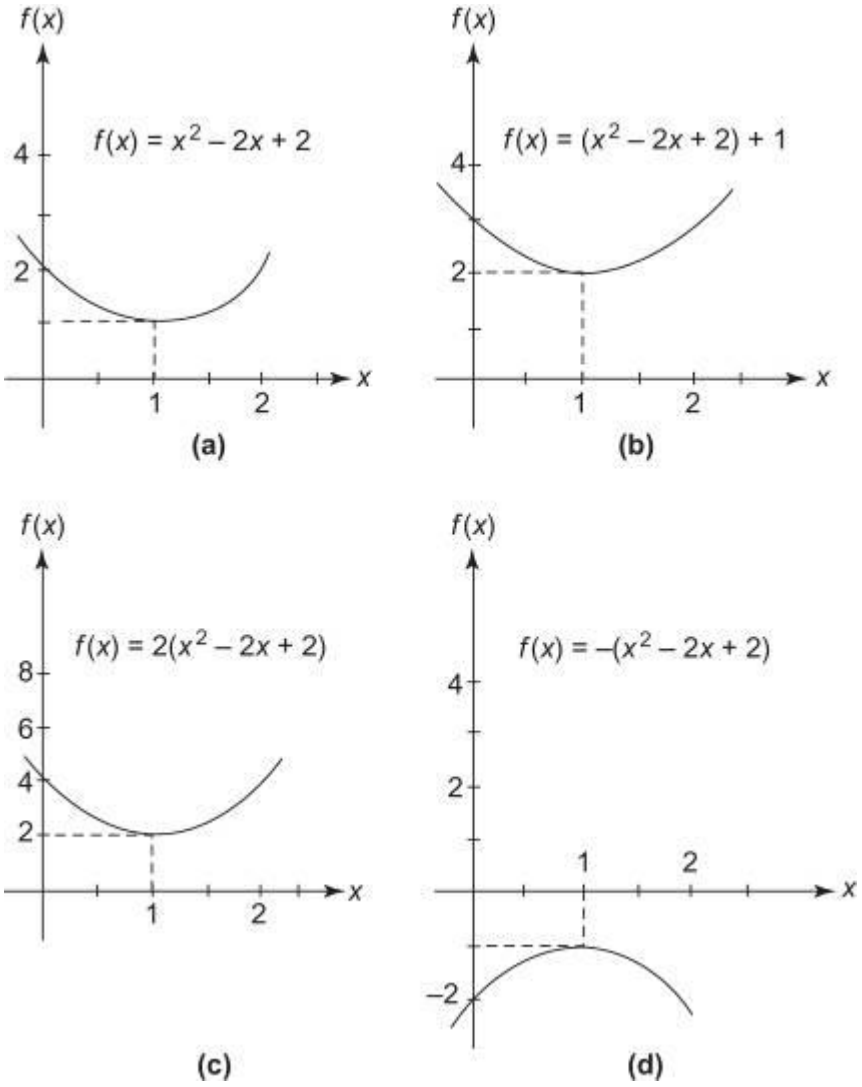


Figure 4.9 Graphs for Example 4.19. Effects of scaling or of adding a constant to a function.
(a) A graph of $f(x)=x^2-2x+2$.
(b) The effect of addition of a constant to $f(x)$.
(c) The effect of multiplying $f(x)$ by a positive constant.
(d) Effect of multiplying $f(x)$ by -1.

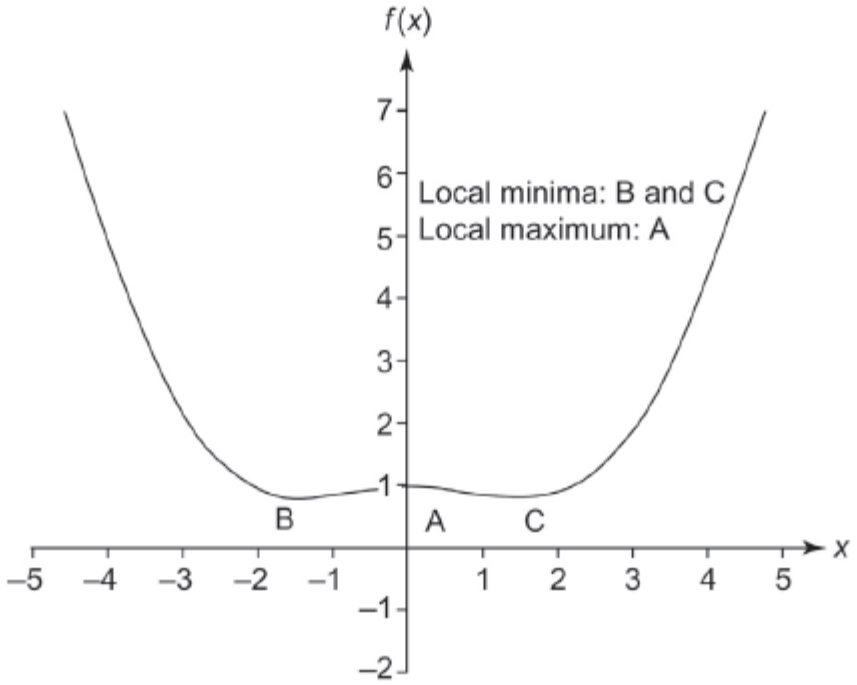
EXAMPLE 4.22 NUMERICAL SOLUTION FOR THE FIRST-ORDER NECESSARY CONDITIONS

Find stationary points for the following function and check the sufficiency conditions for them:

$$f(x) = \frac{1}{3}x^2 + \cos x \tag{a}$$

Solution

The function is plotted in Fig. 4.11. It is seen that there are three stationary points: $x = 0$ (point A), x between 1 and 2 (point C), and x between -1 and -2 (point B). The point $x = 0$ is a local maximum for the function, and the other two are local minima.



The first-order necessary condition is

$$f'(x) = \frac{2}{3}x - \sin x = 0 \tag{b}$$

It is seen that $x = 0$ satisfies Eq. (b), so it is a stationary point. We must find other roots of Eq. (b). Finding an analytical solution for the equation is difficult, so we must use numerical methods.

We can either plot $f'(x)$ versus x and locate the point where $f'(x) = 0$, or use a numerical method for solving nonlinear equations, such as the Newton–Raphson method. By either of the two methods, we find that $x^* = 1.496$ and $x^* = -1.496$ satisfy $f'(x) = 0$ in Eq. (b). Therefore, these are additional stationary points.

To determine whether they are local minimum, maximum, or inflection points, we must determine f'' at the stationary points and use the sufficient conditions of Theorem 4.4. Since $f'' = 2/3 - \cos x$, we have

1. $x^* = 0$; $f''(0) = -1/3 < 0$, so this is a local maximum with $f(0) = 1$.
2. $x^* = 1.496$; $f''(1.496) = 0.592 > 0$, so this is a local minimum with $f(1.496) = 0.821$.
3. $x^* = -1.496$; $f''(-1.496) = 0.592 > 0$, so this is a local minimum with $f(-1.496) = 0.821$.

These results agree with the graphical solutions observed in Fig. 4.11.

Global optima: Note that $x^* = 1.496$ and $x^* = -1.496$ are actually global minimum points for the function, although the function is unbounded and the feasible set is not closed. Therefore, although the conditions of the Weierstrass Theorem 4.1 are not met, the function has global minimum points. This shows that Theorem 4.1 is not an "if-and-only-if" theorem. Note also that there is no global maximum point for the function since the function is unbounded and x is allowed to have any value.

EXAMPLE 4.23 LOCAL MINIMA FOR A FUNCTION OF TWO VARIABLES USING OPTIMALITY CONDITIONS

Find a local minimum point for the function

$$f(\mathbf{x}) = x_1 + \frac{(4\times 10^6)}{x_1 x_2} + 250 x_2 \tag{a}$$

Solution

The first order necessary conditions for optimality are

$$\frac{\partial f}{\partial x_1} = 0; \quad 1 - \frac{(4\times 10^6)}{x_1^2 x_2} = 0 \tag{b}$$

$$\frac{\partial f}{\partial x_2} = 0; \quad 250 - \frac{(4\times 10^6)}{x_1 x_2^2} = 0 \tag{c}$$

Equations (b) and (c) give

$$x_1^2 x_2 - (4\times 10^6) = 0;\ 250 x_1 x_2^2 - (4\times 10^6) = 0 \tag{d}$$

These equations give

$$x_1^2 x_2 = 250 x_1 x_2^2, \text{ or } x_1 x_2 (x_1 - 250 x_2) = 0 \tag{e}$$

Since neither x_1 nor x_2 can be zero (the function has singularity at $x_1 = 0$ or $x_2 = 0$), the preceding equation gives $x_1 = 250x_2$. Substituting this into Eq. (c), we obtain $x_2 = 4$. Therefore, $x_1^* = 1000$, and $x_2^* = 4$ is a stationary point for the function $f(\mathbf{x})$.

Using Eqs. (b) and (c), the second partial derivatives of the cost function are given as

$$\frac{\partial^2 f}{\partial x_1^2} = 2 \times \frac{4 \times 10^6}{x_1^3 x_2}; \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{4 \times 10^6}{x_1^2 x_2^2}; \quad \frac{\partial^2 f}{\partial x_2^2} = 2 \times \frac{4 \times 10^6}{x_1 x_2^3} \tag{f}$$

Using these derivatives, Hessian matrix for $f(\mathbf{x})$ at the point $\mathbf{x}^*$ is assembled as

$$\mathbf{H} = \frac{(4 \times 10^6)}{x_1^2 x_2^2} \begin{bmatrix} \frac{2x_2}{x_1} & 1 \\ 1 & \frac{2x_1}{x_2} \end{bmatrix}; \quad \mathbf{H}(1000, 4) = \frac{(4 \times 10^6)}{(4000)^2} \begin{bmatrix} 0.008 & 1 \\ 1 & 500 \end{bmatrix} \tag{f}$$

Eigenvalues of the Hessian **H** are calculated by defining the following eigenvalue problem (note that the factor of ¼ has been taken inside the matrix):

$$\begin{vmatrix} 0.002-\lambda & 0.25 \\ 0.25 & 125-\lambda \end{vmatrix} = (0.002-\lambda)(125-\lambda) - 0.25^2 = 0 \qquad \text{(g)}$$

Roots of this equation give the two eigenvalues as $\lambda_1 = 0.0015$ and $\lambda_2 = 125$. Since both eigenvalues are positive: the Hessian of $f(\mathbf{x})$ at the point $\mathbf{x}^*$ is positive definite (also, both the principal minors are positive, $M_1 = 0.002, M_2 = 0.4375$). Therefore: $\mathbf{x}^* = (1000, 4)$ satisfies the sufficiency condition for a local minimum point with $f(\mathbf{x}^*) = 3000$. Fig. 4.12 shows some isocost curves for the function of this problem. It is seen that $x_1 = 1000$ and $x_2 = 4$ is the minimum point. (Note that the horizontal and vertical scales are quite different in Fig. 4.12; this is done to obtain reasonable isocost curves.)