OPTIMUM DESIGN

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Lecture 8

Fall 2017

General Design Optimization Model

Design variable vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$

Cost function $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ (4.35)

Equality constraints $h_i(\mathbf{x}) = 0; \quad i = 1 \text{ to } p$ (4.36)

Inequality constraints $g_i(\mathbf{x}) \leq 0; \quad i = 1 \text{ to } m$ (4.37)

MV Optimization- CONSTRAINED

For **x*** to be a local minimum:

$$MINIMIZE: \quad f(\mathbf{x})$$

Subject To:

$$h_j(\mathbf{x}) = 0 \qquad j = 1 \ldots p$$

$$x_i^{(U)} \geq x_i \geq x_i^{(L)} \qquad i = 1 \ldots n$$

LaGrange Function

If we let x* be the minimum $f(\mathbf{x}^*)$ in the feasible region:
All $\mathbf{x}^*$ satisfy the equality constraints (i.e. $h_j = 0$)

Let's create the LaGrange Function by augmenting the objective function with "0's"
Using parameters, known as LaGrange multipliers, and the equality constraints

$$L(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + v_1 h_1(\mathbf{x}) + v_2 h_2(\mathbf{x}) + \ldots v_p h_p(\mathbf{x})$$

or in summation notation

$$L(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x})$$

and using vectors

$$L(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + \mathbf{v}^T \mathbf{h}(\mathbf{x})$$

$$"v" = upsilon = scalar$$

$$"\mathbf{v}" = nu = vector$$

Necessary Condition

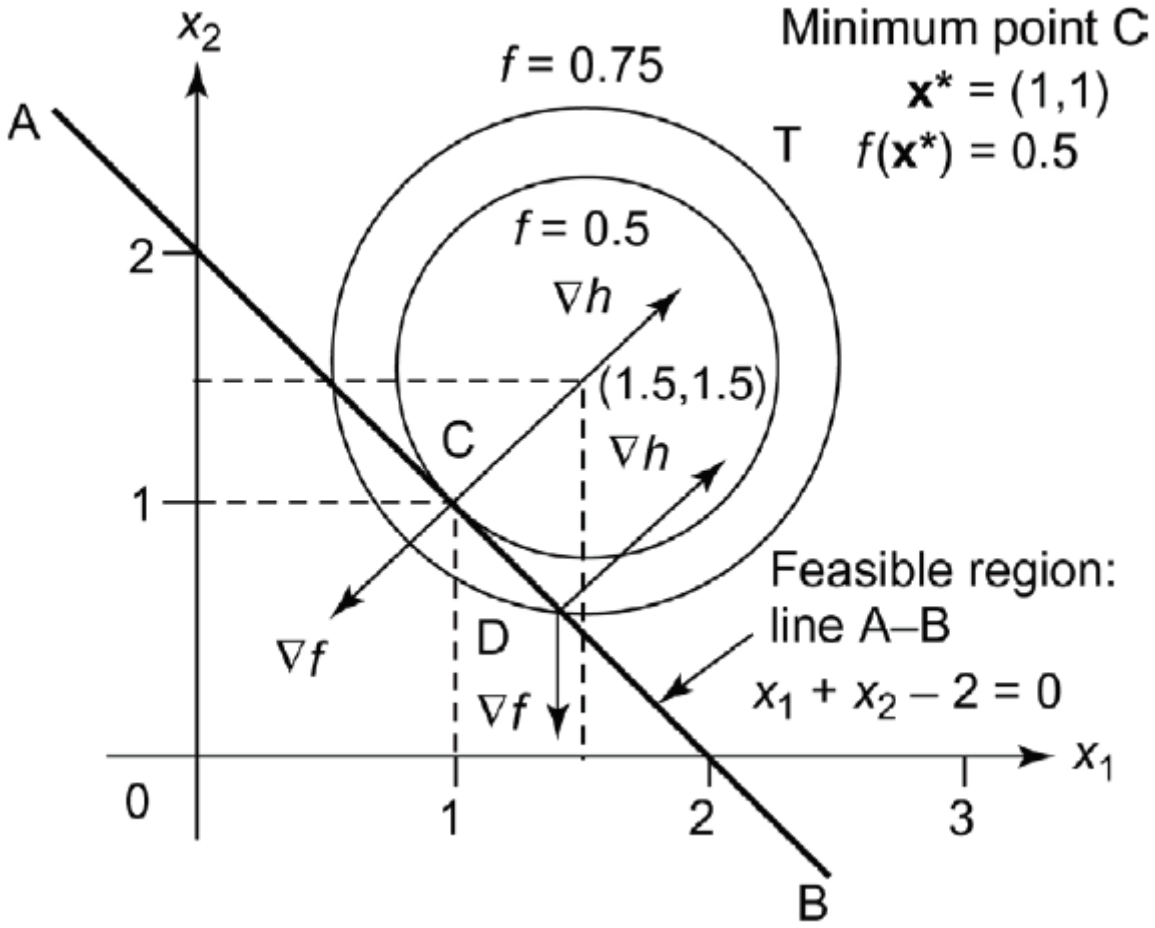
Necessary condition for a stationary point
Given $f(\mathbf{x})$, one equality constraint, and $n=2$

$$\nabla L(\mathbf{x}, \mathbf{v}) = \mathbf{0}$$
$$\nabla\{ f(\mathbf{x}) + v_1 h_1(\mathbf{x})\} = \mathbf{0}$$
$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} + v_1 \frac{\partial h_1(\mathbf{x})}{\partial x_1} = 0$$
$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} + v_1 \frac{\partial h_1(\mathbf{x})}{\partial x_2} = 0$$
$$\frac{\partial L}{\partial v_1} = h_1(\mathbf{x}) = 0$$

$$\nabla L(\mathbf{x}, \mathbf{v}) = \mathbf{0}$$
$$\frac{\partial f}{\partial x_1} + v_1 \frac{\partial h_1(\mathbf{x})}{\partial x_1} = 0$$
$$\frac{\partial f}{\partial x_2} + v_1 \frac{\partial h_1(\mathbf{x})}{\partial x_2} = 0$$
$$h_1(\mathbf{x}) = 0$$

2D Example

$$\text{Minimize } f(x_1, x_2) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$
$$\text{subject to: } h((x_1, x_2) = x_1 + x_2 - 2 = 0$$



Example cont'd

$$L(\mathbf{x}, \upsilon) = f(\mathbf{x}) + \upsilon h(\mathbf{x})$$
$$L(\mathbf{x}, \upsilon) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + \upsilon(x_1 + x_2 - 2)$$

$$\nabla L(\mathbf{x}, \mathbf{v}) = \mathbf{0}$$

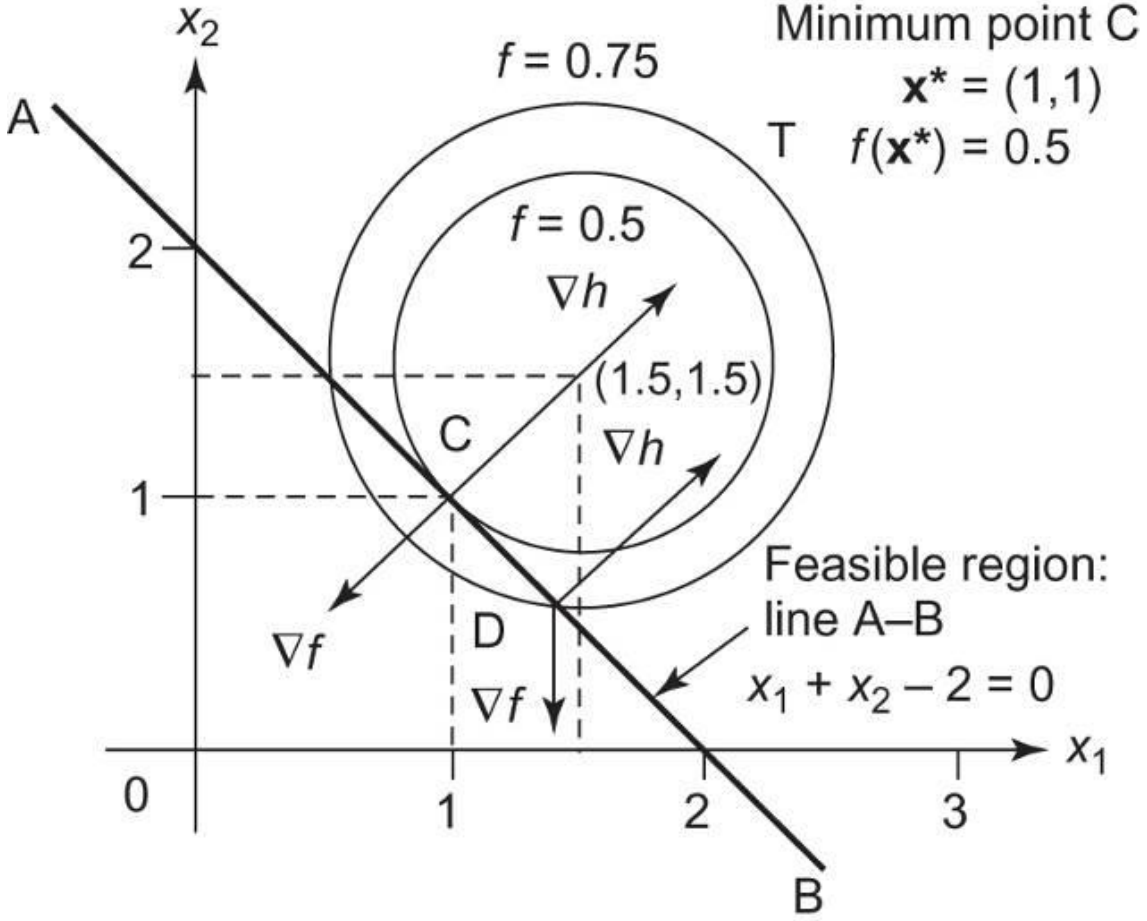
$$\frac{\partial f}{\partial x_1} + \upsilon \frac{\partial h(\mathbf{x})}{\partial x_1} = 2(x_1 - 1.5) + \upsilon = 0$$
$$\frac{\partial f}{\partial x_2} + \upsilon \frac{\partial h(\mathbf{x})}{\partial x_2} = 2(x_2 - 1.5) + \upsilon = 0$$
$$h_1(\mathbf{x}) = x_1 + x_2 - 2 = 0$$

system of 3 eqns and 3 unkns

$$2(x_1 - 1.5) + \upsilon = 0$$
$$2(x_2 - 1.5) + \upsilon = 0$$
$$x_1 + x_2 - 2 = 0$$

simultaneous solution gives :

$$\upsilon = 1$$
$$x_1^* = 1$$
$$x_2^* = 1$$



optimal solution

$$v = 1 \quad x_1^* = 1 \quad x_2^* = 1$$

optimal value

$$f(1,1) = (1-1.5)^2 + (1-1.5)^2 = 0.5$$

$$\nabla f(\mathbf{x}^*) = \begin{bmatrix} 2(x_1 - 1.5) \\ 2(x_2 - 1.5) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\nabla h(\mathbf{x}^*) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

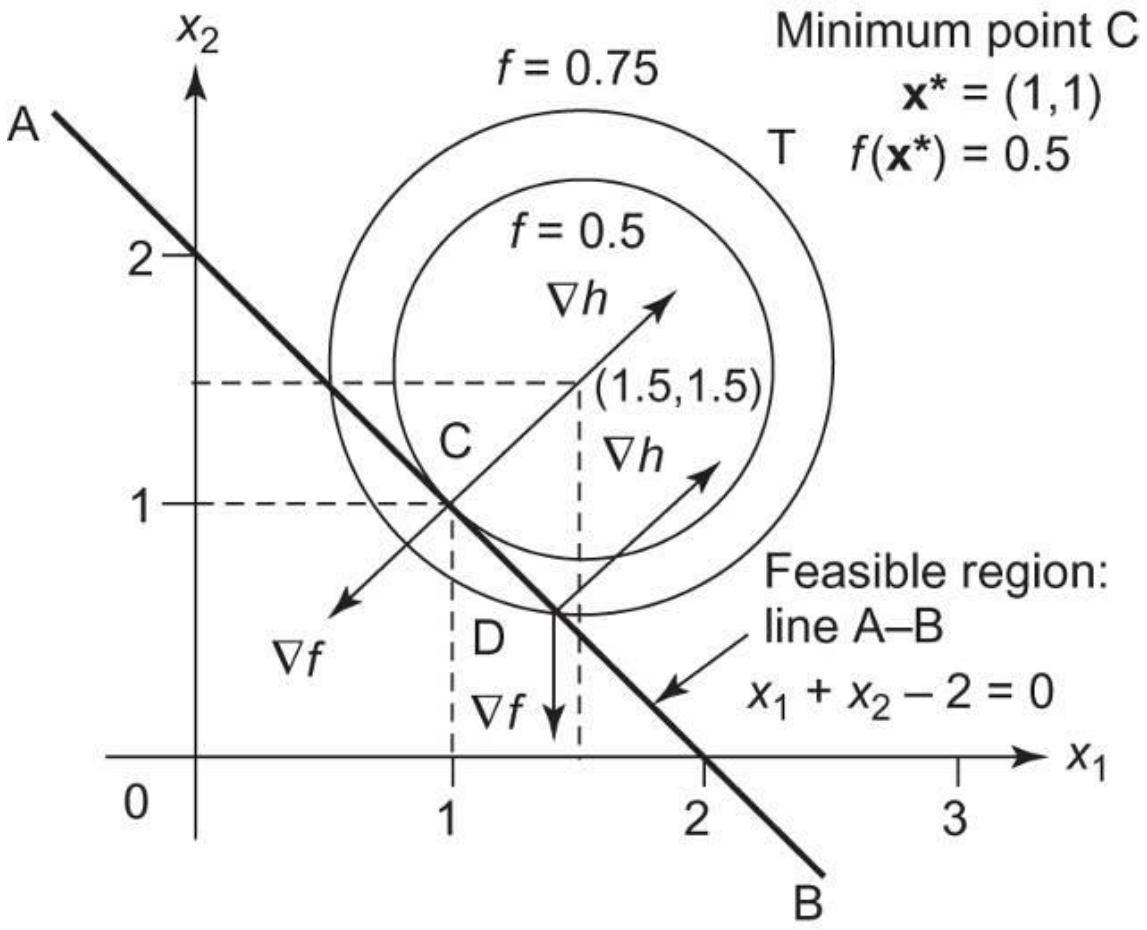
Geometric meaning?

$$\nabla f(\mathbf{x}^*) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\nabla h(\mathbf{x}^*) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\nabla f(\mathbf{x}^*) + \upsilon \nabla h(\mathbf{x}^*) = \mathbf{0}$$

$$\nabla f(\mathbf{x}^*) = -\upsilon \nabla h(\mathbf{x}^*)$$



Lagrange Multiplier Theorem

THEOREM 4.5

Lagrange Multiplier Theorem

Consider the optimization problem defined in Eqs. (4.35) and (4.36):
Minimize $f(\mathbf{x})$
subject to equality constraints

$$h_i(\mathbf{x}) = 0,\ i = 1 \text{ to } p$$

Let $\mathbf{x}^*$ be a regular point that is a local minimum for the problem. Then there exist unique Lagrange multipliers $v_j^*,\ j = 1 \text{ to } p$ such that

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=i}^{p} v_j^* \frac{\partial h_j(\mathbf{x}^*)}{\partial x_i} = 0;\ i = 1 \text{ to } n \tag{4.38}$$

$$h_j(\mathbf{x}^*) = 0;\ j = 1 \text{ to } p \tag{4.39}$$

It is convenient to write these conditions in terms of a Lagrange function, defined as

$$\begin{aligned} L(\mathbf{x},\mathbf{v}) &= f(\mathbf{x}) + \sum_{j=1}^{p} v_j h_j(\mathbf{x}) \\ &= f(\mathbf{x}) + \mathbf{v}^{\mathrm{T}}\mathbf{h}(\mathbf{x}) \end{aligned} \tag{4.40}$$

Then Eq. (4.38) becomes

$$\nabla L(\mathbf{x}^*, \mathbf{v}^*) = \mathbf{0}, \text{ or } \frac{\partial L(\mathbf{x}^*, \mathbf{v}^*)}{\partial x_i} = 0;\quad i = 1 \text{ to } n \tag{4.41}$$

Differentiating $L(\mathbf{x}, \mathbf{v})$ with respect to v_j, we recover the equality constraints as

$$\frac{\partial L(\mathbf{x}^*, \mathbf{v}^*)}{\partial v_j} = 0 \Rightarrow h_j(\mathbf{x}^*) = 0;\quad j = 1 \text{ to } p \tag{4.42}$$

EXAMPLE 4.25 CYLINDRICAL TANK DESIGN

We will resolve the cylindrical storage tank problem (Example 4.21) using the Lagrange multiplier approach. The problem is to find radius R and length H of the cylinder to

Minimize

$$\overline{f} = R^2 + RH \tag{a}$$

subject to

$$h = \pi R^2 H - V = 0 \tag{b}$$

EXAMPLE 4.25 CYLINDRICAL TANK DESIGN

Solution

The Lagrange function L of Eq. (4.40) for the problem is defined as

$$L = R^2 + RH + v(\pi R^2 H - V) \tag{c}$$

The necessary conditions of the Lagrange Multiplier Theorem 4.5 give

$$\frac{\partial L}{\partial R} = 2R + H + 2\pi vRH = 0 \tag{d}$$

$$\frac{\partial L}{\partial H} = R + \pi vR^2 = 0 \tag{e}$$

$$\frac{\partial L}{\partial H} = \pi R^2 H - V = 0 \tag{f}$$

EXAMPLE 4.25 CYLINDRICAL TANK DESIGN

These are three equations in three unknowns v, R, and H. Note that they are nonlinear. However, for the present problem, they can be solved by the elimination process, giving the solution to the necessary conditions as

$$R^* = \left(\frac{V}{2\pi}\right)^{1/3}; \quad H^* = \left(\frac{4V}{\pi}\right)^{1/3}; \quad v^* = -\frac{1}{\pi R} = -\left(\frac{2}{\pi^2 V}\right)^{1/3}; \quad f^* = 3\left(\frac{V}{2\pi}\right)^{2/3} \tag{g}$$

The Role of Inequalities

EXAMPLE 4.26 ACTIVE INEQUALITY

Minimize

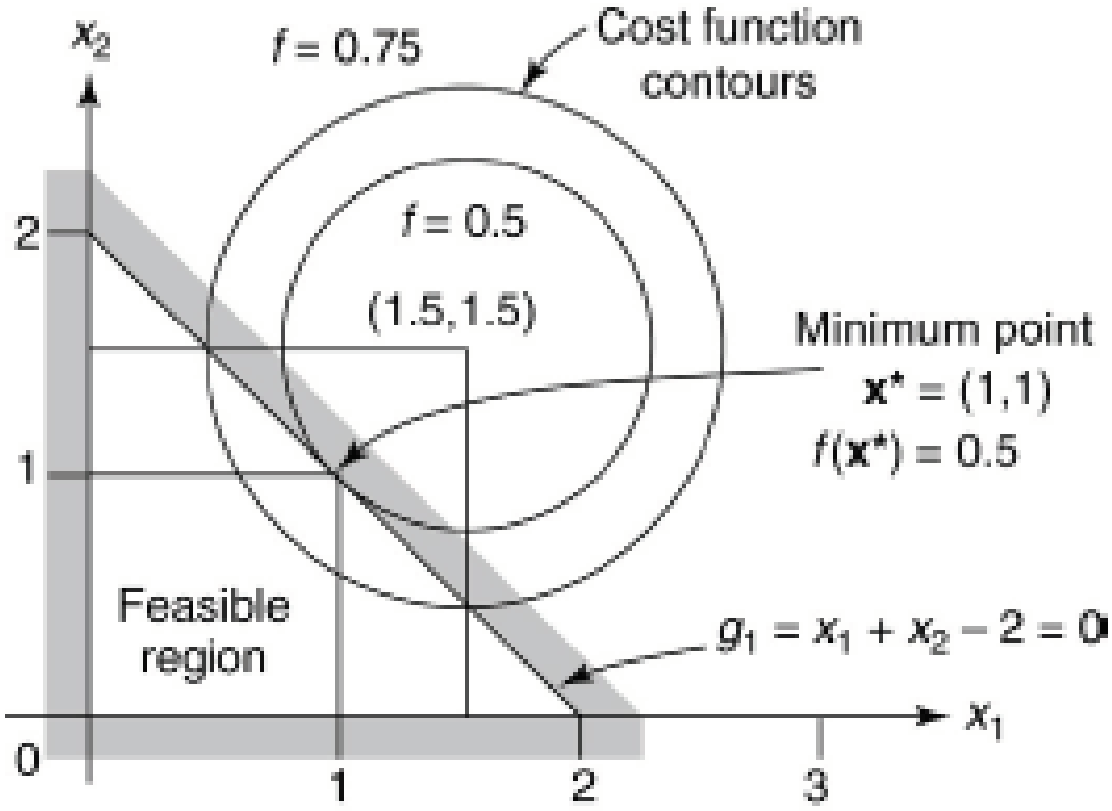
$$f(x) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2 \quad \text{(a)}$$

subject to

$$g_1(x) = x_1 + x_2 - 2 \leq 0 \quad \text{(b)}$$

$$g_2(x) = -x_1 \leq 0;\ g_3(x) = -x_2 \leq 0 \quad \text{(c)}$$

The Role of Inequalities



Example 4.26 graphical representation. Constrained optimum point.

The Role of Inequalities

EXAMPLE 4.27 INACTIVE INEQUALITY

Minimize

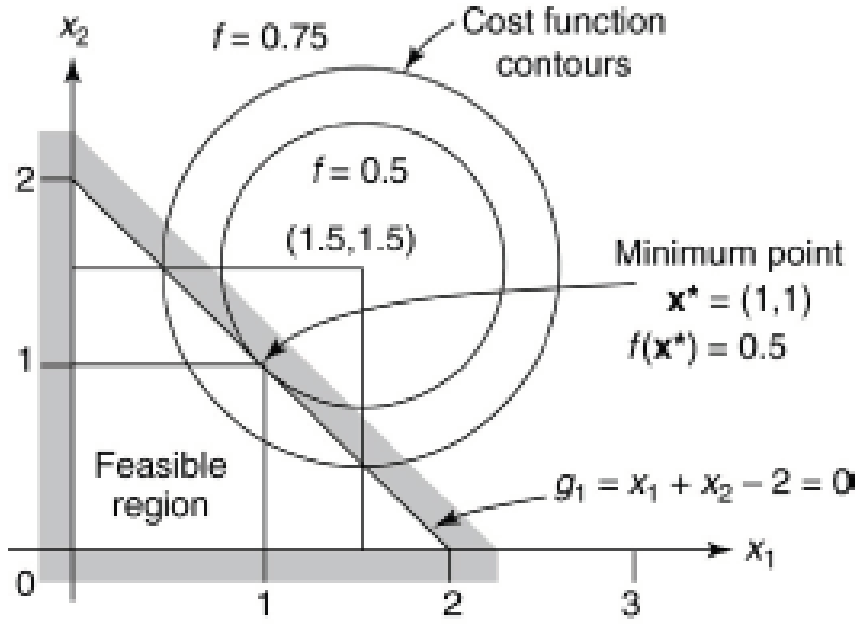
$$f(\mathbf{x}) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2$$

subject to

$$g_1(\mathbf{x}) = x_1 + x_2 - 2 \le 0 \qquad \text{(b)}$$

$$g_2(\mathbf{x}) = -x_1 \le 0;\ g_3(\mathbf{x}) = -x_2 \le 0 \qquad \text{(c)}$$

The Role of Inequalities



Example 4.26 graphical representation. Constrained optimum point.

The Role of Inequalities

EXAMPLE 4.28 INFEASIBLE PROBLEM

Minimize

$$f(\mathbf{x}) = (x_1 - 2)^2 + (x_2 - 2)^2 \tag{a}$$

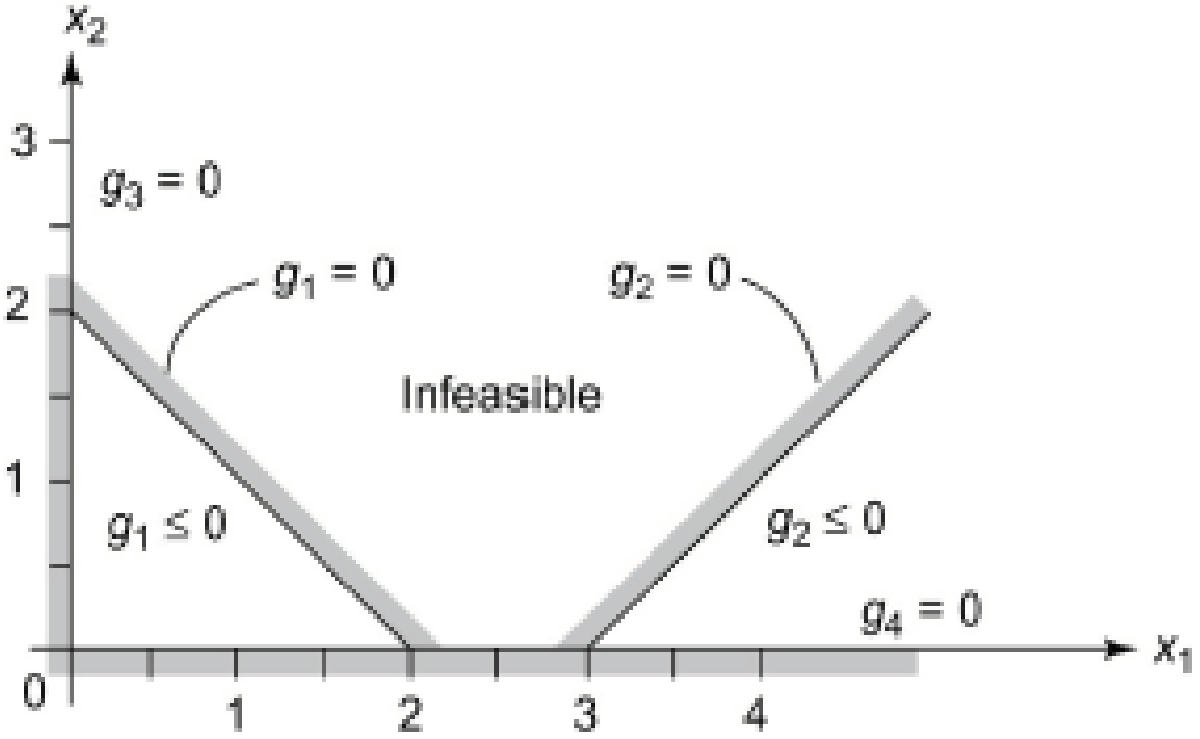
subject to the constraints:

$$g_1(\mathbf{x}) = x_1 + x_2 - 2 \le 0 \tag{b}$$

$$g_2(\mathbf{x}) = -x_1 + x_2 + 3 \le 0 \tag{c}$$

$$g_3(\mathbf{x}) = -x_1 \le 0; \quad g_4(\mathbf{x}) = -x_2 \le 0 \tag{d}$$

The Role of Inequalities



Plot of constraints for Example 4.28. Infeasible problem.

Karush–Kuhn–Tucker Necessary Conditions

We now include inequality constraints $g_i(\mathbf{x}) < 0$ and consider the general design optimization model defined in E qs. (4.35)–(4.37).

$$g_i + s_i^2 = 0$$

if $s_i^2 = 0$, the constraint is active (ie, $g_i = 0$)
if $s_i^2 > 0$, the constraint is satisfied/inactive (ie, $g_i < 0$)
if $s_i^2 < 0$, the constraint is violated and the given point is infeasible.

There is an additional necessary condition for the Lagrange multipliers of "≤type" constraints given as

$$u_j^* \geq 0; \quad j = 1 \text{ to } m \tag{4.45}$$

EXAMPLE 4.29

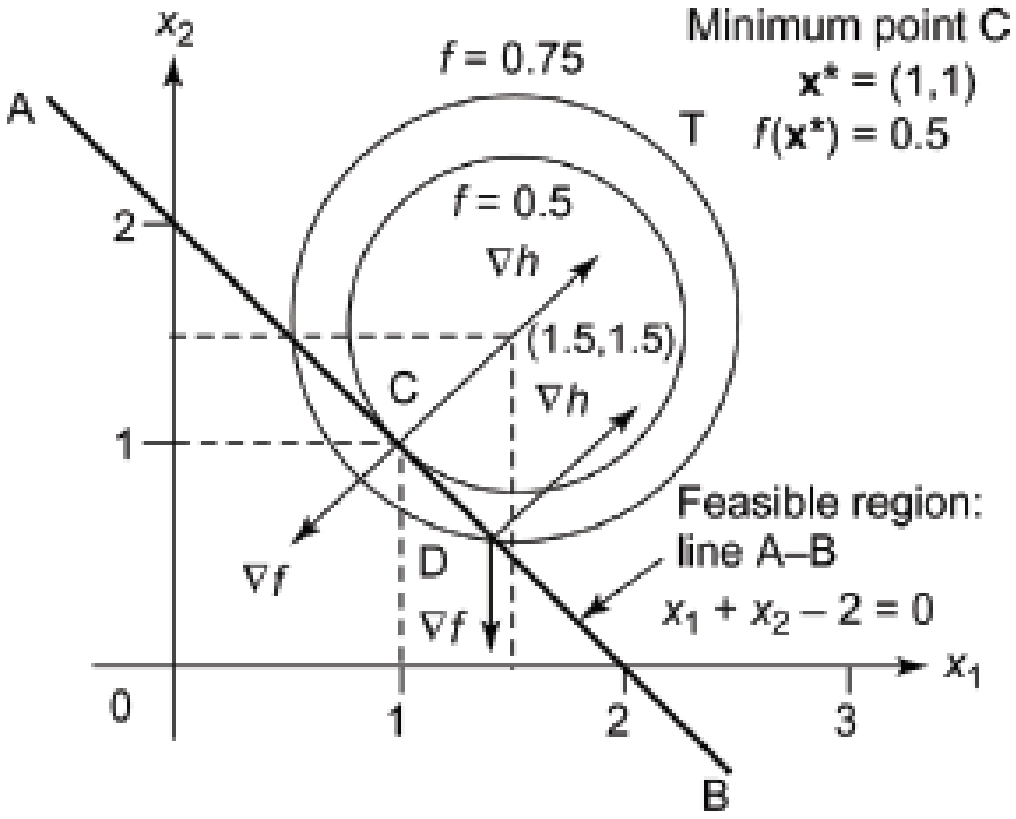
Minimize

$$f(x_1, x_2) = (x_1 - 1.5)^2 + (x_2 - 1.5)^2$$

subject to

$$g(x) = x_1 + x_2 - 2 \leq 0$$

EXAMPLE 4.29


x_2
$f = 0.75$
Minimum point C
$\mathbf{x}^* = (1,1)$
$f(\mathbf{x}^*) = 0.5$
A
T
$f = 0.5$
∇h
2
(1.5,1.5)
C
∇h
1
Feasible region:
line A–B
$x_1 + x_2 - 2 = 0$
∇f
D
∇f
x_1
0
1
2
3
B

EXAMPLE 4.29

$$L = (x_1 - 1.5)^2 + (x_2 - 1.5)^2 + u(x_1 + x_2 - 2 + s^2) \tag{c}$$

where u is the Lagrange multiplier for the inequality constraint. The necessary conditions of the Lagrange theorem give (treating x_1, x_2, u and s as unknowns):

$$\frac{\partial L}{\partial x_1} = 2(x_1 - 1.5) + u = 0 \tag{d}$$

$$\frac{\partial L}{\partial x_2} = 2(x_2 + 1.5) + u = 0 \tag{e}$$

$$\frac{\partial L}{\partial u} = x_1 + x_2 - 2 + s^2 = 0 \tag{f}$$

$$\frac{\partial L}{\partial s} = 2us = 0 \tag{g}$$

$$u \geq 0; \quad s^2 \geq 0 \tag{h}$$

EXAMPLE 4.29

These are four equations for four unknowns x_1, x_2, u, and s. The equations must be solved simultaneously for all of the unknowns. *Note that the equations are nonlinear. Therefore, they can have multiple roots.*

One solution can be obtained by setting s to zero to satisfy the so-called *switching condition* $2us = 0$ in Eq. (g); that is, by assuming that the inequality is active. Solving the remaining equations (d) through (f), we get $x_1^* = x_2^* = 1$, $u^* = 1 > 0$, $s = 0$. This is a stationary point of L, so it is a candidate minimum point. Note from Fig. 4.13 that it is actually a minimum point, since any small feasible move away from x^* either violates the constraint or increases the cost function.

The second stationary point is obtained by setting $u = 0$ to satisfy the switching condition of Eq. (g) and solving the remaining equations for x_1, x_2, and s. This gives $x_1^* = x_2^* = 1.5$, $u^* = 0$, $s^2 = -1$. This is not a valid solution, as the constraint is violated at the point x^* because $g = -s^2 = 1 > 0$.

Karush–Kuhn–Tucker Necessary Conditions

THEOREM 4.6

Karush-Kuhn-Tucker Optimality Conditions

Let $\mathbf{x}^*$ be a regular point of the feasible set that is a local minimum for $f(\mathbf{x})$, subject to $h_i(\mathbf{x}) = 0; i = 1$ to p; $g_j(\mathbf{x}) \le 0; j = 1$ to m. Then there exist Lagrange multipliers $\mathbf{v}^*$ (a p-vector) and $\mathbf{u}^*$ (an m-vector) such that the Lagrangian function is stationary with respect to x_i, v_i, u_j, and s_j at the point $\mathbf{x}^*$.

1. Lagrangian function for the problem written in the standard form:

$$\begin{aligned} L(\mathbf{x},\mathbf{v},\mathbf{u},\mathbf{s}) &= f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x}) + \sum_{j=1}^{m} u_j (g_j(\mathbf{x}) + s_j^2) \\ &= f(\mathbf{x}) + \mathbf{v}^{\mathrm{T}}\mathbf{h}(\mathbf{x}) + \mathbf{u}^{\mathrm{T}}(\mathbf{g}(\mathbf{x}) + \mathbf{s}^2) \end{aligned} \tag{4.46}$$

2. Gradient conditions:

$$\frac{\partial L}{\partial x_k} = \frac{\partial f}{\partial x_k} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i}{\partial x_k} + \sum_{j=1}^{m} u_j^* \frac{\partial g_j}{\partial x_k} = 0; \quad k = 1 \text{ to } n \tag{4.47}$$

$$\frac{\partial L}{\partial v_i} = 0 \Rightarrow h_i(\mathbf{x}^*) = 0; \quad i = 1 \text{ to } p \tag{4.48}$$

$$\frac{\partial L}{\partial u_j} = 0 \Rightarrow (g_j(\mathbf{x}^*) + s_j^2) = 0; \quad j = 1 \text{ to } m \tag{4.49}$$

3. Feasibility check for inequalities:

$$s_j^2 \ge 0; \text{ or equivalently } g_j \le 0; \quad j = 1 \text{ to } m \tag{4.50}$$

Karush–Kuhn–Tucker Necessary Conditions

4. Switching conditions:

$$\frac{\partial L}{\partial s_j} = 0 \Rightarrow 2u_j^* s_j = 0; \quad j = 1 \text{ to } m \tag{4.51}$$

5. Nonnegativity of Lagrange multipliers for inequalities:

$$u_j^* \geq 0; \quad j = 1 \text{ to } m \tag{4.52}$$

6. *Regularity check*: Gradients of the active constraints must be linearly independent. In such a case the Lagrange multipliers for the constraints are unique.

Example 4.30

Write the KKT necessary conditions and solve them for the problem

Minimize

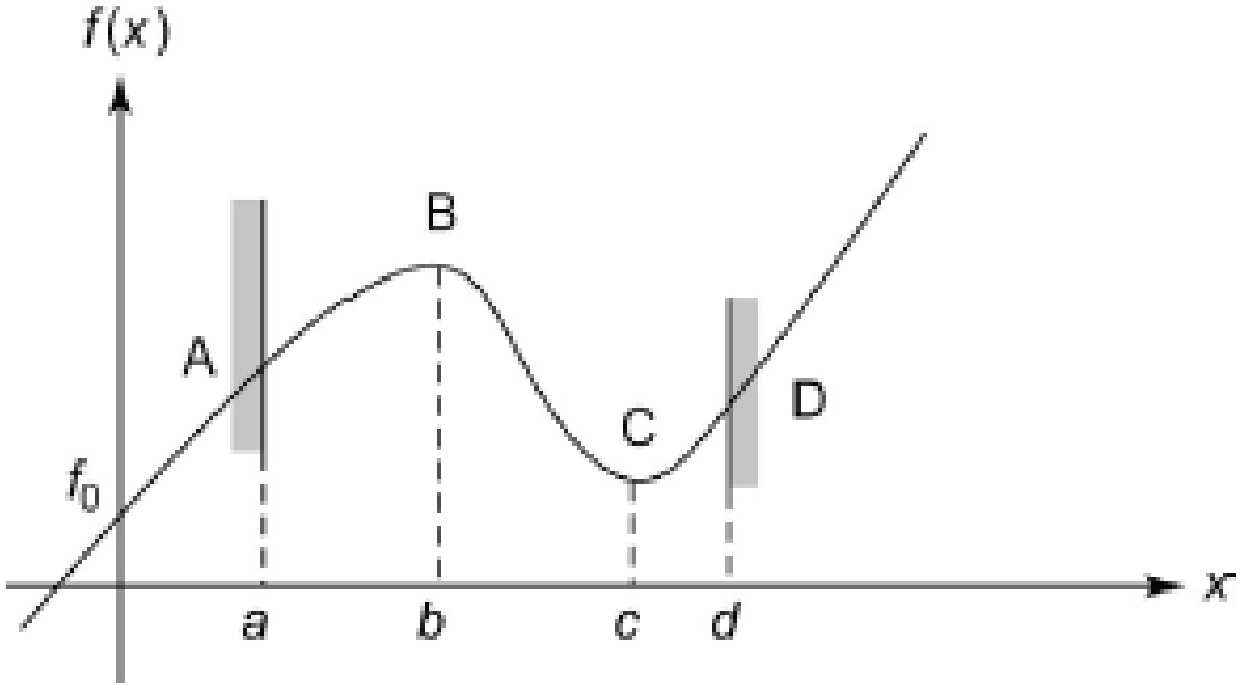
$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}(b+c)x^2 + bcx + f_0 \qquad \text{(a)}$$

subject to

$$a \le x \le d \qquad \text{(b)}$$

where $0 < a < b < c < d$ and f_0 are specified constants (created by Y.S. Ryu).

Example 4.30



Example 4.30

There are two inequality constraints in Eq. (b) that are written in the standard form as

$$g_1 = a - x \le 0; \quad g_2 = x - d \le 0 \tag{c}$$

The Lagrangian function of Eq. (4.46) for the problem is given as

$$L = \frac{1}{3}x^3 - \frac{1}{2}(b+c)x^2 + bcx + f_0 + u_1(a - x + s_1^2) + u_2(x - d + s_2^2) \tag{d}$$

where u_1 and u_2 are the Lagrange multipliers and s_1 and s_2 are the slack variables for the two inequalities g_1 and g_2 in Eq. (c). The KKT conditions give

$$\frac{\partial L}{\partial x} = x^2 - (b+c)x + bc - u_1 + u_2 = 0 \tag{e}$$

$$(a - x) + s_1^2 = 0, \; s_1^2 \ge 0; \quad (x - d) + s_2^2 = 0, \; s_2^2 \ge 0 \tag{f}$$

$$u_1 s_1 = 0; \quad u_2 s_2 = 0 \tag{g}$$

$$u_1 \ge 0; \quad u_2 \ge 0 \tag{h}$$

The switching conditions in Eq. (g) give four cases for the solution to the KKT conditions. Each case will be considered separately and solved.

Example 4.30

There are two inequality constraints in Eq. (b) that are written in the standard form as

$$g_1 = a - x \le 0; \quad g_2 = x - d \le 0 \tag{c}$$

The Lagrangian function of Eq. (4.46) for the problem is given as

$$L = \frac{1}{3}x^3 - \frac{1}{2}(b+c)x^2 + bcx + f_0 + u_1(a - x + s_1^2) + u_2(x - d + s_2^2) \tag{d}$$

where u_1 and u_2 are the Lagrange multipliers and s_1 and s_2 are the slack variables for the two inequalities g_1 and g_2 in Eq. (c). The KKT conditions give

$$\frac{\partial L}{\partial x} = x^2 - (b+c)x + bc - u_1 + u_2 = 0 \tag{e}$$

$$(a - x) + s_1^2 = 0, \; s_1^2 \ge 0; \quad (x - d) + s_2^2 = 0, \; s_2^2 \ge 0 \tag{f}$$

$$u_1 s_1 = 0; \quad u_2 s_2 = 0 \tag{g}$$

$$u_1 \ge 0; \quad u_2 \ge 0 \tag{h}$$

The switching conditions in Eq. (g) give four cases for the solution to the KKT conditions. Each case will be considered separately and solved.

Example 4.30

Case 1: $u_1 = 0$ (g_1 inactive) and $u_2 = 0$ (g_2 inactive).

For this case, Eq. (e) gives two solutions as $x = b$ and $x = c$. For these points both the inequalities are strictly satisfied because slack variables calculated from Eq. (f) are

$$\text{for } x = b: \; s_1^2 = b - a > 0; \quad s_2^2 = d - b > 0 \tag{i}$$

$$\text{for } x = c: \; s_1^2 = c - a > 0; \quad s_2^2 = d - c > 0 \tag{j}$$

Thus, all of the KKT conditions are satisfied, and these are the candidate minimum points. Since the points are unconstrained, they are actually stationary points. We can check the sufficient conditions by calculating the curvature of the cost function at the two candidate points:

$$x = b; \quad \frac{d^2 f}{dx^2} = 2x - (b + c) = b - c < 0 \tag{k}$$

Since $b < c$, d^2f/dx^2 is negative. Therefore, the sufficient condition for a local minimum is not met. Actually, the second-order necessary condition of Eq. (4.29) for a local minimum is violated, so the point cannot be a local minimum for the function. It is actually a local maximum point because it satisfies the sufficient condition for that, as also seen in Fig. 4.16.

$$x = c; \quad \frac{d^2 f}{dx^2} = c - b > 0 \tag{l}$$

Example 4.30

Since $b < c$, d^2f/dx^2 is positive. Therefore, the second-order sufficient condition of Eq. (4.28) is satisfied, and this is an isolated local minimum point, as also seen in Fig. 4.16. It cannot be a local maximum point since second order necessary condition for local maximum is violated.

Case 2: $u_1 = 0$ (g_1 inactive) and $s_2 = 0$ (g_2 active).

g_2 is active for this case, and because $s_2 = 0$, $x = d$ from Eq. (c). Eq. (e) gives

$$u_2 = -[d^2 - (b+c)d + bc] = -(d-c)(d-b) \tag{m}$$

Since $d > c > b$, u_2 is < 0. Actually the term within the square brackets is also the slope of the function at $x = d$, which is positive, so $u_2 < 0$ in Eq. (m). The KKT necessary condition is violated, so there is no solution for this case; that is, $x = d$ is not a candidate minimum point. This is true, as can be observed for point D on the graph of $f(x)$ in Fig. 4.16. Actually, it can be checked that the point satisfies the KKT necessary conditions for the local maximum point.

Example 4.30

Case 3: $s_1 = 0$ (g_1 active) and $u_2 = 0$ (g_2 inactive).

$s_1 = 0$ implies that g_1 is active and therefore $x = a$ from Eq. (c). Eq. (e) gives

$$u_1 = a^2 - (b+c)a + bc = (a-b)(a-c) > 0 \qquad \text{(n)}$$

Also, since u_1 = the slope of the function at $x = a$, it is positive and all of the KKT conditions are satisfied. Thus, $x = a$ is a candidate local minimum point. Actually $x = a$ is a local minimum point because a feasible move from the point increases the cost function. This is a sufficient condition, which we will discuss in chapter: More on Optimum Design Concepts: Optimality Conditions.

Case 4: $s_1 = 0$ (g_1 active) and $s_2 = 0$ (g_2 active).

This case, for which both constraints are active, does not give any valid solution since x cannot be simultaneously equal to a and d; that is, one variable and two equations.