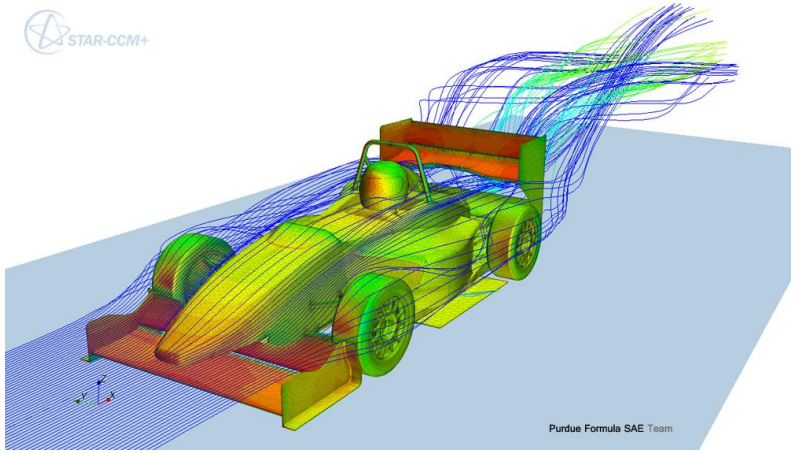
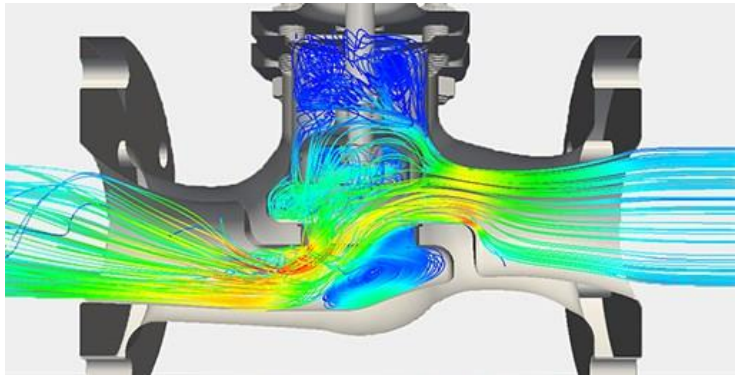
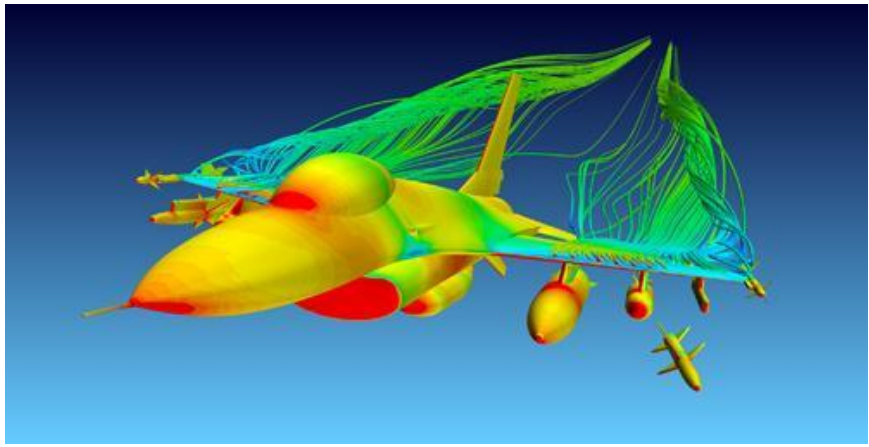
Computer Aided Design
Lecture 5
Finite Difference







Dr./ Ahmed Nagib Elmekawy

Leonhard Euler (1707-1783)

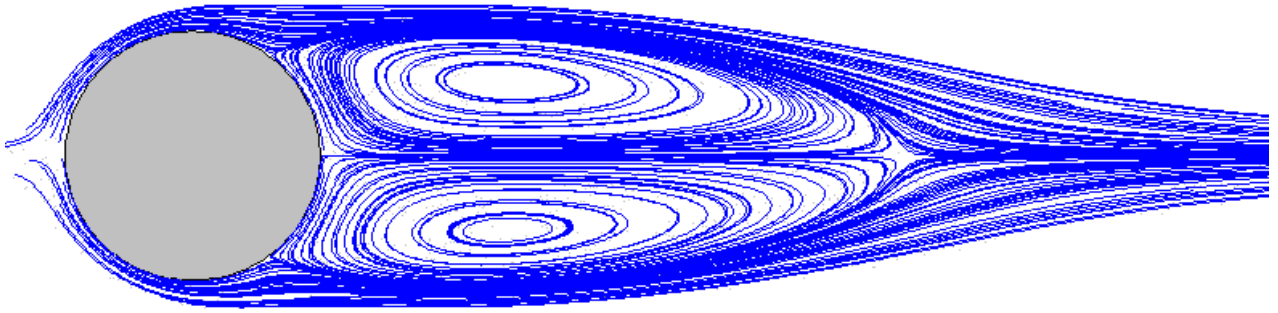
- In 1768, Leonhard Euler introduced the finite difference technique based on Taylor series expansion.

Lewis Fry Richardson (1881-1953)

- In 1922, Lewis Fry Richardson developed the first numerical weather prediction system.
  - Division of space into grid cells and the finite difference approximations of Bjerknes's "primitive differential equations."
  - His own attempt to calculate weather for a single eight-hour period took six weeks and ended in failure.
- His model's enormous calculation requirements led Richardson to propose a solution he called the "forecast-factory."
  - The "factory" would have filled a vast stadium with 64,000 people.
  - Each one, armed with a mechanical calculator, would perform part of the calculation.
  - A leader in the center, using colored signal lights and telegraph communication, would coordinate the forecast.

1930s to 1950s

- Earliest numerical solution: for flow past a cylinder (1933).
    - A.Thom, 'The Flow Past Circular Cylinders at Low Speeds', Proc. Royal Society, A141, pp. 651-666, London, 1933
- Kawaguti obtained a solution for flow around a cylinder, in 1953 by using a mechanical desk calculator, working 20 hours per week for 18 months, citing: "*a considerable amount of labour and endurance*."
    - M. Kawaguti, 'Numerical Solution of the NS Equations for the Flow Around a Circular Cylinder at Reynolds Number 40', Journal of Phy. Soc. Japan, vol. 8, pp. 747-757, 1953.



1960s and 1970s

- During the 1960s the theoretical division at Los Alamos contributed many numerical methods that are still in use today, such as the following methods:
  - Particle-In-Cell (PIC).
  - Marker-and-Cell (MAC).
  - Vorticity-Streamfunction Methods.
  - Arbitrary Lagrangian-Eulerian (ALE).
  - k-ε turbulence model.
- During the 1970s a group working under D. Brian Spalding, at Imperial College, London, develop:
  - Parabolic flow codes (GENMIX).
  - Vorticity-Streamfunction based codes.
  - The SIMPLE algorithm and the TEACH code.
  - The form of the k-ε equations that are used today.
  - Upwind differencing.
  - ‘Eddy break-up’ and ‘presumed pdf’ combustion models.
- In 1980 Suhas V. Patankar publishes *Numerical Heat Transfer and Fluid Flow*, probably the most influential book on CFD to date.

1980s and 1990s

- Previously, CFD was performed using academic, research and in-house codes. When one wanted to perform a CFD calculation, one had to write a program.
- This is the period during which most commercial CFD codes originated that are available today:
  - Fluent (UK and US).
  - CFX (UK and Canada).
  - Fidap (US).
  - Polyflow (Belgium).
  - Phoenix (UK).
  - Star CD (UK).
  - Flow 3d (US).
  - ESI/CFDRC (US).
  - SCRYU (Japan).
  - and more, see www.cfdreview.com.

Navier-Stokes Equation Derivation

- Refer to
- Ch. 3 and Appendix A of
  - Jiyuan Tu, Computational Fluid Dynamics -A Practical Approach, Second Edition, 2013.
- Ch. 2
  - Wendt, Anderson, Computational Fluid Dynamics - An Introduction, 3rd edition 2009.

LAGRANGIAN AND EULERIAN DESCRIPTIONS

Kinematics: The study of motion.

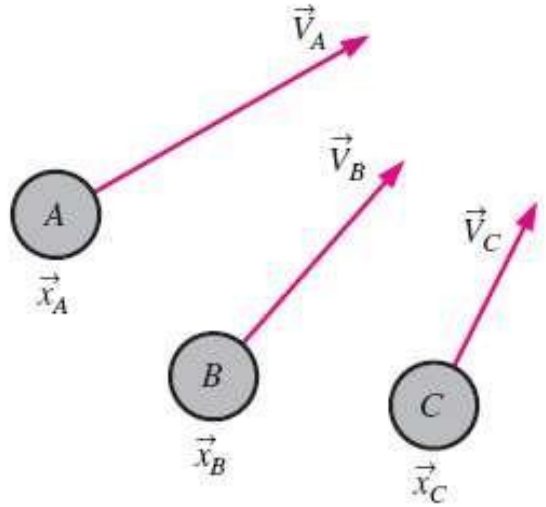
Fluid kinematics: The study of how fluids flow and how to describe fluid motion.

There are two distinct ways to describe motion: Lagrangian and Eulerian

Lagrangian description: To follow the path of individual objects.

This method requires us to track the position and velocity of each individual fluid parcel (fluid particle) and take to be a parcel of fixed identity.

With a small number of objects, such as billiard balls on a pool table, individual objects can be tracked.



In the Lagrangian description, one must keep track of the position and velocity of individual particles.

- Amore common method is Eulerian description of fluid motion.
- In the Eulerian description of fluid flow, a finite volume called a flow domain or control volume is defined, through which fluid flows in and out.
- Instead of tracking individual fluid particles, we define field variables, functions of space and time, within the control volume.
- The field variable at a particular location at a particular time is the value of the variable for whichever fluid particle happens to occupy that location at that time.
- For example, the pressure field is a scalar field variable. We define the velocity field as a vector field variable.

Pressure field:	$P = P(x, y, z, t)$
Velocity field:	$\vec{V} = \vec{V}(x, y, z, t)$
Acceleration field:	$\vec{a} = \vec{a}(x, y, z, t)$

Collectively, these (and other) field variables define the flow field. The velocity field can be expanded in Cartesian coordinates as

$$\vec{V} = (u, v, w) = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k}$$

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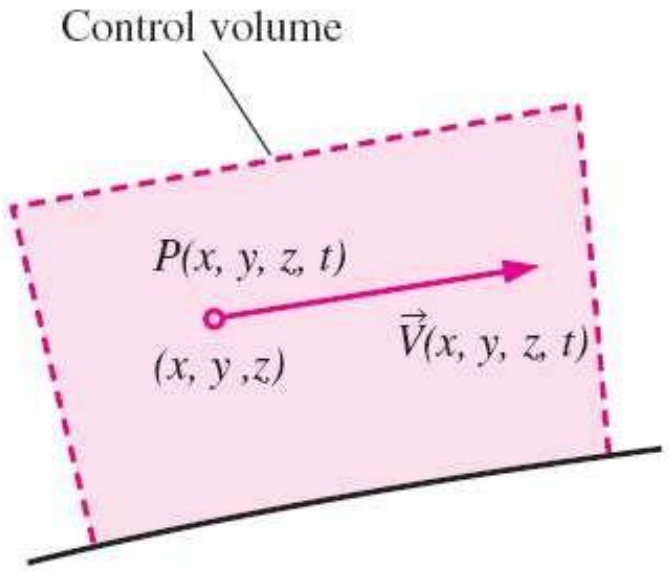


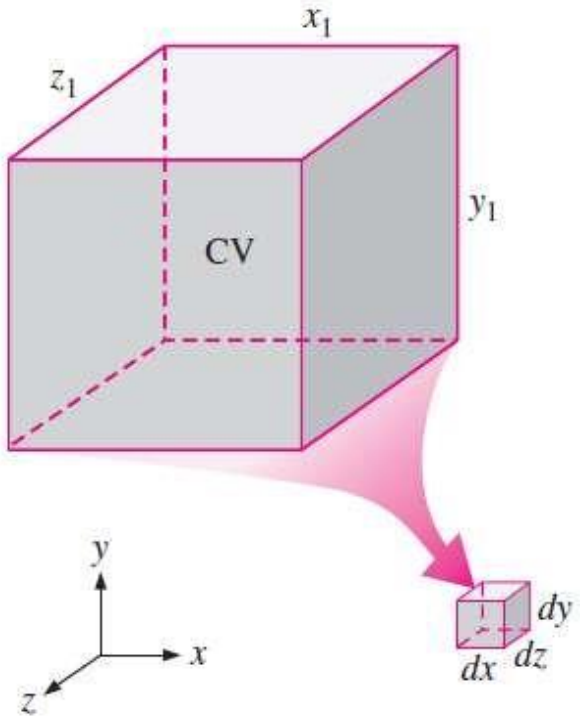
In the Eulerian description, one defines field variables, such as the pressure field and the velocity field, at any location and instant in time.

- In the Eulerian description we don't really care what happens to individual fluid particles; rather we are concerned with the pressure, velocity, acceleration, etc., of whichever fluid particle happens to be at the location of interest at the time of interest.
- While there are many occasions in which the Lagrangian description is useful, the Eulerian description is often more convenient for fluid mechanics applications.
- Experimental measurements are generally more suited to the Eulerian description.

CONSERVATION OF MASS—THE CONTINUITY EQUATION

$$\frac{dm}{dt} = \sum_{in} \dot{m} - \sum_{out} \dot{m} \qquad (3.6)$$

The net rate of change of mass within the control volume is equal to the rate at which mass flows into the control volume minus the rate at which mass flows out of the control volume.



To derive a differential conservation equation, we imagine shrinking a control volume to infinitesimal size.

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The rate at which mass enters the control volume through the surface perpendicular to x may be expressed as $(\rho u)\,\Delta y\,\Delta z$, where ρ is the local density of the fluid, and similarly through the surfaces perpendicular to y and z as $(\rho v)\,\Delta x\Delta z$ and $(\rho w)\,\Delta x\Delta y$, respectively. The rate at which the mass leaves the surface at $x + \Delta x$ may be expressed through Taylor expansion as

$$\left[(\rho u) + \frac{\partial(\rho u)}{\partial x}\Delta x\right]\Delta y\Delta z + O(\Delta x, \Delta V) \quad \text{where} \quad \Delta V = \Delta x\Delta y\Delta z \qquad (3.7)$$

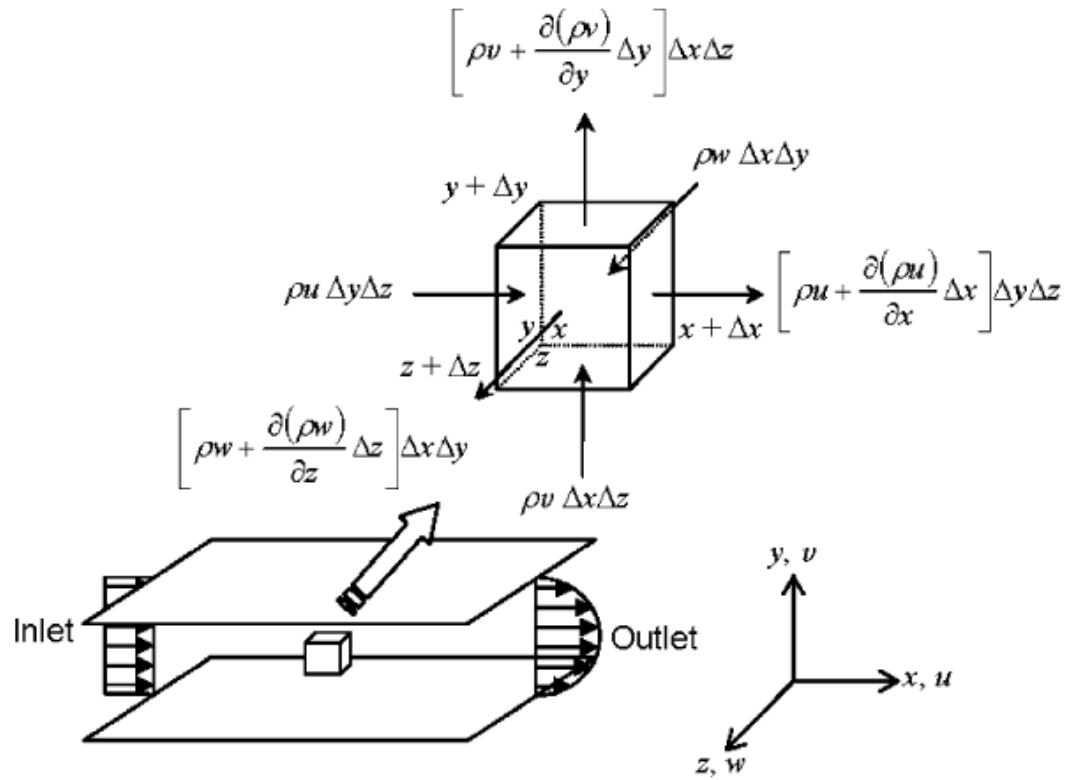


FIGURE 3.2 The conservation of mass in an infinitesimal control volume of a fluid flow between two stationary parallel plates.

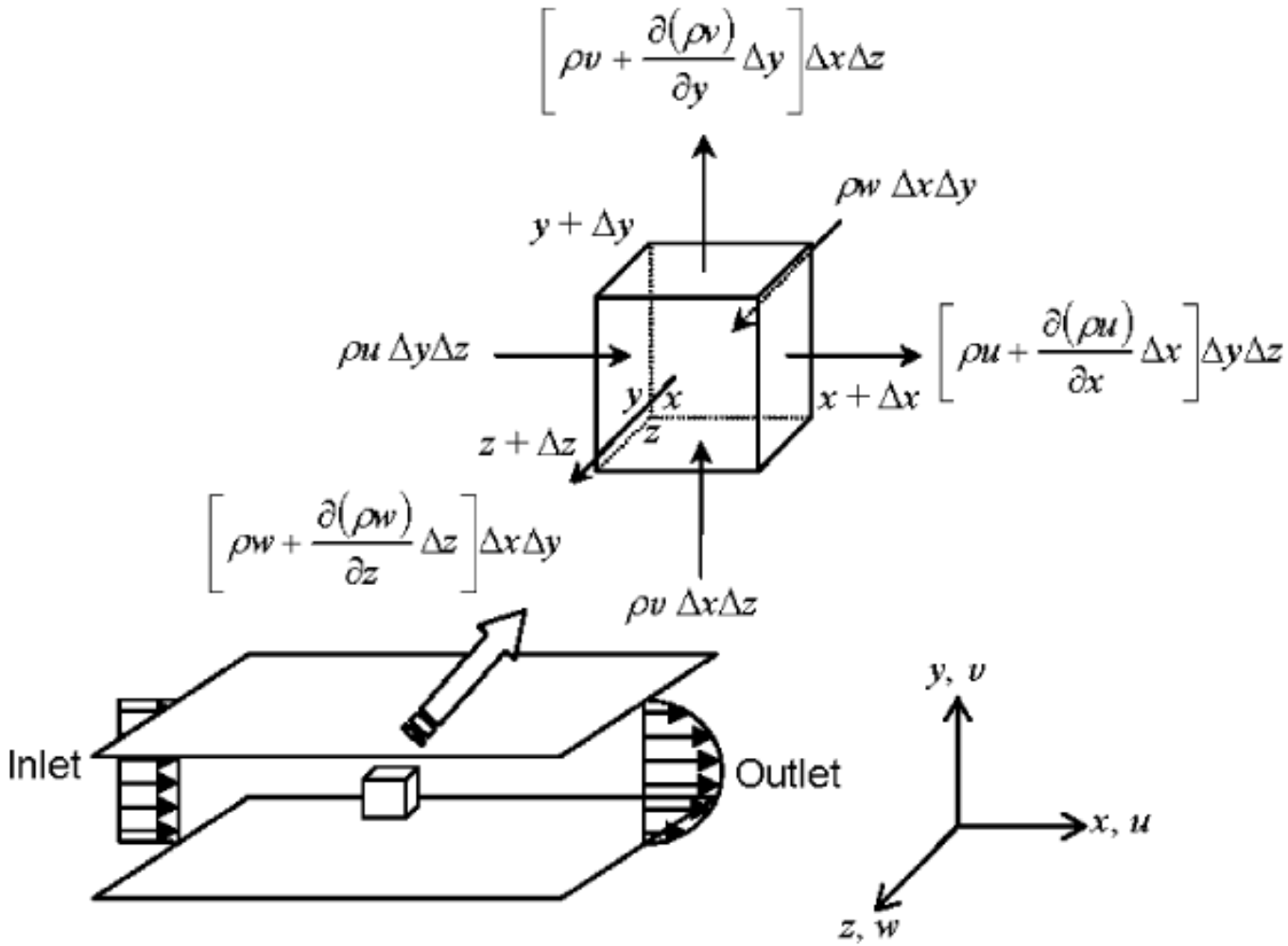


FIGURE 3.2 The conservation of mass in an infinitesimal control volume of a fluid flow between two stationary parallel plates.

Similarly, the rate at which mass leaves the surfaces at $y + \Delta y$ and $z + \Delta z$ may also be expressed as

$$\left[(\rho v) + \frac{\partial(\rho v)}{\partial y}\Delta y\right]\Delta x\Delta z + O(\Delta y, \Delta V)$$

and

$$\left[(\rho w) + \frac{\partial(\rho w)}{\partial z}\Delta z\right]\Delta x\Delta y + O(\Delta z, \Delta V) \tag{3.8}$$

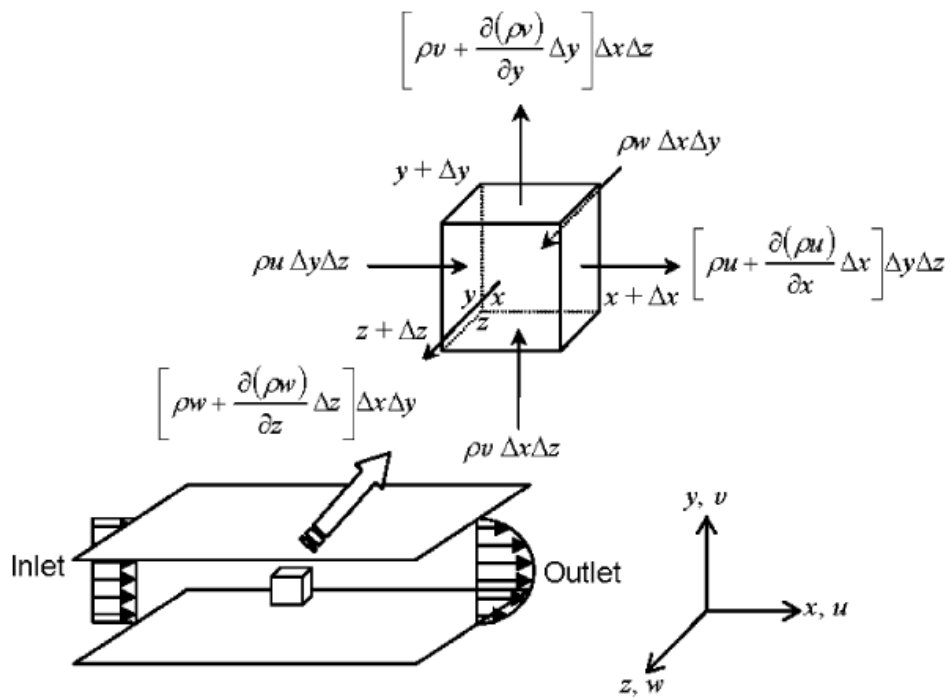


FIGURE 3.2 The conservation of mass in an infinitesimal control volume of a fluid flow between two stationary parallel plates.

Since the mass of the fluid element m is given by $\rho\,\Delta x\Delta y\Delta z$, Eq. (3.6) becomes

$$\frac{\partial(\rho\Delta x\Delta y\Delta z)}{\partial t} = (\rho u)\Delta y\Delta z + (\rho v)\Delta x\Delta z + (\rho w)\Delta x\Delta y$$

$$-\left[(\rho u) + \frac{\partial(\rho u)}{\partial x}\Delta x\right]\Delta y\Delta z - \left[(\rho v) + \frac{\partial(\rho v)}{\partial y}\Delta y\right]\Delta x\Delta z$$

$$-\left[(\rho w) + \frac{\partial(\rho w)}{\partial z}\Delta z\right]\Delta x\Delta y + \Delta V\ \ O(\Delta x, \Delta y, \Delta z) \tag{3.9}$$

In the limit, canceling terms and dividing by the constant-size $\Delta x\Delta y\Delta z$, we obtain

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{3.10}$$

Continuity equation in Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

Gradient or del operator:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V} \right) = 0$$

Conservation of Mass: *Alternative forms*

- Use product rule on divergence term

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = \frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{V} = 0$$

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Conservation of Mass: *Cylindrical coordinates*

- There are many problems which are simpler to solve if the equations are written in cylindrical-polar coordinates
- Easiest way to convert from Cartesian is to use vector form and definition of divergence operator in cylindrical coordinates

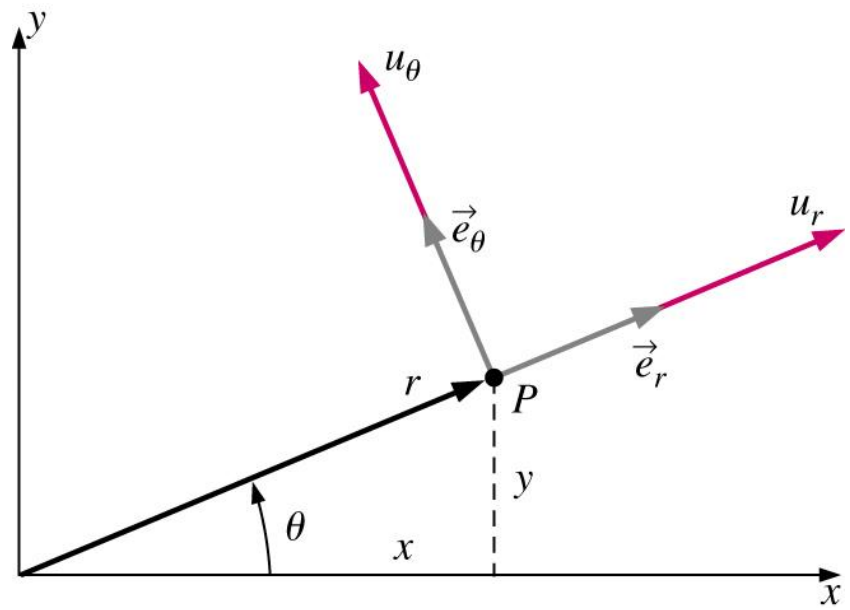
The Divergence Operation

Cartesian coordinates:

$$\vec{\nabla}\cdot(\rho\vec{V}) = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)$$

Cylindrical coordinates:

$$\vec{\nabla}\cdot(\rho\vec{V}) = \frac{1}{r}\frac{\partial(r\rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial \theta} + \frac{\partial(\rho u_z)}{\partial z}$$



Conservation of Mass: *Cylindrical coordinates*

$$\vec{\nabla} = \frac{1}{r}\frac{\partial(r)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$$

$$\vec{V} = U_r\hat{e}_r + U_\theta\hat{e}_\theta + U_z\hat{e}_z$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho U_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho U_\theta)}{\partial \theta} + \frac{\partial (\rho U_z)}{\partial z} = 0$$

Conservation of Mass: *Special Cases*

- Steady compressible flow

$$\cancel{\frac{\partial \rho}{\partial t}} + \vec{\nabla} \cdot \left(\rho \vec{V} \right) = 0$$

$$\boxed{\vec{\nabla} \cdot \left(\rho \vec{V} \right) = 0}$$

Cartesian

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

Cylindrical

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r}\frac{\partial (r \rho U_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho U_\theta)}{\partial \theta} + \frac{\partial (\rho U_z)}{\partial z} = 0$$

Conservation of Mass: *Special Cases*

- **Incompressible flow**

ρ = constant, and hence

$$\frac{\partial \rho}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{V} = 0$$

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Cartesian

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Cylindrical

$$\frac{1}{r}\frac{\partial (r U_r)}{\partial r} + \frac{1}{r}\frac{\partial (U_\theta)}{\partial \theta} + \frac{\partial (U_z)}{\partial z} = 0$$

Conservation of Mass

- In general, continuity equation cannot be used by itself to solve for flow field, however it can be used to

1. Determine if a velocity field represents a flow.

Example

For an incompressible flow

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z$$

$$w = ?$$

Determine: w, required to satisfy the continuity equation.

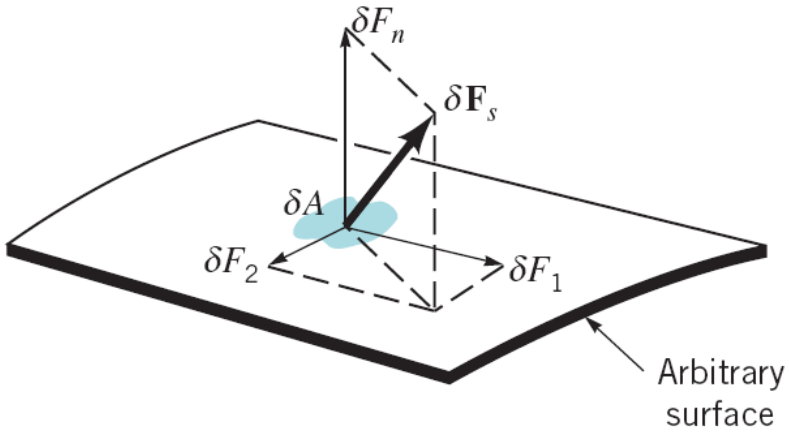
$$\text{Solution}: w = -3xz - \frac{z^2}{2} + c(x, y)$$

Conservation of Momentum

Types of forces:

1. Surface forces: include all forces acting on the boundaries of a medium though direct contact such as pressure, friction,…etc.
2. Body forces are developed without physical contact and distributed over the volume of the fluid such as gravitational and electromagnetic.

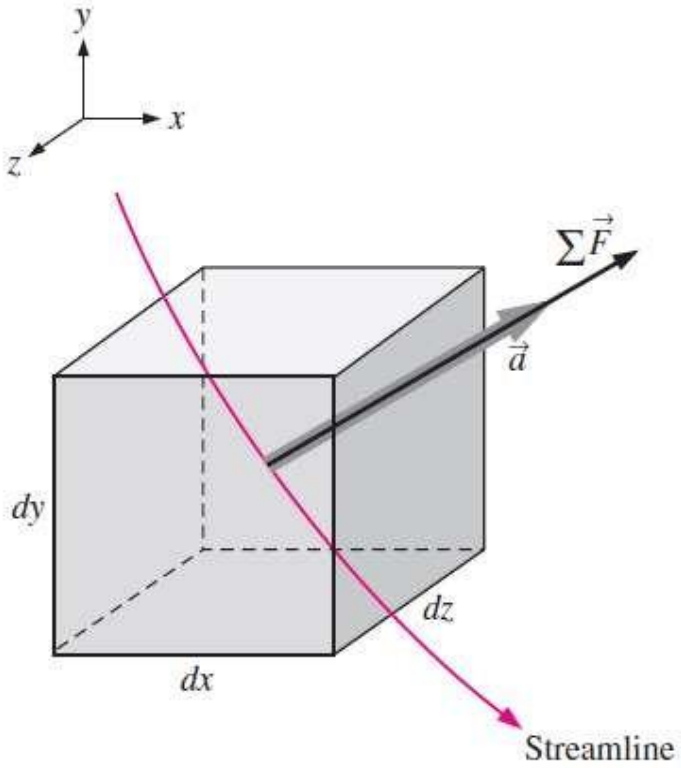
- The force δF acting on δA may be resolved into two components, one normal and the other tangential to the area.



$$\sum \vec{F} = m\vec{a} = m\frac{D\vec{V}}{Dt} = \rho\, dx\, dy\, dz\frac{D\vec{V}}{Dt}$$

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij}$$

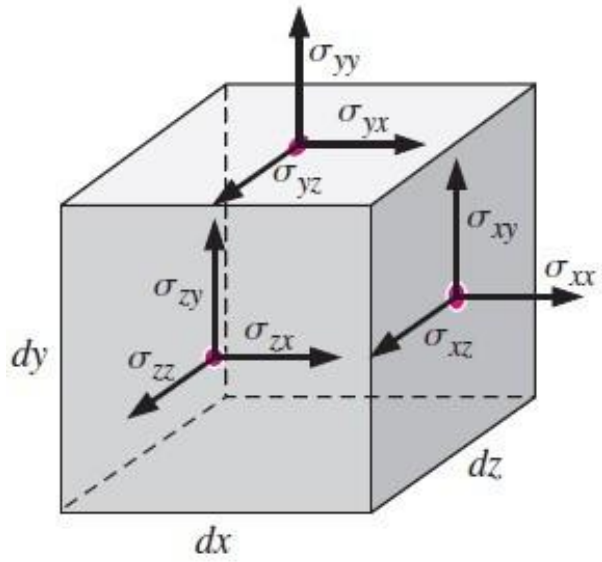
If the differential fluid element is a material element, it moves with the flow and Newton's second law applies directly.



Body Forces

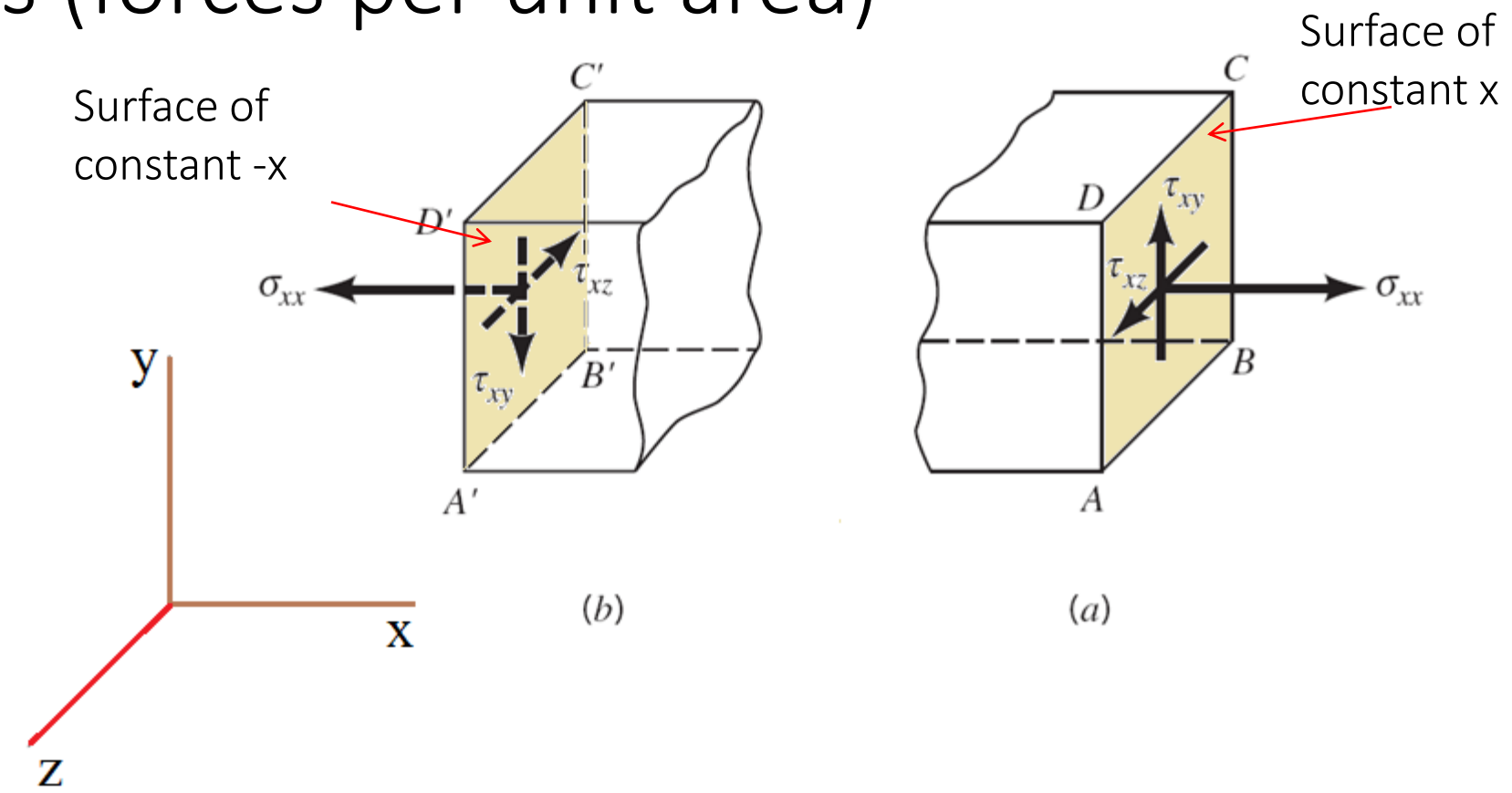
$$\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}}$$

$$\vec{g} = g_x\vec{i} + g_y\vec{j} + g_z\vec{k} \qquad \sum F_{x,\,\text{body}} = \sum F_{x,\,\text{gravity}} \equiv \rho g_x\, dx\, dy\, dz$$



Positive components of the stress tensor in Cartesian coordinates on the positive (right, top, and front) faces of an infinitesimal rectangular control volume. The blue dots indicate the center of each face. Positive components on the negative (left, bottom, and back) faces are in the opposite direction of those shown here.

Stresses (forces per unit area)



Double subscript notation for stresses.

- First subscript refers to the surface
- Second subscript refers to the direction
- Use σ for normal stresses and τ for tangential stresses

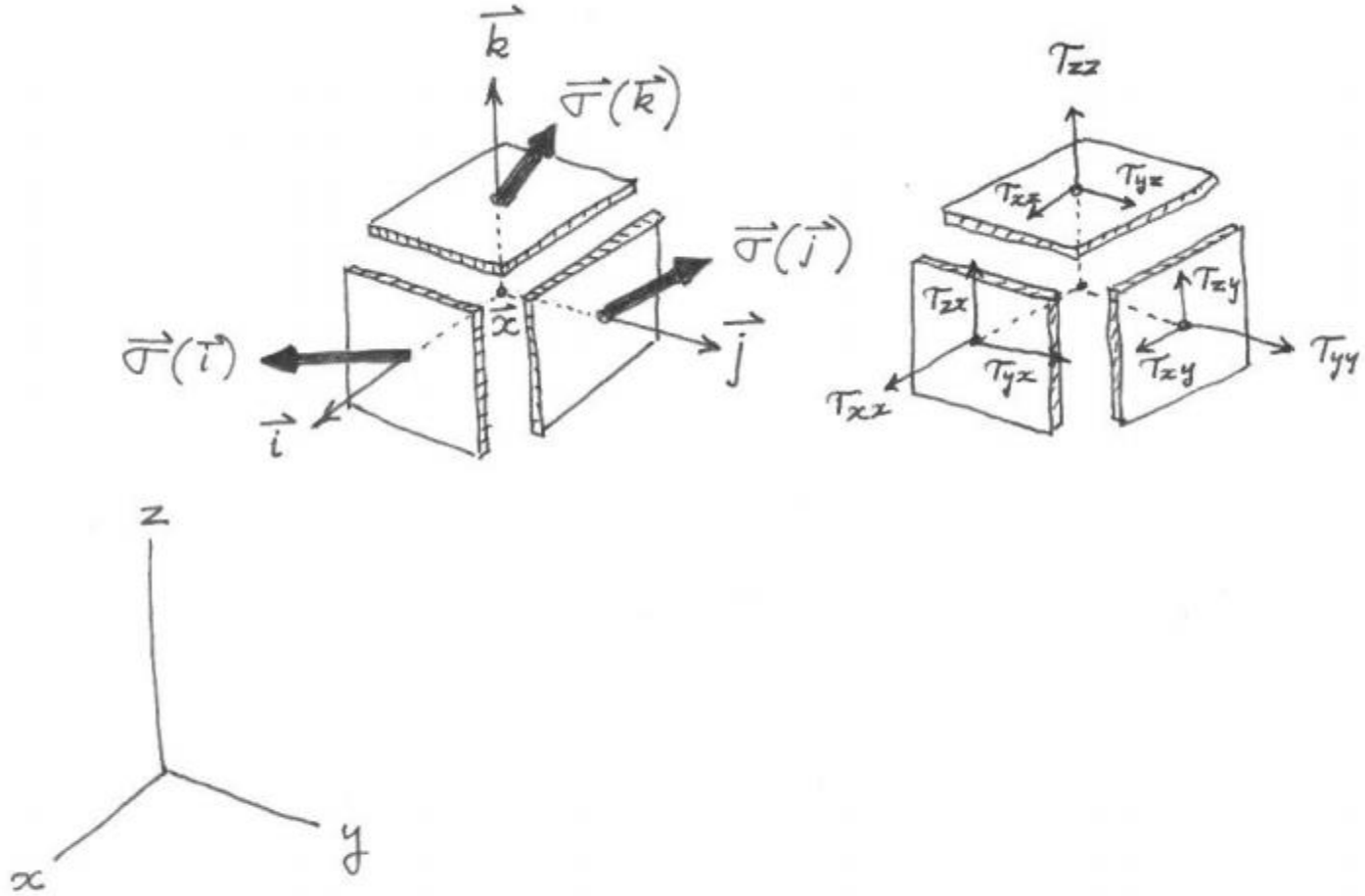


Figure 3: Reference stresses at a point in the continuum.

3.3 THE MOMENTUM EQUATION

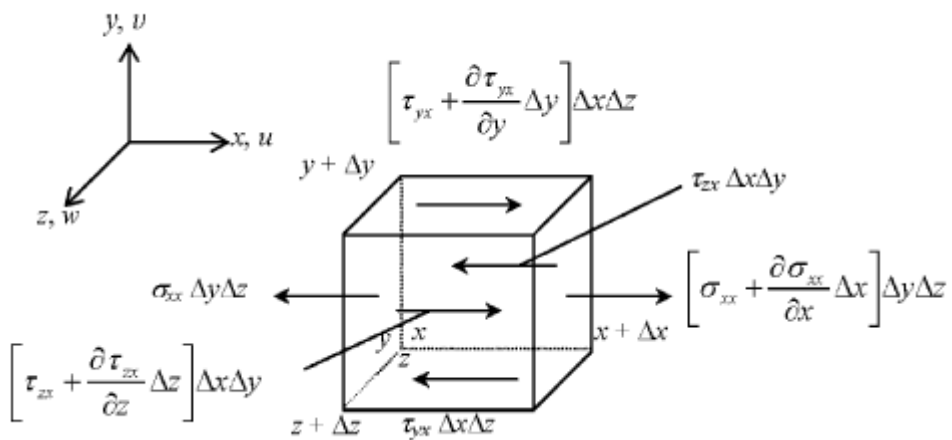
3.3.1 Force Balance

In deriving this physical law, we begin by considering the fluid element as described in Figure 3.2 for mass conservation. *Newton's second law of motion* states that the sum of forces acting on the fluid element, as illustrated in Figure 3.4, equals the product of its mass and the acceleration of the element. There are essentially three scalar relations along the x, y, and z directions of the Cartesian frame for which the fundamental law can be invoked. We begin by considering the x component of Newton's second law,

$$\sum F_x = ma_x \tag{3.12}$$

where F_x and a_x are the force and acceleration along the x direction. The acceleration a_x on the right-hand side of Eq. (3.12) is simply the time rate change of u, which is given by the substantial derivative (see Appendix A for a more detailed description). Thus,

$$a_x = \frac{Du}{Dt}$$



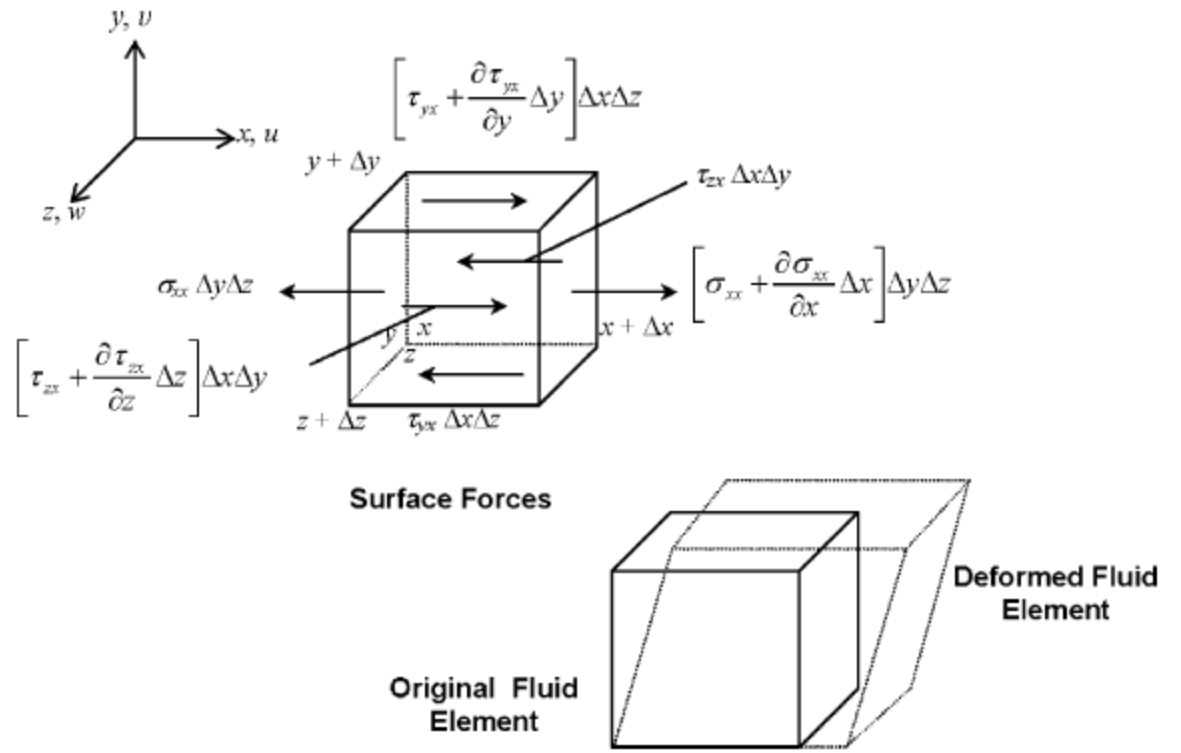


FIGURE 3.4 Surface forces acting on the infinitesimal control volume for the velocity component *u*. Deformed fluid element due to the action of the surface forces.

Recalling that the mass of the fluid element m is $\rho\,\Delta x\Delta y\Delta z$, the rate of increase of x-momentum is

$$\rho\frac{Du}{Dt}\Delta x\Delta y\Delta z \tag{3.14}$$

On the left-hand side of Eq. (3.12), there are two sources of the force that the moving fluid element experiences. They are *body forces* and *surface forces*. The body forces that may influence the rate of change of the fluid momentum are

gravity, centrifugal, Coriolis, and electromagnetic forces. These effects are usually incorporated by introducing them into the momentum equations as additional source terms to the contribution of the surface forces. The surface forces for the velocity component u, as seen in Figure 3.4, that deform the fluid element are due to the normal stress σ_{xx} and tangential stresses τ_{yx} and τ_{zx} acting on the surfaces of the fluid element.

For the conservation of momentum, the net force in the x direction is the sum of the force components acting on the fluid element. Considering the velocity component u as seen in Figure 3.4, the surface forces are due to the normal stress σ_{xx} and the tangential stresses τ_{yx} and τ_{zx} acting on the surfaces of the fluid element. The net force in the normal x direction is

$$\left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\Delta x\right]\Delta y\Delta z - \sigma_{xx}\Delta y\Delta z \tag{A.7}$$

while the net tangential forces acting along the x direction are given by

$$\left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right]\Delta x\Delta z - \tau_{yx}\Delta x\Delta z \tag{A.8}$$

and

$$\left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z\right]\Delta x\Delta y - \tau_{zx}\Delta x\Delta y \tag{A.9}$$

The total net force per unit volume on the fluid due to these surface stresses should be equal to the sum of Eqs. (A.7), (A.8), and (A.9) divided by the control volume $\Delta x\Delta y\Delta z$:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{A.10}$$

It is not too difficult to verify that the total net forces per unit volume on the rest of the control volume surfaces in the y direction and z direction are given by

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{A.11}$$

and

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \tag{A.12}$$

Combining Eq. (A.10) with the substantial derivative of the horizontal velocity component u and body forces, the x-momentum equation becomes

$$\rho \frac{Du}{Dt} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \sum F_x^{body\ forces} \tag{A.13}$$

In a similar fashion, the y-momentum and z-momentum equations, using Eqs. (A.11) and (A.12), can be obtained through

$$\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \sum F_y^{body\ forces} \tag{A.14}$$

and

$$\rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \sum F_z^{body\ forces} \tag{A.15}$$

If the fluid is taken to be Newtonian and isotropic—since all gases and the majority of liquids are isotropic—the normal stresses σ_{xx}, σ_{yy}, and σ_{zz} appearing in Eqs. (A.13)–(A.15) can be formulated in terms of pressure p and normal viscous stress components τ_{xx}, τ_{yy}, and τ_{zz} acting perpendicular on the control volume. The remaining terms contain the tangential viscous stress components as also described in Eqs. (A.13)–(A.15). In many fluid flows, a suitable model for the viscous stresses is introduced, which can be expressed as a function of the local deformation rate (or strain rate). Assuming that the fluid is Newtonian and isotropic, the rate of linear deformation on the control volume $\Delta x \Delta y \Delta z$ caused by the motion of fluid can usually be expressed in terms of the velocity gradients. The normal stress relationships can be expressed as

$$\sigma_{xx} = -p + \tau_{xx} \qquad \sigma_{yy} = -p + \tau_{yy} \qquad \sigma_{zz} = -p + \tau_{zz} \tag{A.16}$$

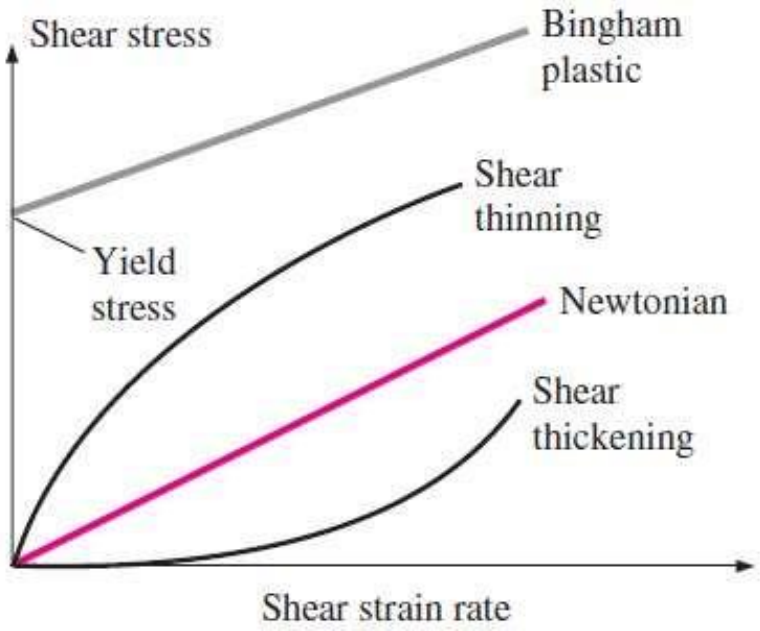
Complete Navier–Stokes equations

$$x\text{-component}: \quad \frac{\partial(\rho u)}{\partial t} + \nabla\cdot(\rho u\vec{V}) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$y\text{-component}: \quad \frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v\vec{V}) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$$

$$z\text{-component}: \quad \frac{\partial(\rho w)}{\partial t} + \nabla\cdot(\rho w\vec{V}) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$$

Newtonian versus Non-Newtonian Fluids



Rheological behavior of fluids—shear stress as a function of shear strain rate.

In some fluids a finite stress called the yield stress is required before the fluid begins to flow at all; such fluids are called Bingham plastic fluids.

Rheology: The study of the deformation of flowing fluids.

Newtonian fluids: Fluids for which the shear stress is linearly proportional to the shear strain rate.

Non-Newtonian fluids: Fluids for which the shear stress is not linearly related to the shear strain rate.

Viscoelastic: A fluid that returns (either fully or partially) to its original shape after the applied stress is released.

Some non-Newtonian fluids are called shear thinning fluids or pseudoplastic fluids, because the more the fluid is sheared, the less viscous it becomes.

Plastic fluids are those in which the shear thinning effect is extreme.

According to *Newton's law of viscosity*, the normal and tangential viscous stress components are given by

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] \quad \tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$$

$$\tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad (A.17)$$

The proportionality constants of μ and λ are the (first) dynamic viscosity that relates stresses to linear deformation and the second viscosity that relates stresses to the volumetric deformation,

Derivation of the Navier–Stokes Equation for Incompressible, Isothermal Flow

For a fluid flow that is both incompressible and isothermal:

- ρ = constant
- μ = constant

And therefore:

- ν = constant

The incompressible flow approximation implies constant density, and the isothermal approximation implies constant viscosity.

Viscous stress tensor for an incompressible Newtonian fluid with constant properties:

$$\tau_{ij} = 2\mu\varepsilon_{ij} \tag{9–55}$$

$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} 2\mu\dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right) & 2\mu\dfrac{\partial v}{\partial y} & \mu\left(\dfrac{\partial v}{\partial z}+\dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right) & 2\mu\dfrac{\partial w}{\partial z} \end{pmatrix}$$

respectively. To the present day, not much is known about the second viscosity. Nevertheless, the Stokes hypothesis that $\lambda = -2/3\mu$ is frequently used and it has been found for gases to be a good working approximation.

When we combine Eqs. (A.16) and (A.17) with Eqs. (A.13)–(A.15), the equations for the velocity components u, v, and w in three dimensions can be rewritten as

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\sum F_x^{body\ forces} \tag{A.18}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\sum F_y^{body\ forces} \tag{A.19}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}+\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\sum F_z^{body\ forces} \tag{A.20}$$

Navier-Stokes Equations

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}-\frac{2}{3}\vec{\nabla}.\vec{V}\right)\right]$$

$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] \qquad (a)$$

x-momentum

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]$$

$$+\frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y}-\frac{2}{3}\vec{\nabla}.\vec{V}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] \qquad (b)$$

y-momentum

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]$$

$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z}-\frac{2}{3}\vec{\nabla}.\vec{V}\right)\right] \qquad (c)$$

z-momentum

Navier-Stokes Equations

- For incompressible fluids, constant μ:
- Continuity equation: $\nabla.V = 0$

$$\frac{\partial}{\partial x}[\mu(2\frac{\partial u}{\partial x}-\frac{2}{3}\vec{\nabla}.\vec{V})]+\frac{\partial}{\partial y}[\mu(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})]+\frac{\partial}{\partial z}[\mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]=$$

$$\mu\{\frac{\partial}{\partial x}[(2\frac{\partial u}{\partial x})]+\frac{\partial}{\partial y}[(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})]+\frac{\partial}{\partial z}[(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]\}=$$

$$\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z})=$$

$$\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+\mu\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z})=$$

$$\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})=\mu\nabla^2 u$$

Navier-Stokes Equations

- For incompressible flow with constant dynamic viscosity:

- x- momentum

$$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad (a)$$

- Y- momentum

$$\rho\frac{Dv}{Dt}=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad (b)$$

- z-momentum

$$\rho\frac{Dw}{Dt}=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \qquad (c)$$

- In vector form, the three equations are given by:

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}\ -\nabla p+\mu\nabla^2\vec{V}$$

Incompressible NSE written in vector form

Navier-Stokes Equations

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}-\frac{2}{3}\vec{\nabla}.\vec{V}\right)\right]$$

$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] \qquad (a)$$

x-momentum

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]$$

$$+\frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y}-\frac{2}{3}\vec{\nabla}.\vec{V}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] \qquad (b)$$

y-momentum

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]$$

$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z}-\frac{2}{3}\vec{\nabla}.\vec{V}\right)\right] \qquad (c)$$

z-momentum

Navier-Stokes Equations

- For incompressible fluids, constant μ:
- Continuity equation: $\nabla.V = 0$

$$\frac{\partial}{\partial x}[\mu(2\frac{\partial u}{\partial x}-\frac{2}{3}\vec{\nabla}.\vec{V})]+\frac{\partial}{\partial y}[\mu(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})]+\frac{\partial}{\partial z}[\mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]=$$

$$\mu\{\frac{\partial}{\partial x}[(2\frac{\partial u}{\partial x})]+\frac{\partial}{\partial y}[(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})]+\frac{\partial}{\partial z}[(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]\}=$$

$$\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z})=$$

$$\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+\mu\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z})=$$

$$\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})=\mu\nabla^2 u$$

Navier-Stokes Equations

- For incompressible flow with constant dynamic viscosity:

- x- momentum

$$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad (a)$$

- Y- momentum

$$\rho\frac{Dv}{Dt}=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad (b)$$

- z-momentum

$$\rho\frac{Dw}{Dt}=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \qquad (c)$$

- In vector form, the three equations are given by:

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}\ -\nabla p+\mu\nabla^2\vec{V}$$

Incompressible NSE written in vector form

Navier-Stokes Equation

- The Navier-Stokes equations for incompressible flow in vector form:

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

$$\nabla \cdot \vec{V} = 0$$

Incompressible NSE written in vector form

- This results in a *closed* system of equations!
  - 4 equations (continuity and 3 momentum equations)
  - 4 unknowns (u, v, w, p)
- In addition to vector form, incompressible N-S equation can be written in several other forms including:
  - Cartesian coordinates
  - Cylindrical coordinates
  - Tensor notation

Euler Equations

- For inviscid flow (μ = 0) the momentum equations are given by:

- x- momentum $$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x} \qquad (a)$$

- Y- momentum $$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y} \qquad (b)$$

- z-momentum $$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z} \qquad (c)$$

- In vector form, the three equations are given by:

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}\ -\nabla p$$

Euler equations written in vector form

Differential Analysis of Fluid Flow Problems

- Now that we have a set of governing partial differential equations, there are 2 problems we can solve
  - Calculate pressure (*P*) for a known velocity field
  - Calculate velocity (*U, V, W*) and pressure (*P*) for known geometry, boundary conditions (BC), and initial conditions (IC)
- There are about 80 known exact solutions to the NSE
- Solutions can be classified by type or geometry, for example:
  1. Couette shear flows
  2. Steady duct/pipe flows (Poisseulle flow)

Exact Solutions of the NSE

Procedure for solving continuity and NSE

1. Set up the problem and geometry, identifying all relevant dimensions and parameters
2. List all appropriate assumptions, approximations, simplifications, and boundary conditions
3. Simplify the differential equations as much as possible
4. Integrate the equations
5. Apply BCs to solve for constants of integration
6. Verify results

- Boundary conditions are critical to exact, approximate, and computational solutions.
  - BC's used in analytical solutions are
    - No-slip boundary condition
    - Interface boundary condition

Summary of Fluid Dynamic Equations in CFD Analysis

3D Compressible Navier–Stokes Equations

C.E. in conservative form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

Complete Navier–Stokes equations in conservation form

$$x\text{-component}: \quad \frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$y\text{-component}: \quad \frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \vec{V}) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$$

$$z\text{-component}: \quad \frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \vec{V}) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$$

3D Compressible Navier–Stokes Equations

C.E. in conservative form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

By expanding

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

3D Compressible Navier–Stokes Equations

Momentum Equations in conservative form

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}$$
$$=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]$$
$$+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\rho f_x$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho vw)}{\partial z}$$
$$=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left(\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial v}{\partial y}\right)$$
$$+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+\rho f_y$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho uw)}{\partial x}+\frac{\partial(\rho vw)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}$$
$$=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]$$
$$+\frac{\partial}{\partial z}\left(\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial w}{\partial z}\right)+\rho f_z$$

3D Compressible Navier–Stokes Equations

when expanded using Stokes' hypothesis (λ= - 2/3 μ) gives

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}-\frac{2}{3}\vec{\nabla}.\vec{V}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] \qquad (a)$$

x-momentum

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y}-\frac{2}{3}\vec{\nabla}.\vec{V}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right] \qquad (b)$$

y-momentum

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]$$
$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z}-\frac{2}{3}\vec{\nabla}.\vec{V}\right)\right] \qquad (c)$$

z-momentum

3D Incompressible Navier–Stokes Equations

Continuity Equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

X-Momentum $$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

Y-Momentum $$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

Z-Momentum $$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

2D Incompressible Navier–Stokes Equations

Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

X-Momentum

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Y-Momentum

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g_y - \frac{\partial p}{\partial y} + +\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Euler Equations

Continuity Equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

X-Momentum $$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}$$

Y-Momentum $$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}$$

Z-Momentum $$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}$$

Poisson Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

or

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = f(x, y)$$

Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

2D Viscous Burgers' Equation (Convection)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \upsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

2D Heat Equation (Diffusion)

$$\frac{\partial u}{\partial t} = \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

2D Inviscid Burgers' Equation (Convection)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0$$

2D Wave Equation (Linear Convection)

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = 0$$

1D Viscous Burgers' Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \upsilon\frac{\partial^2 u}{\partial x^2}$$

1D Heat Equation (Diffusion)

$$\frac{\partial u}{\partial t} = \upsilon\frac{\partial^2 u}{\partial x^2}$$

1D Inviscid Burgers' Equation (Convection)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

1D Wave Equation (Linear Convection)

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

Basics of Finite Difference Formulations

Refer to

Ch. 2

Hoffmann, A., Chiang, S., Computational Fluid Dynamics for Engineers, Vol. I, 4th ed., Engineering Education System, 2000.

Ch. 3, 4 and 5

Pletcher, R. H., Tannehill, J. C., Anderson, D., Computational Fluid Mechanics and Heat Tranfer, 3rd ed., CRC Press, 2011.

Ch. 5

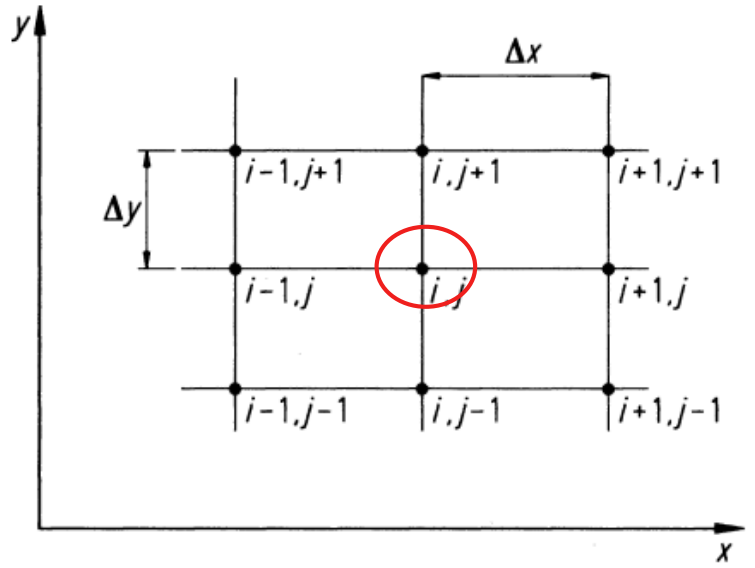
Wendt, Anderson, Computational Fluid Dynamics - An Introduction, 3rd edition 2009.

Discretization methods (Finite Difference)

- First step in obtaining a numerical solution is to discretize the geometric domain→ to define a numerical grid
- Each node has one unknown and needs one algebraic equation, which is a relation between the variable value at that node and those at some of the neighboring nodes.
- The approach is to replace each term of the PDE at the particular node by a finite-difference approximation.
- Numbers of equations and unknowns must be equal

Discretization (Grid Generation)

- Numerical solutions can give answers at only discrete points in the domain, called grid points.



- If the PDEs are totally replaced by a system of algebraic equations which can be solved for the values of the flow-field variables at the discrete points only, in this sense, the original PDEs have been discretized.
- Moreover, this method of discretization is called the method of finite differences.

Taylor's series expansion

- A partial derivative replaced with a suitable algebraic difference quotient is called **finite difference**.
- Most finite-difference representations of derivatives are based on Taylor's series expansion.
- Taylor's series expansion:

  Consider a continuous function of x, namely, f(x), with all derivatives defined at *x*. Then, the value of f at a location $x+\Delta x$ can be estimated from a Taylor series expanded about point x, that is,

$$f(x+\Delta x) = f(x) + \frac{\partial f}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}(\Delta x)^3 + \ldots + \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(\Delta x)^n + \ldots$$

$$f(x_{i+1}) = f(x_i) + \frac{\partial f}{\partial x}(x_{i+1}-x_i) + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x_{i+1}-x_i)^2 + \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}(x_{i+1}-x_i)^3 + \ldots + \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(x_{i+1}-x_i)^n + \ldots$$

- In general, to obtain more accuracy, additional higher-order terms must be included.

Taylor's series expansion

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \ldots$$

$$+ \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n + R_n$$

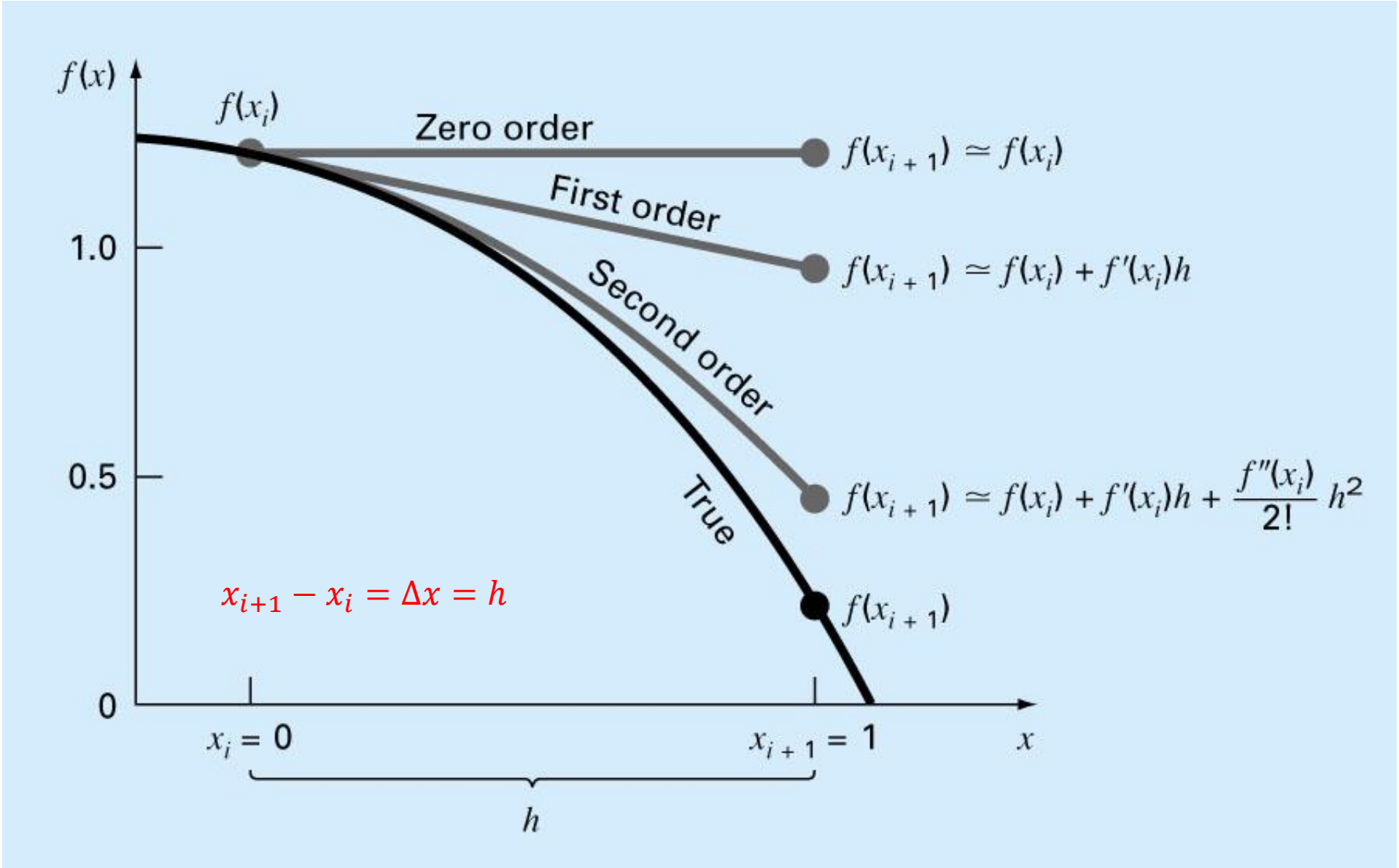
$(x_{i+1}-x_i)= \Delta x$ *step size* (define first)

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}\Delta x^2 + \ldots$$

$$+ \frac{f^{(n)}(x_i)}{n!}\Delta x^n + R_n = f(x_i) + \sum_{1}^{\infty} \frac{(\Delta x)^n}{n!} f^n(x_i)$$

- The term, R_n, accounts for all terms from (n+1) to infinity, Truncation error.

Taylor's series expansion

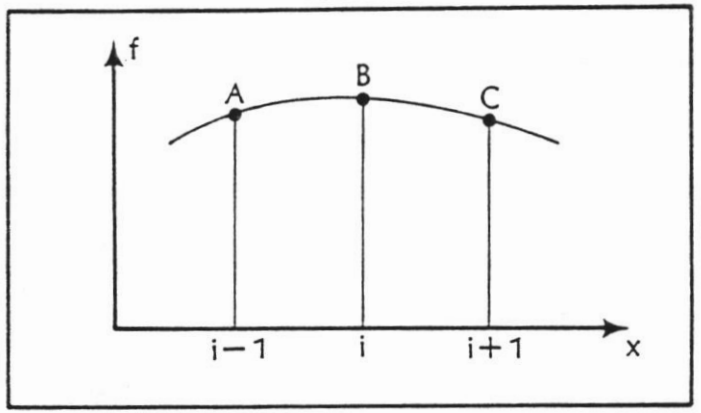


Truncation Error

- Need to determine $f^{n+1}(x)$, to do this you need f'(x).
- If we knew f(x), there wouldn't be any need to perform the Taylor series expansion.
- However, $R=O(\Delta x^{n+1})$, $(n+1)^{th}$ order, the order of truncation error is Δx^{n+1}.
- $O(\Delta x)$, halving the step size will halve the error.
- $O(\Delta x^2)$, halving the step size will quarter the error.

Forward, Backward and Central Differences:

(1) Forward difference:



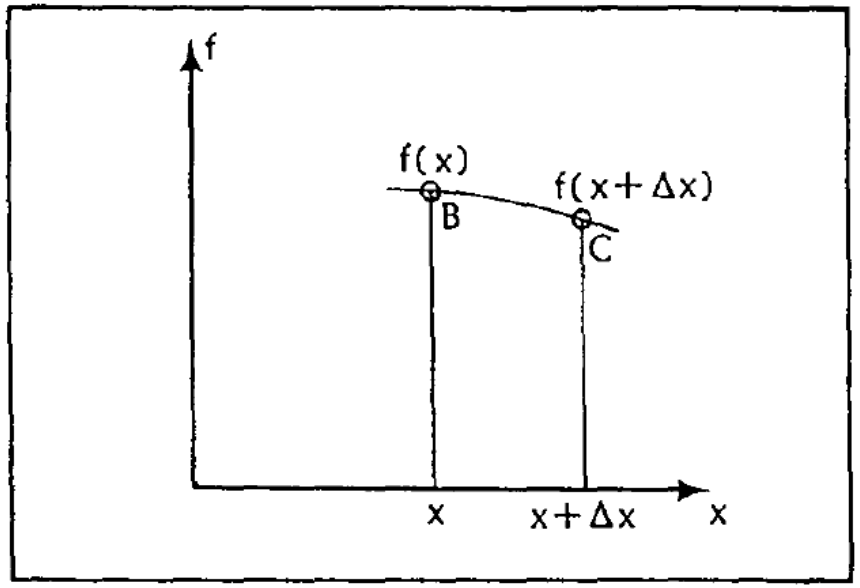
Neglecting higher-order terms,

we can get

$$f(x_{i+1}) = f(x_i) + (\frac{\partial f}{\partial x})_i (x_{i+1} - x_i) + \frac{(x_{i+1} - x_i)^2}{2!} (\frac{\partial^2 f}{\partial x^2})_i + \frac{(x_{i+1} - x_i)^3}{3!} (\frac{\partial^3 f}{\partial x^3})_i$$

$$+ \ldots + \frac{(x_{i+1} - x_i)^n}{n!} (\frac{\partial^n f}{\partial x^n})_i + \ldots$$

Solve for $(\frac{\partial f}{\partial x})_i$, we get

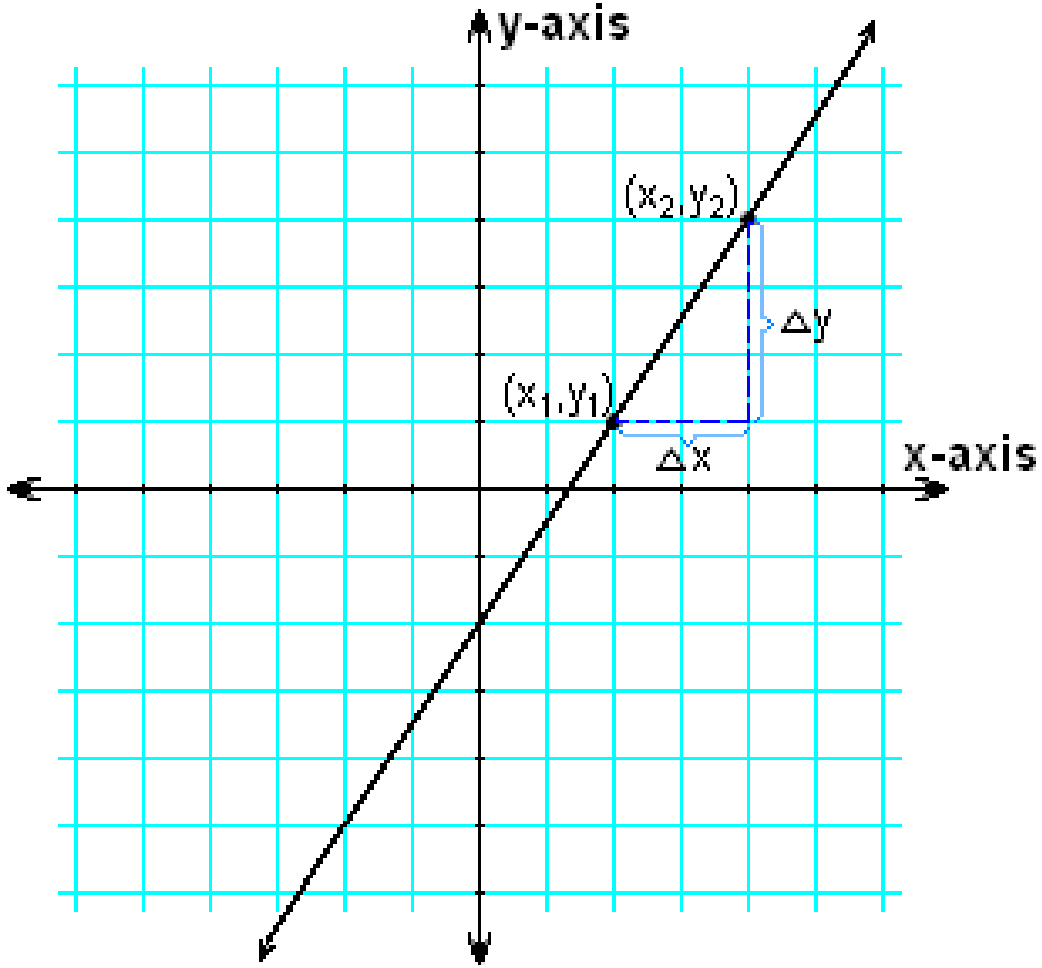


Finite Differences:

Recall the Definition of a derivative:

$$\frac{\partial y}{\partial x} = \lim_{\Delta x \to 0} \frac{y_2 - y_1}{x_2 - x_1}$$

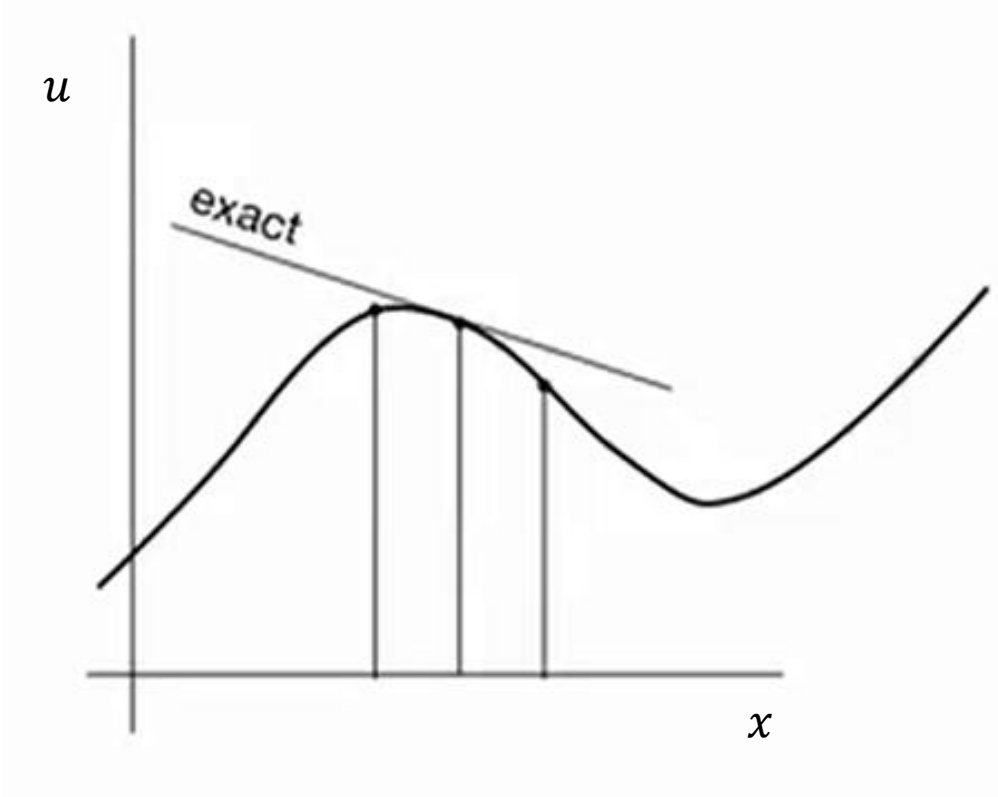
$$\frac{\partial y}{\partial x} = \lim_{\Delta x \to 0} \frac{y_2 - y_1}{\Delta x}$$



Finite Differences:

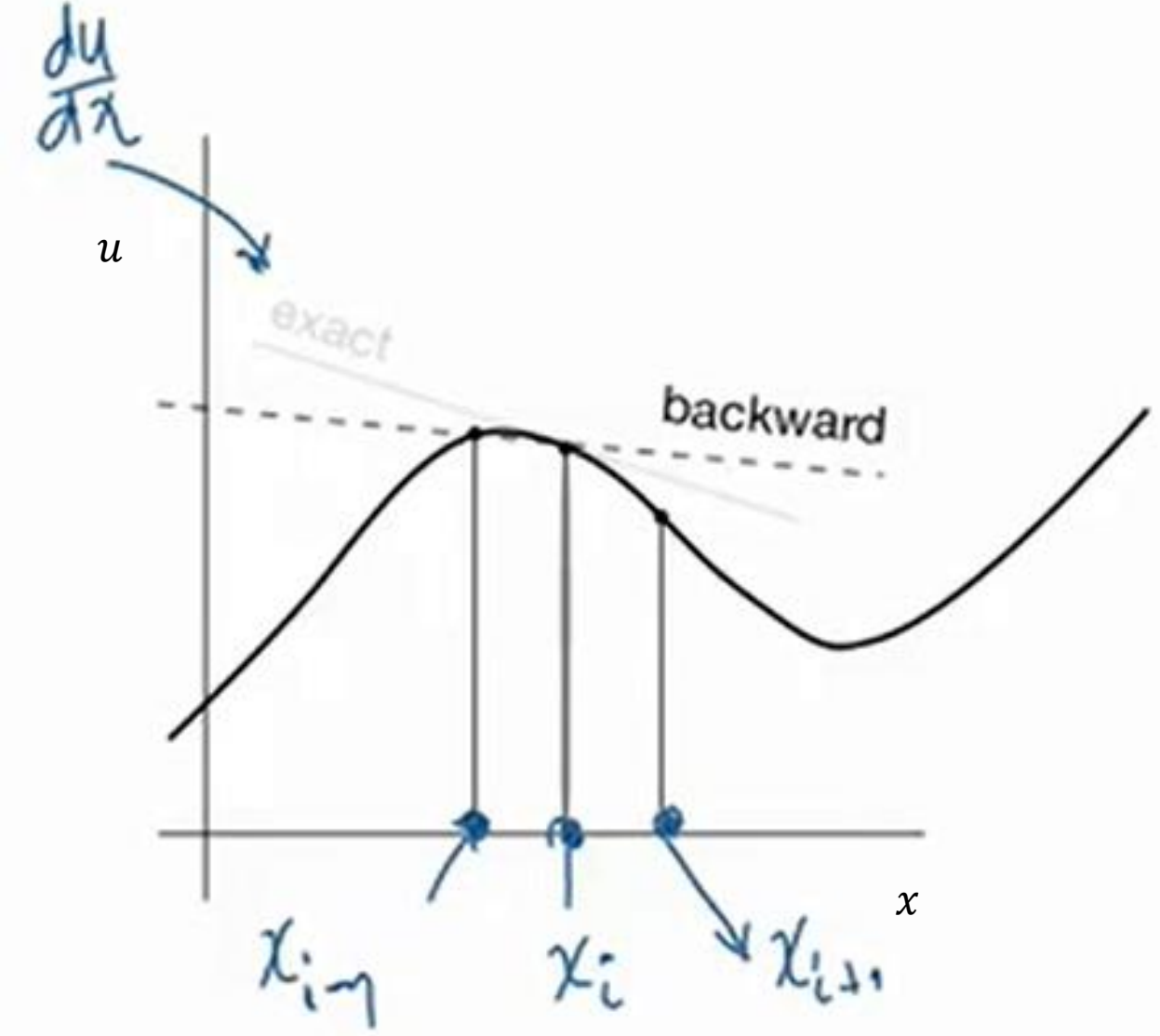
Recall the Definition of a derivative:

$$\left(\frac{\partial u}{\partial x}\right)_{x_i} = \lim_{\Delta x \to 0} \frac{u(x_i+\Delta x)-u(x_i)}{\Delta x}$$



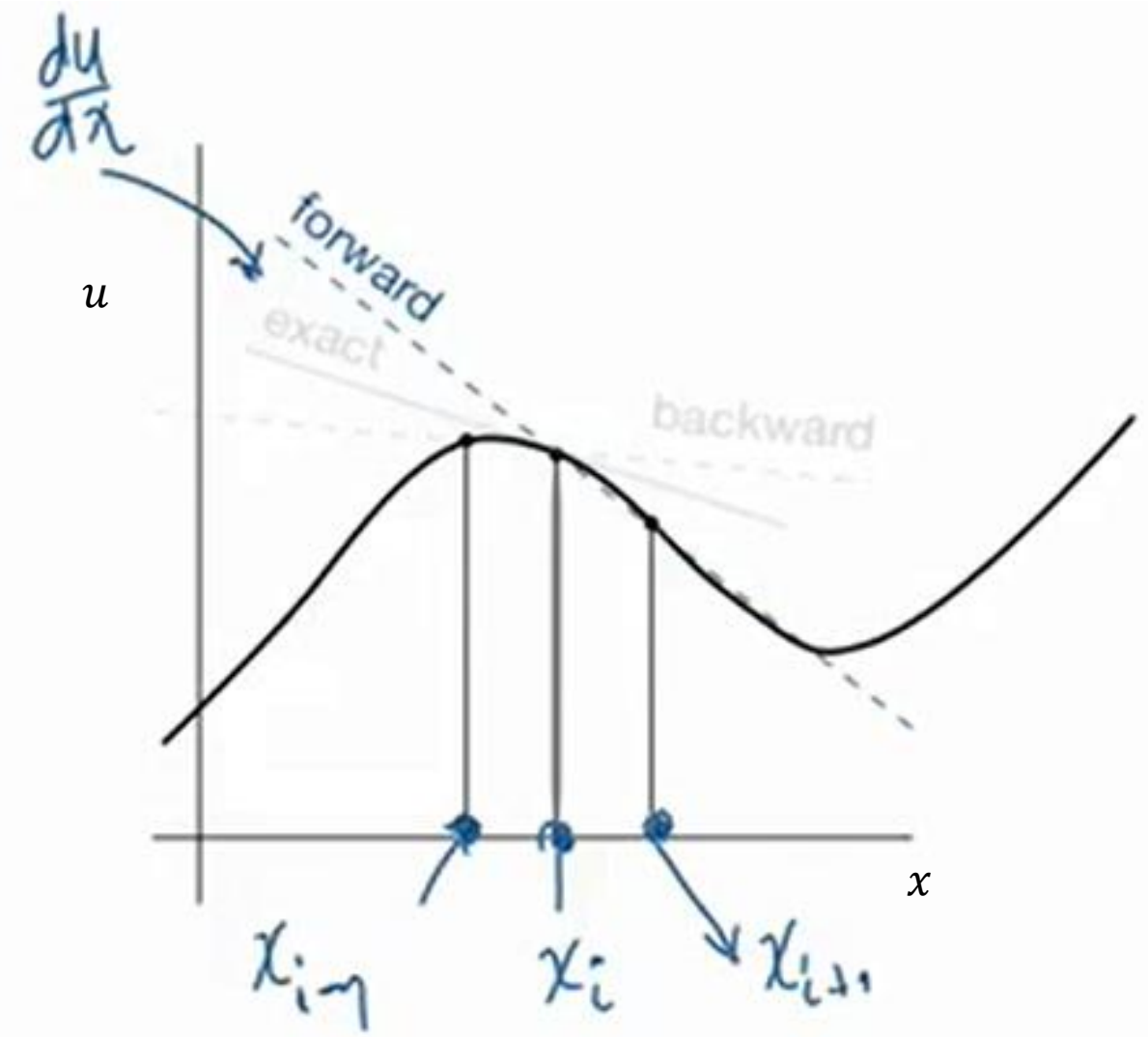
Forward, Backward and Central Differences:

(1) Backward difference:



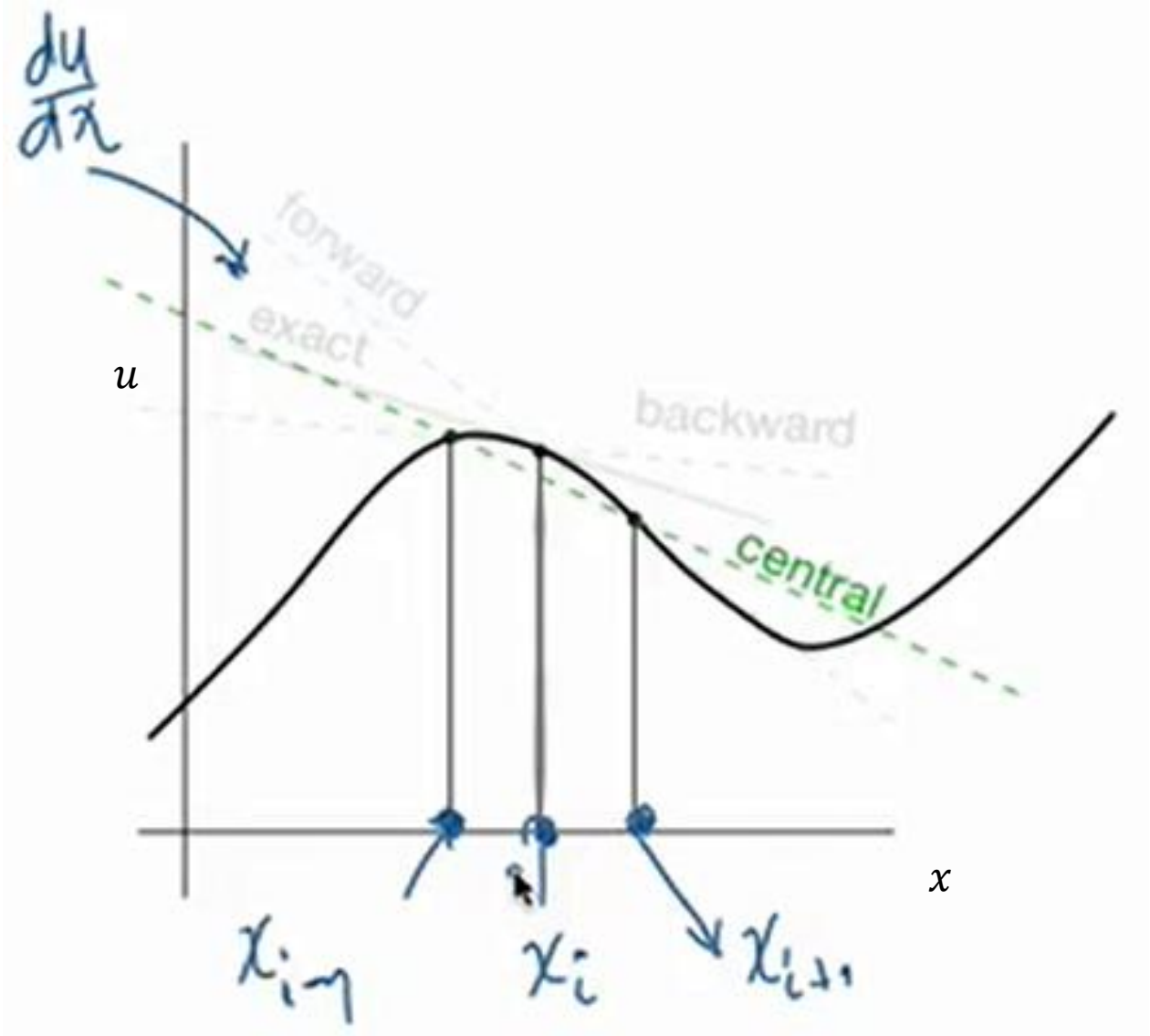
Forward, Backward and Central Differences:

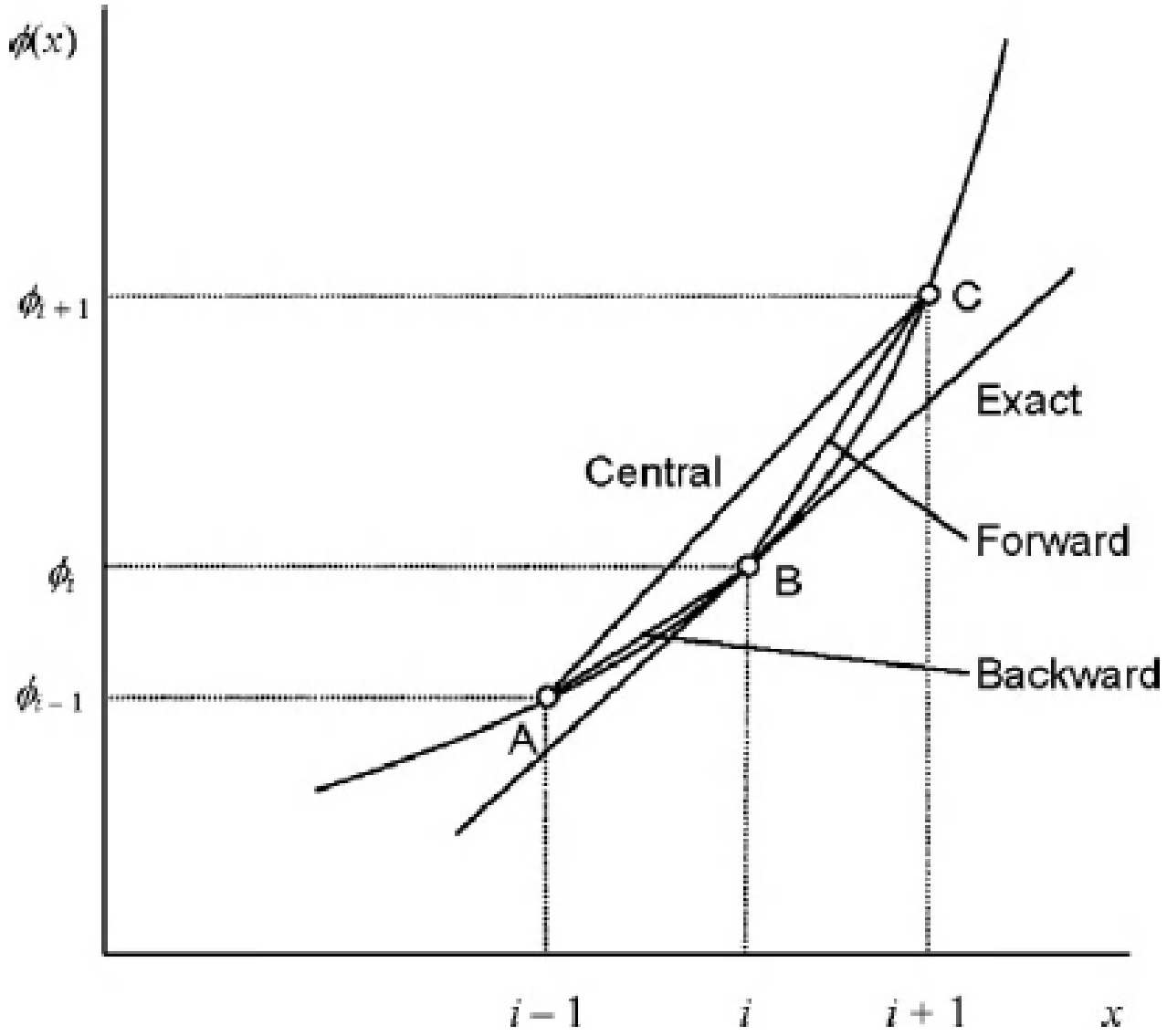
(2) Forward difference:

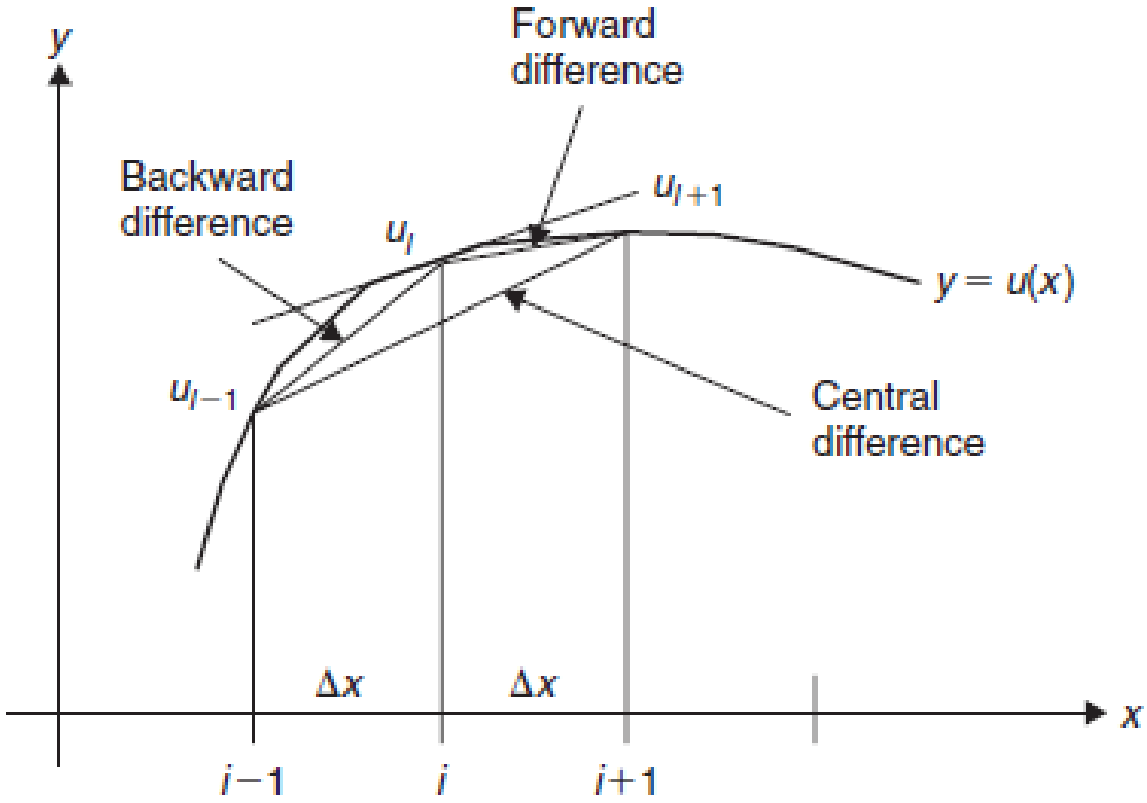


Forward, Backward and Central Differences:

(3) Central difference:







Finite-difference representation of the first-order derivative for $\partial\phi/\partial x$.

(1) Forward difference:

$$\left(\frac{\partial f}{\partial x}\right)_i = \frac{f(x_{i+1}) - f(x_i)}{(x_{i+1} - x_i)} - \frac{(x_{i+1} - x_i)^2}{2!(x_{i+1} - x_i)}\left(\frac{\partial^2 f}{\partial x^2}\right)_i - \frac{(x_{i+1} - x_i)^3}{3!(x_{i+1} - x_i)}\left(\frac{\partial^3 f}{\partial x^3}\right)_i$$

$$- \ldots - \frac{(x_{i+1} - x_i)^n}{n!(x_{i+1} - x_i)}\left(\frac{\partial^n f}{\partial x^n}\right)_i - \ldots$$

$$\left(\frac{\partial f}{\partial x}\right)_i = \frac{f_{i+1} - f_i}{\Delta x} - \frac{\Delta x}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_i - \frac{\Delta x^2}{6}\left(\frac{\partial^3 f}{\partial x^3}\right)_i - \ldots - \frac{\Delta x^{n-1}}{n!}\left(\frac{\partial^n f}{\partial x^n}\right)_i - \ldots$$

$$= \frac{f_{i+1} - f_i}{\Delta x} + O(\Delta x) \qquad (a)$$

- This equation is known as the first forward difference approximation of $\frac{\partial f}{\partial x}$ of order (Δx).
- It is obvious that as the step size decreases, the error term is reduced and therefore the accuracy of the approximation is increased.

(2) Backward difference

Taylor series expansion:

Neglecting higher-order terms, we can get

$$f(x_{i-1}) = f(x_i) - (\frac{\partial f}{\partial x})_i (x_i - x_{i-1}) + \frac{(x_i - x_{i-1})^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i - \frac{(x_i - x_{i-1})^3}{3!}(\frac{\partial^3 f}{\partial x^3})_i$$

$$+ \ldots + (-1)^n \frac{(x_i - x_{i-1})^n}{n!}(\frac{\partial^n f}{\partial x^n})_i + \ldots = f(x) + \sum_{n=1}^{\infty}\left[(-1)^n \frac{(\Delta x)^n}{n!}\right]\frac{\partial^n f}{\partial x^n}$$

Solve for $(\frac{\partial f}{\partial x})_i$, we get

$$(\frac{\partial f}{\partial x})_i = \frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})} + \frac{(x_i - x_{i-1})}{2}(\frac{\partial^2 f}{\partial x^2})_i - \frac{(x_i - x_{i-1})^2}{6}(\frac{\partial^3 f}{\partial x^3})_i$$

$$+ \ldots + (-1)^n \frac{(x_i - x_{i-1})^{n-1}}{n!}(\frac{\partial^n f}{\partial x^n})_i + .. = \frac{f_i - f_{i-1}}{\Delta x} + O(\Delta x) \qquad (b)$$

(2) Backward difference

- which represents the slope of the function at *B* using the values of the function at points *A* and *B,* as shown in Figure 2-2.
- Equation (2-6) is the first backward difference approximation

of $\frac{\partial f}{\partial x}$ of order (Δx).

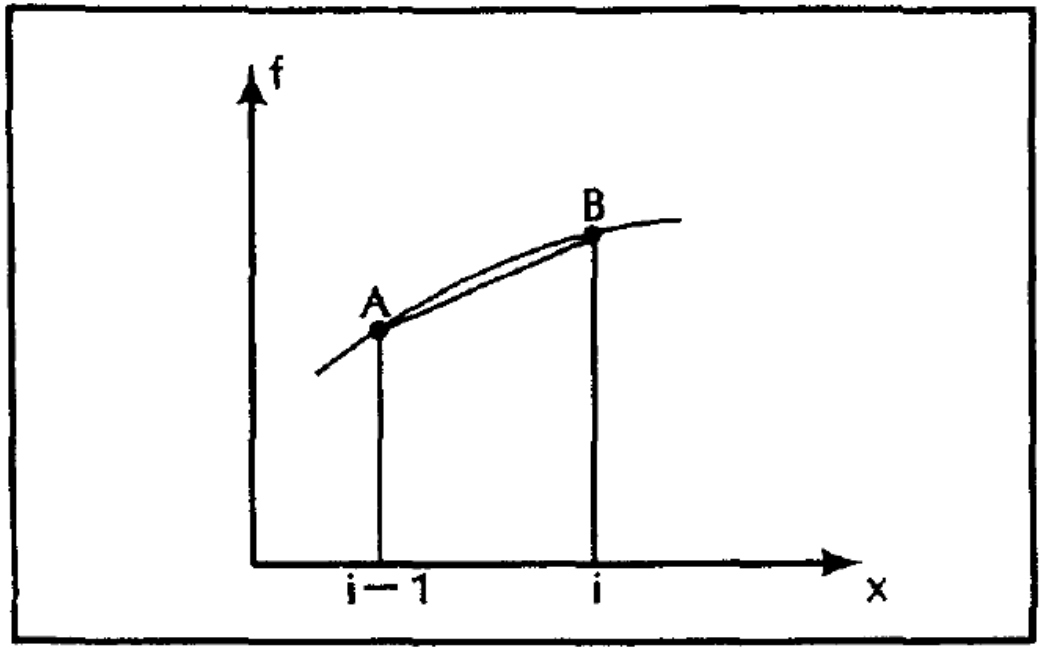


Figure 2-2. Illustration of grid points used in Equation (2-6).

(3) Central difference:

$$\left(\frac{\partial f}{\partial x}\right)_i = \frac{f_{i+1} - f_i}{\Delta x} - \frac{\Delta x}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_i - \frac{\Delta x^2}{6}\left(\frac{\partial^3 f}{\partial x^3}\right)_i - \ldots - \frac{\Delta x^{n-1}}{n!}\left(\frac{\partial^n f}{\partial x^n}\right)_i - \ldots$$

$$= \frac{f_{i+1} - f_i}{\Delta x} + O(\Delta x) \qquad (a)$$

$$\left(\frac{\partial f}{\partial x}\right)_i = \frac{f_i - f_{i-1}}{\Delta x} + \frac{\Delta x}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_i - \frac{\Delta x^2}{6}\left(\frac{\partial^3 f}{\partial x^3}\right)_i + \ldots + (-1)^n \frac{(\Delta x)^{n-1}}{n!}\left(\frac{\partial^n f}{\partial x^n}\right)_i + ..$$

$$= \frac{f_i - f_{i-1}}{\Delta x} + O(\Delta x) \qquad (b)$$

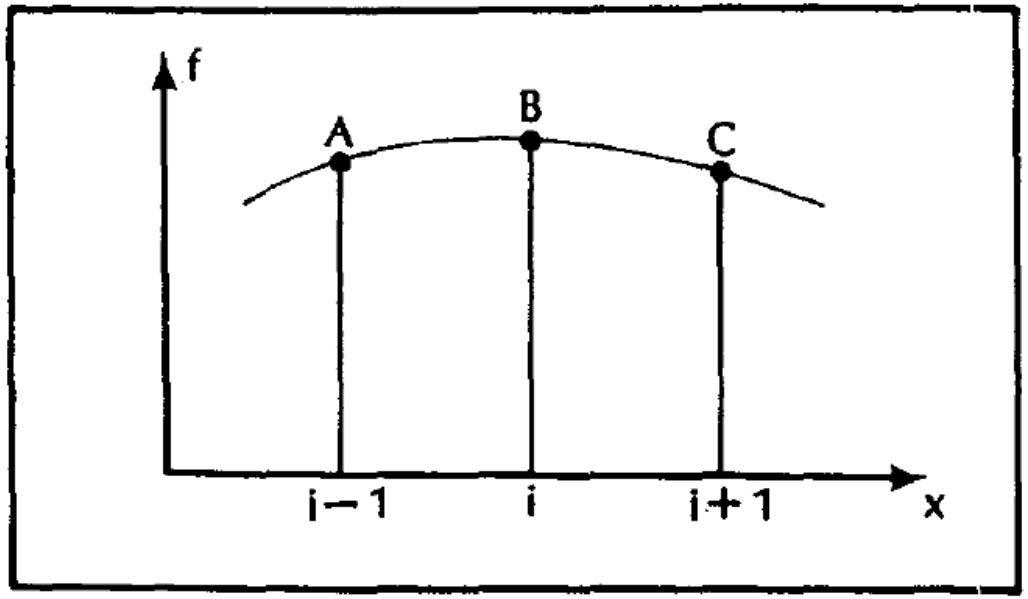
- Adind (a)+(b) and neglecting higher-order terms, we can get

$$2\left(\frac{\partial f}{\partial x}\right)_i = \frac{f_{i+1} - f_i + f_i - f_{i-1}}{\Delta x} - \frac{\Delta x^2}{3}\frac{\partial^3 f}{\partial x^3} + HOT$$

$$\left(\frac{\partial f}{\partial x}\right)_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + O(\Delta x)^2 \qquad (c)$$

(3) Central difference:

- which represents the slope of the function *f* at point *B* using the values of the function at points *A* and *C,* as shown in Figure 2-3.
- This representation of $\frac{\partial f}{\partial x}$ is known as the central difference approximation of order $(\Delta x)^2$•



Truncation error:

The higher-order term neglecting in Eqs. (a), (b), (c) constitute the truncation error.

Forward: $$(\frac{\partial f}{\partial x})_i = \frac{f_{i+1} - f_i}{\Delta x} + O(\Delta x)$$

Backward: $$(\frac{\partial f}{\partial x})_i = \frac{f_i - f_{i-1}}{\Delta x} + O(\Delta x)$$

Central: $$(\frac{\partial f}{\partial x})_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + O(\Delta x)^2$$

Second derivatives:

$$f_{i+1} = f_i + (\frac{\partial f}{\partial x})_i \Delta x + \frac{(\Delta x)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i + \frac{(\Delta x)^3}{3!}(\frac{\partial^3 f}{\partial x^3})_i + \ldots + \frac{(\Delta x)^n}{n!}(\frac{\partial^n f}{\partial x^n})_i + \ldots$$

$$f_{i-1} = f_i - (\frac{\partial f}{\partial x})_i (\Delta x) + \frac{(\Delta x)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i - \frac{(\Delta x)^3}{3!}(\frac{\partial^3 f}{\partial x^3})_i + \ldots + (-1)^n \frac{(\Delta x)^n}{n!}(\frac{\partial^n f}{\partial x^n})_i + \ldots$$

if $\Delta x_i = \Delta x_{i+1} = \Delta x$, then (a)+(b) becomes

* Central difference:

$$f_{i+1} + f_{i-1} = 2 f_i + (\Delta x)^2 (\frac{\partial^2 f}{\partial x^2})_i + O(\Delta x)^4 + HOT$$

$$(\frac{\partial^2 f}{\partial x^2})_i = \frac{f_{i+1} - 2 f_i + f_{i-1}}{(\Delta x)^2} + O(\Delta x)^2$$

Second derivatives:

$$f_{i+1} = f_i + (\frac{\partial f}{\partial x})_i \Delta x + \frac{(\Delta x)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i + \frac{(\Delta x)^3}{3!}(\frac{\partial^3 f}{\partial x^3})_i + \ldots + \frac{(\Delta x)^n}{n!}(\frac{\partial^n f}{\partial x^n})_i + \ldots$$

$$f_{i+2} = f_i + (\frac{\partial f}{\partial x})2\Delta x + \frac{(2\Delta x)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i + \frac{(2\Delta x)^3}{3!}(\frac{\partial^3 f}{\partial x^3}) + \ldots + \frac{(2\Delta x)^n}{n!}(\frac{\partial^n f}{\partial x^n}) + \ldots$$

If $\Delta x_i = \Delta x_{i+1} = \Delta x$, then (b)-2(a) becomes

$$f_{i+2} - 2f_{i+1} = f_i - 2f_i + \frac{(2\Delta x)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i - 2\frac{(\Delta x)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i + \frac{(2\Delta x)^3}{3!}(\frac{\partial^3 f}{\partial x^3}) - 2\frac{(\Delta x)^3}{3!}(\frac{\partial^3 f}{\partial x^3}) + HOT$$

$$f_{i+2} - 2f_{i+1} + f_i = \Delta x^2 (\frac{\partial^2 f}{\partial x^2})_i + O(\Delta x)^3$$

* Forward difference:

$$(\frac{\partial^2 f}{\partial x^2})_i = \frac{f_{i+2} - 2f_{i+1} + f_i}{(\Delta x)^2} + O(\Delta x)$$

Second derivatives:

$$f_{i-1} = f_i - (\frac{\partial f}{\partial x})_i (\Delta x) + \frac{(\Delta x)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i - \frac{(\Delta x)^3}{3!}(\frac{\partial^3 f}{\partial x^3})_i + \ldots + (-1)^n \frac{(\Delta x)^n}{n!}(\frac{\partial^n f}{\partial x^n})_i + \ldots$$

$$f_{i-2} = f_i - (\frac{\partial f}{\partial x})2\Delta x + \frac{(2\Delta x)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i - \frac{(2\Delta x)^3}{3!}(\frac{\partial^3 f}{\partial x^3}) + \ldots + \frac{(2\Delta x)^n}{n!}(\frac{\partial^n f}{\partial x^n}) + \ldots$$

If $\Delta x_i = \Delta x_{i+1} = \Delta x$, then (b)-2(a) becomes

$$f_{i-2} - 2f_{i-1} = f_i - 2f_i + \frac{(2\Delta x)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i - 2\frac{(\Delta x)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i - \frac{(2\Delta x)^3}{3!}(\frac{\partial^3 f}{\partial x^3}) + 2\frac{(\Delta x)^3}{3!}(\frac{\partial^3 f}{\partial x^3}) + HOT$$

$$f_{i-2} - 2f_{i-1} + f_i = \Delta x^2 (\frac{\partial^2 f}{\partial x^2})_i + O(\Delta x)^3$$

* Backward difference:

$$(\frac{\partial^2 f}{\partial x^2})_i = \frac{f_i - 2f_{i-1} + f_{i-2}}{(\Delta x)^2} + O(\Delta x)$$

1-D Wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

By applying Forward in time and central in space (FTCS)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x} = 0$$

By rearranging

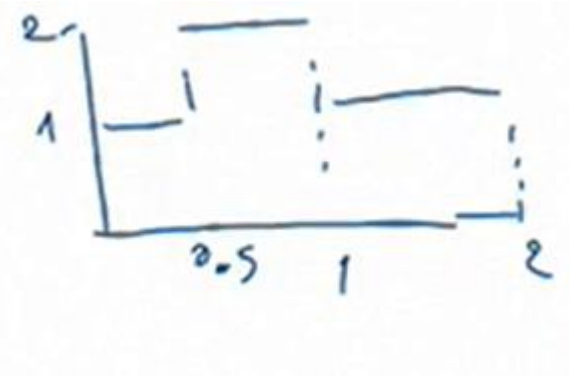
$$u_i^{n+1} = u_i^n + c\frac{\Delta t}{\Delta x} \times (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

1-D Wave equation

$$u_i^{n+1} = u_i^n + c\frac{\Delta t}{\Delta x} \times (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

Assume initial conditions of

$$u = 2 \ @ \ 0.5 \leq x \leq 1$$
$$u = 1 \ @ \ everywhere \ else$$



Assume Boundary conditions

$$u = 1 \ @ \ x = 0, 2$$

1-D Wave equation

$$u_i^{n+1} = u_i^n + c\frac{\Delta t}{\Delta x} \times (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

I.C.

$$u = 2 \ @ \ 0.5 \leq x \leq 1$$

$$u = 1 \ @ \ everywhere\ else$$

B.C.

$$u = 1 \ @ \ x = 0, 2$$

Pseudocode Step 1

```
nx = 20, nt = 50
dt = 0.01, c = 1
dx = 2/(nx-1)

for i = 1:nx
 if 0.5 <= x(i) <= 1
  u(i)= 2
 else
  u(i)= 1
 end
end

for it = 1:nt
 un=u
 for i = 2:nx-1
  u(i) = un(i)- c*dt/dx*...
       ( un(i)- un(i-1) )
 end
end
```

1-D Inviscid Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

By applying Forward in time and central in space (FTCS)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x} = 0$$

By rearranging

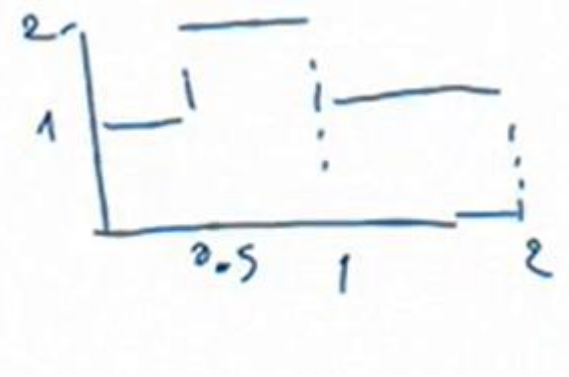
$$u_i^{n+1} = u_i^n + c\frac{\Delta t}{\Delta x} \times u_i^n \times (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

1-D Inviscid Burgers' equation

$$u_i^{n+1} = u_i^n + c\frac{\Delta t}{\Delta x} \times u_i^n \times (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

Assume initial condition of

$$u = 2 \ @ \ 0.5 \leq x \leq 1$$
$$u = 1 \ @ \ everywhere \ else$$



Assume Boundary condition

$$u = 1 \ @ \ x = 0, 2$$

1-D Inviscid Burgers' equation

$$u_i^{n+1} = u_i^n + c\frac{\Delta t}{\Delta x} \times u_i^n \times (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

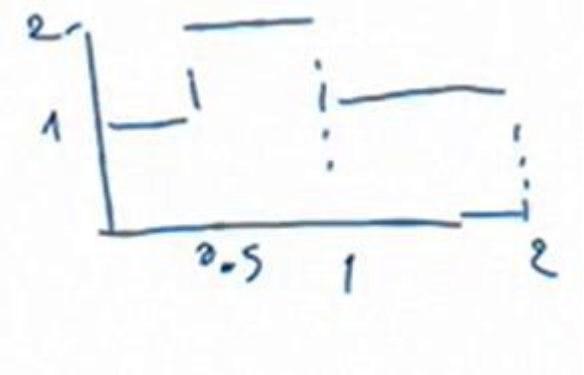
I.C.

$$u = 2 \ @ \ 0.5 \leq x \leq 1$$

$$u = 1 \ @ \ everywhere\ else$$

B.C.

$$u = 1 \ @ \ x = 0, 2$$



Pseudocode Step 2

```
nx = 20, nt = 50
dt = 0.01
dx = 2/(nx-1)

for i = 1:nx
 if 0.5 <= x(i) <= 1
  u(i)= 2
 else
  u(i)= 1
 end
end

for it = 1:nt
 un=u
 for i = 2:nx-1
  u(i) = un(i)- un(i)*dt/dx*.
       ( un(i)- un(i-1) )
 end
end
```

1-D Diffusion equation

$$\frac{\partial u}{\partial t} = \upsilon \frac{\partial^2 u}{\partial y^2}$$

By applying Forward in time and central in space (FTCS)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \upsilon \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

By rearranging

$$u_i^{n+1} = u_i^n + \upsilon \frac{\Delta t}{(\Delta x)^2} \times (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

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By rearranging

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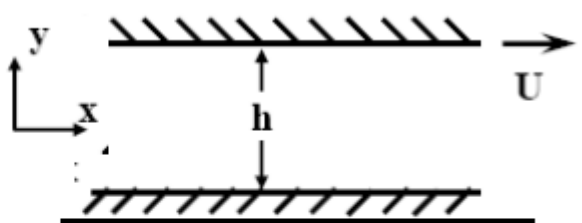
$$u_i^{n+1} = u_i^n + d \times (u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

Application to FTCS-Explicit scheme for the parabolic model equation

Assume J_{max} = 5 and Δt = 0.1 sec, then Δy = 0.01 m and d = 0.2.



h=4 cm , U = 40 m/s

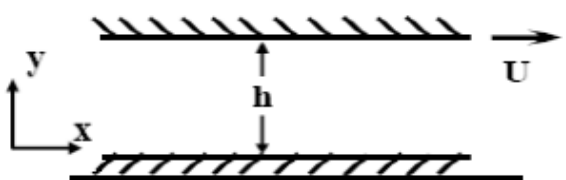
ν = 0.0002 m^2/s

"$d = \nu\Delta t/(\Delta y)^2$".

Distance in "m"	Index of y	t =0 sec	t=0.1 sec	t=0.2 sec	t=0.3 sec	t=0.4 sec	t=0.5 sec	t=∞ (s.s)
y	j	n=0	n=1	n=2	n=3	n=4	n=5	n=∞
0.04	5	40	40	40	40	40	40	40
0.03	4	0	8	12	16	18.304	20.058	30
0.02	3	0	0	1.6	3.52	5.376	7.066	20
0.01	2	0	0	0	0.32	0.896	1.613	10
0	1	0	0	0	0	0	0	0

Application to FTCS-Explicit scheme for the parabolic model equation

Assume J_{max} = 5 and Δt = 0.5 sec, then Δy = 0.01 m and d = 1.0.



h=4 cm , U = 40 m/s

ν = 0.0002 m^2/s

"$d = \nu\Delta t/(\Delta y)^2$".

Distance in "m"	Index of y	t =0 sec	t=0.5 sec	t=1.0 sec	t=1.5 sec	t=2.0 sec	t=2.5 sec	t=∞ (s.s)
y	J	n=0	n=1	n=2	n=3	n=4	n=5	n=∞
0.04	5	40	40	40	40	40	40	40
0.03	4	0	40	0	80	-80	280	?
0.02	3	0	0	40	-40	160	-320	?
0.01	2	0	0	0	40	-80	240	?
0	1	0	0	0	0	0	0	0

Numerical experiments show that if d > 0.5 the FTCS-Explicit scheme solution is unstable

Programming Assignment

Consider a fluid bounded by two parallel plates extended to infinity such that no end effects are encountered. The walls and the fluid are initially at rest. Now, the lower wall is suddenly accelerated in the x-direction. The Navier-Stokes equations for this problem may be expressed as:

$$\frac{\partial u}{\partial t} = \upsilon \frac{\partial^2 u}{\partial y^2}$$

It is required to compute $u(t, y)$.

Programming Assignment 1

Assume initial conditions of

$$u = u_o \ @ \ y = 0$$
$$u = 0 \ @ \ 0 < y \leq h$$

where h is the distance between the two plates and equals 40 mm.

Assume Boundary conditions

$$u = u_o \ @ \ y = 0$$
$$u = 0 \ @ \ y = h$$

Take υ=0.000217 m^2/s, u_o =40 m/s, max time of 1.08 sec. Assume 40 nodes in y direction

Programming Assignment 1

Apply FTCS scheme.

Calculate and plot the velocity distribution by using Matlab by using the following time steps:

- ❑ dt = 0.002 sec
- ❑ dt = 0.00232 sec
- ❑ dt = 0.003 sec

Bonus points will be given to the student who

- **Creates a video of the development of the flow speed with time.**