ME – 733 CFD

Programming Assignment 2

Consider an incompressible and inviscid fluid flow moving around an infinite cylinder. The inlet velocity is 1 m/s. The cylinder radius is 1 m. Consider a mesh starting from r = 1 m with Δr = 0.1 and a final radius of 10 m and $\Delta\theta$=360/100=3.6°.

For polar coordinates

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\ , \qquad v_\theta = -\frac{\partial \psi}{\partial r} \qquad (1)$$

The continuity equation in cartesian coordinates for an incompressible flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

The continuity equation in polar coordinates is

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (v_\theta)}{\partial \theta} = 0 \qquad (2)$$

By substitution of eq(1) in eq(2), we get

$$\frac{1}{r}\frac{\partial \left(r \times \frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)}{\partial r} + \frac{1}{r}\frac{\partial \left(-\frac{\partial \psi}{\partial r}\right)}{\partial \theta} = \frac{1}{r}\frac{\partial \left(\frac{\partial \psi}{\partial \theta}\right)}{\partial r} - \frac{1}{r}\frac{\partial \left(\frac{\partial \psi}{\partial r}\right)}{\partial \theta}$$

$$= \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial^2 \psi}{\partial \theta \partial r} = 0$$

So, the continuity is satisfied.

Assume the following boundary conditions:

$$\psi = 0 \ @ \ r = 1$$

$$\psi = r \sin\theta - \frac{\sin\theta}{r} - \frac{\Gamma}{2\pi}\ln r \ @ \ farfield, r = 10\ m$$

Consider the rotation vector in cartesian coordinates:

$$\omega = \begin{vmatrix} e_x & e_y & e_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

the rotation vector in polar coordinates:

$$\omega = \begin{vmatrix} e_r & e_\theta & e_z \\ \dfrac{1}{r}\dfrac{\partial}{\partial r} & \dfrac{1}{r}\dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ v_r & v_\theta & v_z \end{vmatrix}$$

$$\omega = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)e_r + \left(\frac{1}{r}\frac{\partial v_z}{\partial r} - \frac{\partial v_r}{\partial z}\right)e_\theta + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)e_z$$

For 2d irrotational flow

$$\omega_z = \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = 0$$

Substitute

$$\frac{1}{r}\frac{\partial\left(r - \dfrac{\partial\psi}{\partial r}\right)}{\partial r} - \frac{1}{r}\frac{\partial\left(\dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}\right)}{\partial\theta} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

The Laplace equation for polar coordinates is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

Use the following Finite Difference:

$$\frac{1}{r(j)}\frac{\mathrm{r}(\mathrm{j}+.5)\dfrac{\psi_{\mathrm{i,j+1}} - \psi_{\mathrm{i,j}}}{\Delta\mathrm{r}} - \mathrm{r}(\mathrm{j}-.5)\dfrac{\psi_{\mathrm{i,j}} - \psi_{\mathrm{i,j-1}}}{\Delta\mathrm{r}}}{\Delta\mathrm{r}} + \frac{1}{(r(j))^2}\frac{\psi_{\mathrm{i+1,j}} - 2\psi_{\mathrm{i,j}} + \psi_{\mathrm{i\text{-}1,j}}}{\Delta\theta^2} = 0$$

By rearranging

$$\psi_{i,j} = \frac{1}{r(j+.5) + r(j-.5) + 2\beta}\left[r(j+.5)\psi_{i,j+1} + r(j-.5)\psi_{i,j-1} + \beta\psi_{i+1,j} + \beta\psi_{i-1,j}\right] \qquad (3)$$

Where $\beta = \frac{\Delta r^2}{\Delta\theta^2}$

1. Point Gauss Seidal method (PGS)

In this method we calculate ψ (i,j) from the new values immediately.

For point i,j

Because ψ (i-1,j) and ψ (i,j-1) have been calculated previously so we use their values directly (their values have been overwritten).

We will use the following equation:

Eq. (3) will be

$$\psi(i,j)=\frac{r(j+.5)\psi(i,j+1)+r(j-.5)\psi(i,j-1)+\beta\psi(i+1,j)+\beta\psi(i-1,j)}{r(j+.5)+r(j-.5)+2\beta} \quad (4)$$

2- Point Jacobbi method (PJ)

In this method we calculate ψ (i,j) from the old values of ψ (i,j), so we need two ψ (ψ_{old}, ψ_{new}). We will use the following equation:

$$\psi_{new}(i,j)=\frac{r(j+.5)\psi_{old}(i,j+1)+r(j-.5)\psi_{old}(i,j-1)+\beta\psi_{old}(i+1,j)+\beta\psi_{old}(i-1,j)}{r(j+.5)+r(j-.5)+2\beta} \quad (5)$$

3-Point Successive Over Relaxation method (PSOR)

We will use this method to increase the speed of the program by predicting the new value of ψ by a factor ω.

After calculating ψ^*(i,j) from

$$\psi^*(i,j)=\frac{r(j+.5)\psi(i,j+1)+r(j-.5)\psi(i,j-1)+\beta\psi(i+1,j)+\beta\psi(i-1,j)}{r(j+.5)+r(j-.5)+2\beta} \quad (6)$$

We will calculate the new ψ (i,j) from

$$\psi(i,j)=\psi_{old}(i,j)+\omega(\psi^*(i,j)-\psi_{old}(i,j)) \quad (7)$$

where $1\prec\omega\prec 2$

4-Line Gauss Seidal method (LGS)

In this method we will use the following implicit algorithm to solve the elliptic equation:

$$\frac{1}{r(j)}\;\frac{\mathrm{r(j+.5)}\dfrac{\psi^{k+1}{}_{\mathrm{i,j+1}}-\psi^{K+1}{}_{\mathrm{i,j-1}}}{\Delta\mathrm{r}}-(\mathrm{r(j-.5)}\dfrac{\psi^{k+1}{}_{\mathrm{i,j}}-\psi^{k+1}{}_{\mathrm{i,j-1}}}{\Delta\mathrm{r}}}{\Delta\mathrm{r}}+\frac{1}{(r(j))^2}\frac{\psi^{k}{}_{\mathrm{i+1,j}}-2\psi^{k+1}{}_{\mathrm{i,j}}+\psi^{k}{}_{\mathrm{i-1,j}}}{\Delta\theta^2}=0 \quad (8)$$

Eq. (8) can be rewritten as

$$r(j+.5)\psi_{i,j+1}^{k+1} - \left[2\beta + r(j+.5) + r(j-.5)\right]\psi_{i,j}^{k+1} + r(j-.5)\psi_{i,j-1}^{k+1} = -\beta\psi_{i+1,j}^{k} - \beta\psi_{i-1,j}^{k} \quad (9)$$

which van be written as:

$$a\psi_{i,j+1}^{k+1} - d\psi_{i,j}^{k+1} + b\psi_{i,j-1}^{k+1} = c$$

We will get that

1. At cylinder surface (j = 1)
   a(1) = 0 d(1) = 1 b(1) = 0 c(1) = 0
2. For internal nodes
   a(k) = r(j + .5)
   b(k) = r(j - .5)
   d(k) = - [2 β + r (j + .5)+ r (j - .5)]
   c(k) = - β (ψ (i+1,j) + ψ (i+1,j))
3. For far field (j = j_{max})
   a(j_{max}) = 0 d(j_{max}) = 1 b(j_{max}) = 0
   c(j_{max}) = r (j_{max}) $\sin(\theta(i))$ - $\dfrac{\sin(\theta(i))}{r(j_{max})} - \dfrac{\Gamma}{2\pi}\ln(r(j_{max}))$

5- Line successive over relaxation method (LSOR)

Same procedure as LGS but after we calculate the $\psi(i, j)$ for the same line (j = 1: j_{max}) we modify the value of $\psi(i, j)$ by using Eq. (7)

Take into consideration

$$d\theta = \frac{2*\pi}{i_{max}} (\text{rad}) \qquad \text{where } i_{max} = No.\text{of } \theta \text{ rays} = 100$$

$$dr = 0.1$$

$$r(j) = 1 + (j-1)\,dr$$

$$\theta(i) = \frac{d\theta}{2} + (i-1)\,d\theta$$

Sequence of Programs

For all following methods we will use the following

Initial Conditions:-

$\psi(i,j) = r(j)\sin(\Theta(i))$ for $j=2:j_{max}-1$, $i = 1 : i_{max}$

Boundary Conditions:-

At $j = 1$ $\psi(i,j)$ for $i = 1 : i_{max}$

At $j = j_{max}$ $\psi(i,j) = r(j_{max})\sin(\theta(i)) - \frac{\sin(\theta(i))}{r(j_{max})} - \frac{\Gamma}{2\pi}\ln(r(j_{max}))$

Apply the following schemes:

- Point Jacobbi Method
- Point Gauss Seidal
- Point Successive Over Relaxation (PSOR)
- Line Successive Over Relaxation (LSOR)

Plot the streamlines by using Matlab by using the following: $\Gamma = 0, 2\pi, 4\pi$ and 8π.