EME 304– Mechanical Vibrations
Sheet 2 - Solution
Free Damped Single Degree of Freedom

1. A shock absorber is to be designed to limit its overshoot to 15 percent of its initial displacement when released. Find the damping ratio ζ required. What will be the overshoot if ζ is made equal to (a) $\frac{3}{4}\zeta_0$ and (b) $\frac{5}{4}\zeta_0$.

Equation (2.92) can be expressed as $\delta = \frac{1}{m} \ln\left(\frac{x_0}{x_m}\right)$

For half cycle, $m = \frac{1}{2}$ and hence $\delta = 2 \ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 2 \ln\left(\frac{1}{0.15}\right)$

$= 3.7942$

Necessary damping ratio ζ_0 is

$$\zeta_0 = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{3.7942^2}{\sqrt{4\pi^2 + 3.7942^2}}$$

$$= 0.5169$$

(a)

If $\zeta = \frac{3}{4}\zeta_0 = 0.3877$, the overshoot can be determined by finding δ from Eq.(2.85):

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.3877)}{\sqrt{1-0.3877^2}} = 2.6427 = 2 \ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$$\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 1.32135$$

$$x_{\frac{1}{2}} = x_0 / e^{1.32135} = 0.266775\, x_0$$

$\therefore$ overshoot is 26.6775%

(b)

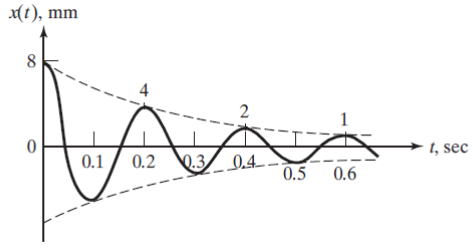
If $\zeta = \frac{5}{4}\zeta_0 = 0.6461$, δ is given by

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.6461)}{\sqrt{1-(0.6461)^2}} = 5.3189 = 2 \ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

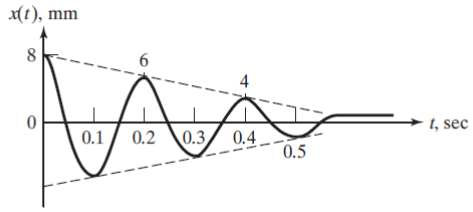
$$\frac{x_0}{x_{\frac{1}{2}}} = 14.2888, \qquad x_{\frac{1}{2}} = 0.0700\, x_0$$

$\therefore$ overshoot = 7%

2. The free-vibration responses of an electric motor of weight 500 N mounted on different types of foundations are shown in Figs. 2.107(a) and (b). Identify the following in each case: (i) the nature of damping provided by the foundation, (ii) the spring constant and damping coefficient of the foundation, and (iii) the undamped and damped natural frequencies of the electric motor.



(a)



(b)

Fig.1.

(i) (a) Viscous damping, (b) Coulomb damping.

(iii) (a) $\tau_d = 0.2$ sec, $f_d = 5$ Hz, $\omega_d = 31.416$ rad/sec.
(b) $\tau_n = 0.2$ sec, $f_n = 5$ Hz, $\omega_n = 31.416$ rad/sec.

(ii) (a) $\dfrac{x_i}{x_{i+1}} = e^{\zeta \omega_n \tau_d}$

$$\ln\left(\frac{x_i}{x_{i+1}}\right) = \ln 2 = 0.6931 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

or $39.9590\,\zeta^2 = 0.4804$ or $\zeta = 0.1096$

Since $\omega_d = \omega_n \sqrt{1-\zeta^2}$, we find

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(31.6065)^2 = 5.0916\,(10^4) \text{ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n}$$

Hence $c = 2\,m\,\omega_n\,\zeta = 2\,(\frac{500}{9.81})\,(31.6065)\,(0.1096) = 353.1164$ N–s/m

(b) From Eq. (2.135):

$$k = m\,\omega_n^2 = \frac{500}{9.81}\,(31.416)^2 = 5.0304\,(10^4) \text{ N/m}$$

Using $N = W = 500$ N,

$$\mu = \frac{0.002\,k}{4\,W} = \frac{(0.002)\,(5.0304\,(10^4))}{4\,(500)} = 0.0503$$

3. A railroad car of mass 2,000 kg traveling at a velocity of $v = 10$ m/s is stopped at the end of the tracks by a spring-damper system, as shown in Fig. 2. If the stiffness of the spring is $k = 80$ N/mm and the damping constant is $c = 20$ N-s/mm, determine (a) the maximum displacement of the car after engaging the springs and damper and (b) the time taken to reach the maximum displacement.

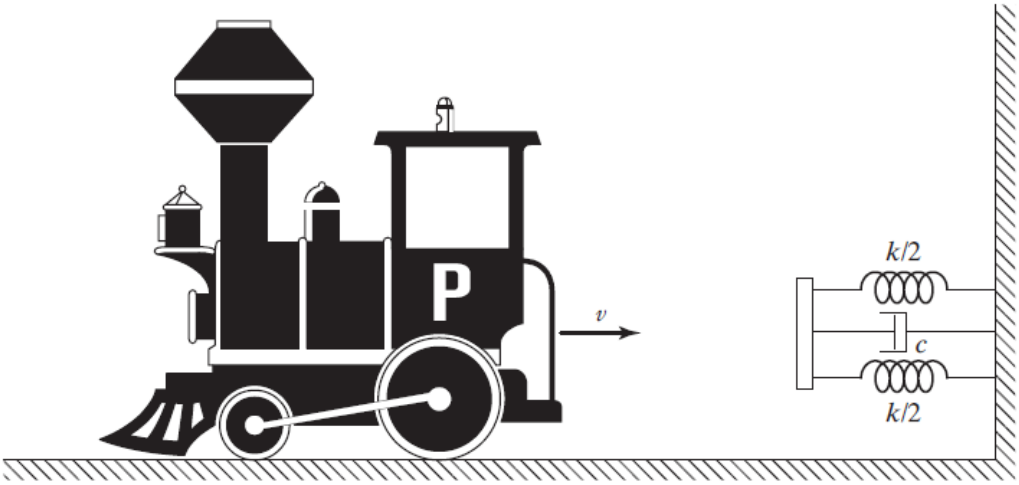


Fig. 2.

$m = 2000$ kg, $v = \dot{x}_0 = 10$ m/sec, $k = 40{,}000$ N/m

$c = 20{,}000$ N-sec/m

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{2000}} = 4.4721$ rad/sec

$c_c = 2\sqrt{km} = 25{,}298.221$ N-sec/m

$\zeta = c/c_c = 0.7906$

$\omega_d = \omega_n \sqrt{1-\zeta^2} = 4.4721\sqrt{1-(0.7906)^2} = 2.7384$ rad/sec

$\tau_d = 2\pi/\omega_d = 2.2945$ sec

(a) For $x_0 = 0$ and $\dot{x}_0 = 10$ m/sec, Eq. (2.72) gives

$$x(t) = e^{-\zeta\omega_n t}\frac{\dot{x}_0}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t$$

At x_{max}, $\omega_n t \simeq \frac{\pi}{2}$ and $\sin\omega_n\sqrt{1-\zeta^2}\,t \simeq 1$

$$\therefore x_{max} \simeq e^{-0.7906(\pi/2)}\cdot\left(\frac{10}{2.7384}\right)\cdot(1) = 1.0548 \text{ m}$$

(b) $t = \tau_d/4 = 2.2945/4 = 0.5736$ sec.

4. A boy riding a bicycle can be modeled as a spring-mass-damper system with an equivalent weight, stiffness, and damping constant of 800 N, 50,000 N/m, and 1,000 N-s/m, respectively. The differential setting of the concrete blocks on the road caused the level surface to decrease suddenly, as indicated in Fig. 3. If the speed of the bicycle is 5 m/s (18 km/hr), determine the displacement of the boy in the vertical direction. Assume that the bicycle is free of vertical vibration before encountering the step change in the vertical displacement.

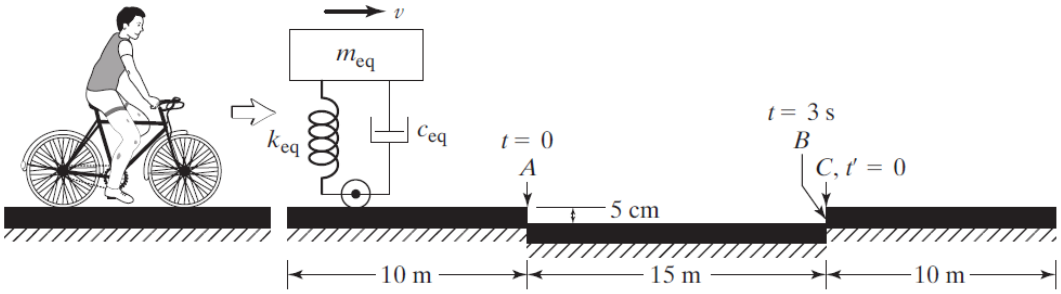


Fig. 3.

Assume that the bicycle and the boy fall as a rigid body by 5 cm at point A. Thus the mass (m_{eq}) will be subjected to an initial downward displacement of 5 cm (t = 0 assumed at point A):

$$x_0 = 0.05 \text{ m}, \dot{x}_0 = 0$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{(50000)\,(9.81)}{800}} = 24.7614 \text{ rad/sec}$$

$$c_c = 2\,m\,\omega_n = 2\left(\frac{800}{9.81}\right)(24.7614) = 4038.5566\ \text{N–s/m}$$

$$\varsigma = \frac{c}{c_c} = \frac{1000.0}{4038.5566} = 0.2476\ \text{(underdamping)}$$

$$\omega_d = \omega_n\sqrt{1-\varsigma^2} = 24.7614\sqrt{1-0.2476^2} = 23.9905\ \text{rad/sec}$$

Response of the system:

$$x(t) = X\,e^{-\varsigma\omega_n t}\sin(\omega_d t + \phi)$$

where $$X = \left\{x_0^2 + \left(\frac{\dot{x}_0 + \varsigma\omega_n x_0}{\omega_d}\right)^2\right\}^{\frac{1}{2}}$$

$$= \left\{(0.05)^2 + \left(\frac{(0.2476)(24.7614)(0.05)}{23.9905}\right)^2\right\}^{\frac{1}{2}} = 0.051607\ \text{m}$$

and $$\phi = \tan^{-1}\left(\frac{x_0\,\omega_d}{\dot{x}_0 + \varsigma\omega_n x_0}\right) = \tan^{-1}\left(\frac{0.05\,(23.9905)}{0.2476\,(24.7614)\,(0.05)}\right) = 75.6645°$$

Thus the displacement of the boy (positive downward) in vertical direction is given by

$$x(t) = 0.051607\,e^{-6.1309\,t}\sin(23.9905\,t + 75.6645°)\ \text{m}$$

5. A body vibrating with viscous damping makes five complete oscillations per second, and in 50 cycles its amplitude diminishes to 10 percent. Determine the logarithmic decrement and the damping ratio. In what proportion will the period of vibration be decreased if damping is removed?

$\tau_d = 0.2\ \text{sec} = \frac{2\pi}{\omega_d}$, $\omega_d = 31.416\ \text{rad/sec}$

From Eq. (2.92) $\delta = \frac{1}{50}\ln 10 = 0.04605$

$$\gamma = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.04605}{\sqrt{(2\pi)^2 + 0.04605^2}} = 0.007329$$

When damping is neglected,

$\omega_n = \omega_d/\sqrt{1-\gamma^2} = 31.417\ \text{rad/sec}$; $\tau_n = \frac{2\pi}{\omega_n} = 0.19999\ \text{sec}$

Proportional decrease in period $= \left(\frac{0.2 - 0.19999}{0.2}\right) = 0.00005$

6. The maximum permissible recoil distance of a gun is specified as 0.5 m. If the initial recoil velocity is to be between 8 m/s and 10 m/s, find the mass of the gun and the spring stiffness of the recoil mechanism. Assume that a critically damped dashpot is used in the recoil mechanism and the mass of the gun has to be at least 500 kg.

For critically damped system

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)t\}\, e^{-\omega_n t} \qquad (E_1)$$

$$\dot{x}(t) = e^{-\omega_n t}\{\dot{x}_0 - \dot{x}_0 \omega_n t - \omega_n^2 x_0 t\} \qquad (E_2)$$

Let t_m = time at which $x = x_{max}$ and $\dot{x} = 0$ occur.

Here $x_0 = 0$ and $\dot{x}_0$ = initial recoil velocity. By setting $\dot{x}(t) = 0$, Eq. (E_2) gives

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} = \frac{\dot{x}_0}{\omega_n \dot{x}_0} = \frac{1}{\omega_n} \qquad (E_3)$$

With Eq. (E_3) for t_m and $x_0 = 0$, (E_1) gives

$$x_{max} = \dot{x}_0 t_m e^{-\omega_n t_m} = \frac{\dot{x}_0 e^{-1}}{\omega_n}$$

i.e. $$\dot{x}_0 = \omega_n x_{max} e = \omega_n (0.5)(2.7183) \qquad (E_4)$$

Using $\dot{x}_0 = 10$ m/sec, $\omega_n = 10/(0.5 * 2.7183) = 7.3575 \frac{rad}{sec}$

When mass of gun is 500 kg, the stiffness of the spring is

$$k = \omega_n^2 m = (7.3575)^2 (500) = 27{,}066.403 \text{ N/m}$$

7. Derive the equation of motion and find the natural frequency of vibration of each of the systems shown in Figs. 4,5,6 and 7.

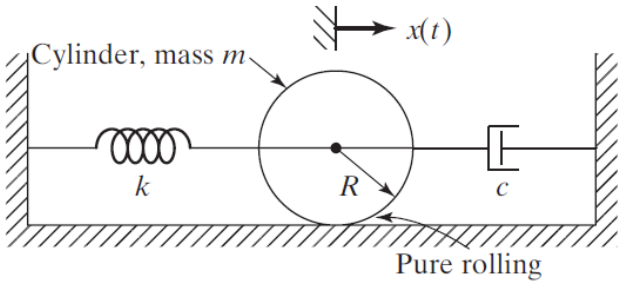


Fig. 4.

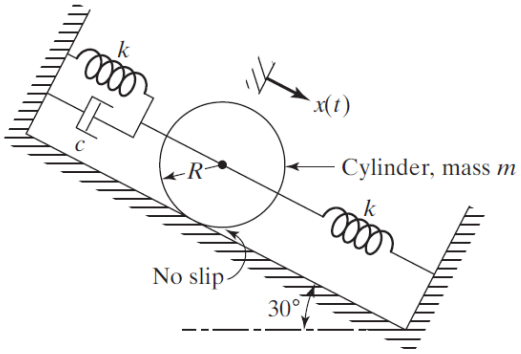


Fig. 5.

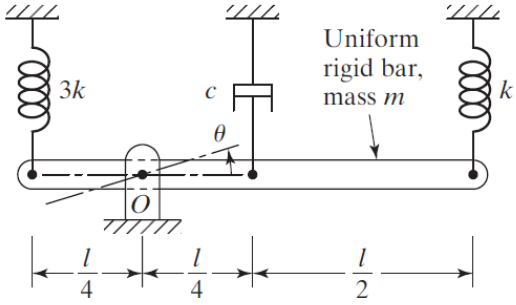
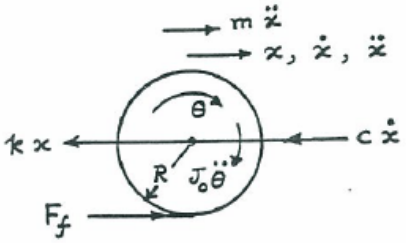


Fig. 6.

Newton's second law of motion:

$$\sum F = m\ddot{x} = -kx - c\dot{x} + F_f \qquad (1)$$

$$\sum M = J_0\ddot{\theta} = -F_f R \qquad (2)$$



where F_f = friction force.

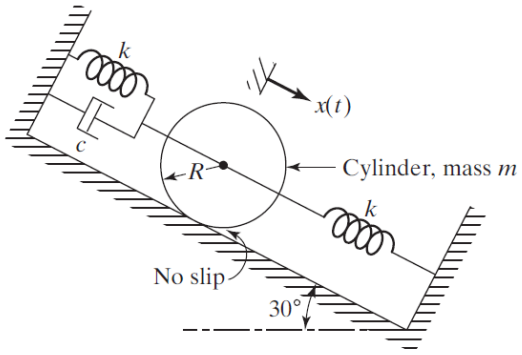
Using $J_0 = \frac{mR^2}{2}$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

$$F_f = -\frac{1}{2R}\left(mR^2\right)\frac{\ddot{x}}{R} = -\frac{1}{2}m\ddot{x} \qquad (3)$$

Substitution of Eq. (3) into (1) yields:

$$\frac{3}{2}\,m\,\ddot{x} + c\,\dot{x} + k\,x = 0 \tag{4}$$

The undamped natural frequency is: $\omega_n = \sqrt{\dfrac{2\,k}{3\,m}}$ (5)



Newton's second law of motion: (measuring x from static equilibrium position of cylinder)

$$\sum F = m\,\ddot{x} = -k\,x - c\,\dot{x} - k\,x + F_f \tag{1}$$

$$\sum M = J_0\,\ddot{\theta} = -F_f\,R \tag{2}$$

where F_f = friction force. Using $J_0 = \frac{1}{2}\,m\,R^2$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

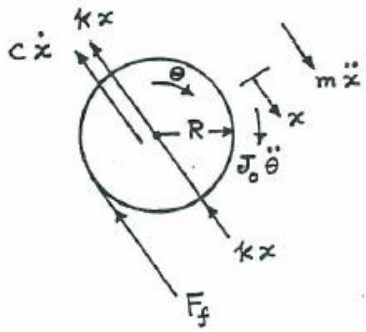
$$F_f = -\frac{1}{2}\,m\,\ddot{x} \tag{3}$$

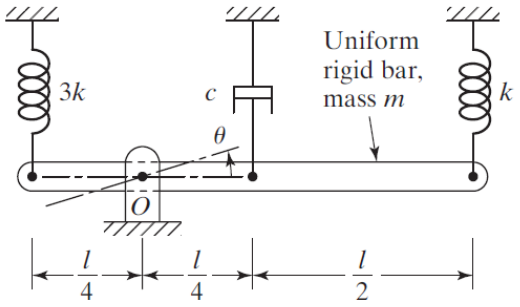
Substitution of Eq. (3) into (1) gives

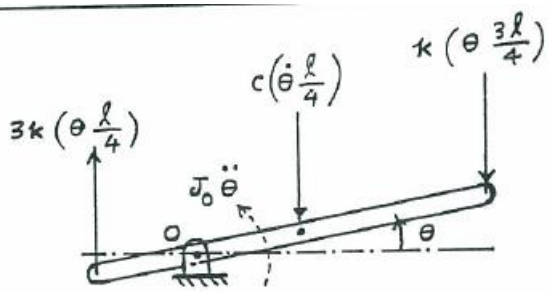
$$\frac{3}{2}\,m\,\ddot{x} + c\,\dot{x} + 2\,k\,x = 0 \tag{4}$$

Undamped natural frequency of the system:

$$\omega_n = \sqrt{\frac{4\,k}{3\,m}} \tag{4}$$







Consider a small angular displacement of the bar θ about its static equilibrium position. Newton's second law gives:

$$\sum M = J_0 \ddot{\theta} = -k\left(\theta \frac{3\ell}{4}\right)\left(\frac{3\ell}{4}\right) - c\left(\dot{\theta}\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right) - 3k\left(\theta\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right)$$

$$\text{i.e.,} \quad J_0 \ddot{\theta} + \frac{c\ell^2}{16}\dot{\theta} + \frac{3}{4}k\ell^2\theta = 0$$

where $J_0 = \frac{7}{48} m\ell^2$. The undamped natural frequency of torsional vibration is given by:

$$\omega_n = \sqrt{\frac{3k\ell^2}{4J_0}} = \sqrt{\frac{36k}{7m}}$$

8. Determine The system shown in Fig. 7 has a natural frequency of 5 Hz for the following data: When the system is disturbed by giving it an initial displacement, the amplitude of free vibration is reduced by 80 percent in 10 cycles. Determine the values of k and c. m = 10 kg, J_0 = 5 kg-m^2, r_1 = 10 cm, r_2= 25 cm.

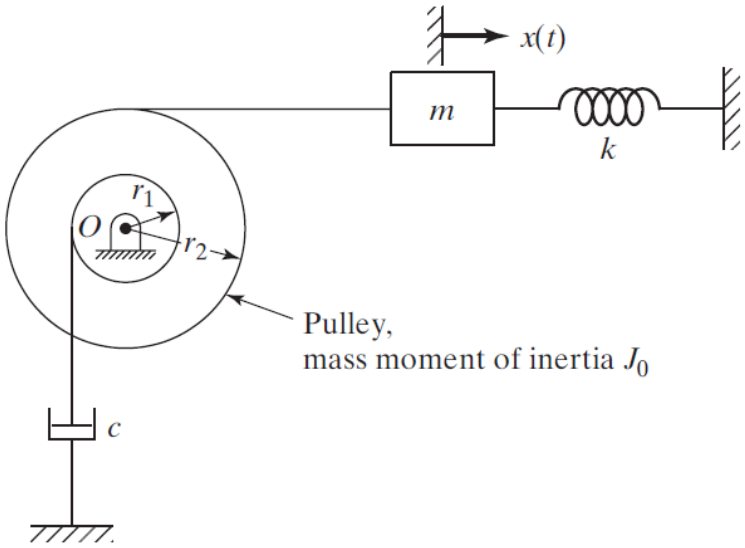
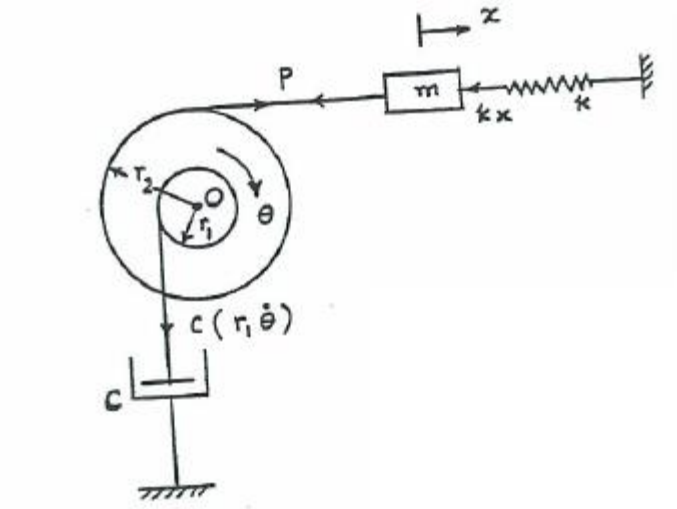


Fig. 7.



Let x = displacement of mass and P = tension in rope on the left of mass.
Equations of motion:

$$\sum F = m\ddot{x} = -k\,x - P \quad \text{or} \quad P = -m\ddot{x} - k\,x \tag{1}$$

$$\sum M = J_0\,\ddot{\theta} = P\,r_2 - c\,(\dot{\theta}\,r_1) \tag{2}$$

Using Eq. (1) in (2), we obtain

$$-(m\ddot{x} + k\,x)\,r_2 - c\,\dot{\theta}\,r_1 = J_0\,\ddot{\theta} \tag{3}$$

With $x = \theta\,r_2$, Eq. (3) can be written as:

$$(J_0 + m\,r_2^2)\,\ddot{\theta} + c\,r_1\,\dot{\theta} + k\,r_2^2\,\theta = 0 \tag{4}$$

For given data, Eq. (4) becomes

$$[5 + 10\,(0.25)^2]\,\ddot{\theta} + c\,(0.1)\,\dot{\theta} + k\,(0.25)^2\,\theta = 0$$

$$\text{or} \quad 5.625\,\ddot{\theta} + 0.1\,c\,\dot{\theta} + 0.0625\,k\,\theta = 0 \tag{5}$$

Since amplitude is reduced by 80% in 10 cycles,

$$\frac{x_1}{x_{11}} = \frac{1.0}{0.2} = 5 = e^{10\,\zeta\,\omega_n\,\tau_d}$$

$$\ln\frac{x_1}{x_{11}} = \ln 5 = 1.6094 = 10\,\zeta\,\omega_n\,\tau_d \tag{6}$$

Since the natural frequency (assumed to be undamped torsional vibration frequency) is 5 Hz, $\omega_n = 2\,\pi\,(5) = 31.416$ rad/sec. Also

$$\tau_d = \frac{1}{f_d} = \frac{2\,\pi}{\omega_d} = \frac{2\,\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{0.2}{\sqrt{1-\zeta^2}} \tag{7}$$

Eq. (6) gives

$$1.6094 = 10\,\zeta\,(31.416)\left(\frac{0.2}{\sqrt{1-\zeta^2}}\right) = \frac{62.832\,\zeta}{\sqrt{1-\zeta^2}}$$

$$\text{i.e.,} \quad \sqrt{1-\zeta^2} = \frac{62.832}{1.6094}\,\zeta = 39.0406\,\zeta$$

$$\text{i.e.,} \quad \zeta = 0.02561$$

Thus we obtain:

$$\omega_n = \sqrt{\frac{0.0625\,k}{5.625}} = 31.416 \quad \text{or} \quad k = 8.8827\,(10^4)\ \text{N/m}$$

$$\zeta = 0.02561 = \frac{c}{c_c} = \frac{c}{2\,m_{eq}\,\omega_n} = \frac{0.10\,c}{2\,(5.625)\,(31.416)}$$

$$\text{or} \quad c = 90.5134\ \text{N–s/m}$$

9. A metal block, placed on a rough surface, is attached to a spring and is given an initial displacement of 10 cm from its equilibrium position. It is found that the natural time period of motion is 1.0 s and that the amplitude reduces by 0.5 cm in each cycle. Find (a) the kinetic coefficient of friction between the metal block and the surface and (b) the number of cycles of motion executed by the block before it stops.

Coulomb damping.

(a) Natural frequency $= \omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{1} = 6.2832$ rad/sec. Reduction in amplitude in each cycle:

$$= \frac{4\mu N}{k} = 4\mu g \frac{m}{k} = \frac{4\mu g}{\omega_n^2} = 4\mu \left(\frac{9.81}{6.2832^2}\right)$$

$$= 0.9940\,\mu = \frac{0.5}{100} = 0.005 \text{ m}$$

Kinetic coefficient of friction $= \mu = 0.00503$

(b) Number of half-cycles executed (r) is:

$$r \geq \frac{(x_0 - \frac{\mu N}{k})}{(\frac{2\mu N}{k})} = \frac{(x_0 - \frac{\mu g}{\omega_n^2})}{(\frac{2\mu g}{\omega_n^2})}$$

$$\geq \frac{\left(0.1 - \frac{0.00503\,(9.81)}{6.2832^2}\right)}{\left(\frac{2\,(0.00503)\,(9.81)}{6.2832^2}\right)}$$

$$\geq 39.5032$$

$$\geq 40$$

Thus the block stops oscillating after 20 cycles.

10. The mass of a spring-mass system vibrates on a dry surface inclined at 30° to the horizontal as shown in Fig. 8.
    a. Derive the equation of motion.
    b. Find the response of the system for the following data:
    m = 20 kg, k = 1,000 N/m, m = 0.1, x_0 = 0.1 m, $\dot{x}_0$ = 5 m/s.

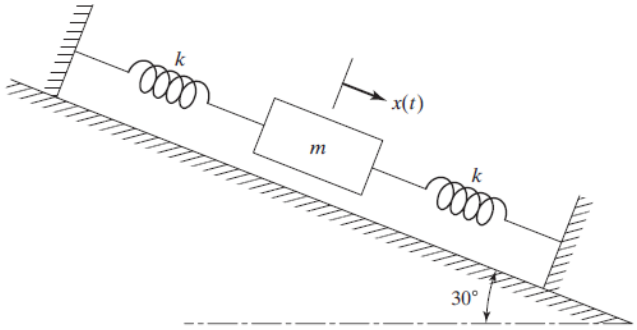
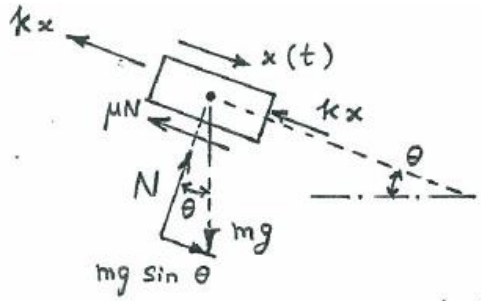


Fig. 8.

(a) $\theta = 30^\circ$

$N = mg \cos\theta$



Case 1: When $x = +$ and $\dot{x} = +$ or $x = -$ and $\dot{x} = +$:

$$m\ddot{x} = -2kx - \mu N + mg\sin\theta$$

or $$m\ddot{x} + 2kx = -\mu mg\cos\theta + mg\sin\theta \qquad (E\cdot1)$$

Case 2: When $x = +$ and $\dot{x} = -$ or $x = -$ and $\dot{x} = -$:

$$m\ddot{x} = -2kx + \mu N + mg\sin\theta$$

or $$m\ddot{x} + 2kx = \mu mg\cos\theta + mg\sin\theta \qquad (E\cdot2)$$

Eqs. (E·1) and (E·2) can be written as a single equation as:

$$m\ddot{x} + \mu mg\cos\theta \; sgn(\dot{x}) + 2kx + mg\sin\theta = 0 \qquad (E\cdot3)$$

(b) $x_0 = 0.1$ m, $\dot{x}_0 = 5$ m/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{20}} = 7.071068 \text{ rad/s}$$

Solution of Eq. (E.1):

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu mg \cos\theta}{k} + \frac{mg \sin\theta}{k} \quad (E.4)$$

Solution of Eq. (E.2):

$$x(t) = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu mg \cos\theta}{k} + \frac{mg \sin\theta}{k} \quad (E.5)$$

Using the initial conditions in each half cycle, the constants A_1 and A_2 or A_3 and A_4 are to be found. For example, in the first half cycle, the motion starts from left toward right with $x_0 = 0.1$ and $\dot{x}_0 = 5$. These values can be used in Eq. (E.4) to find A_1 and A_2.

11. A The mass of a spring-mass system is initially displaced by 10 cm from its unstressed position by applying a force of 25 N, which is equal to five times the weight of the mass. If the mass is released from this position, how long will the mass vibrate and at what distance will it stop from the unstressed position? Assume a coefficient of friction of 0.2.

Friction force $= \mu N = 0.2\ (5) = 1$ N. $k = \frac{25}{0.10} = 250$ N/m. Reduction in amplitude in each cycle $= \frac{4\ \mu N}{k} = \frac{4\ (1)}{250} = 0.016$ m. Number of half-cycles executed before the motion ceases (r):

$$r \geq \left(\frac{x_0 - \frac{\mu N}{k}}{\frac{2\ \mu N}{k}} \right) = \frac{0.1 - 0.004}{0.008} \geq 12$$

Thus after 6 cycles, the mass stops at a distance of $0.1 - 6\ (0.016) = 0.004$ m from the unstressed position of the spring.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{250\ (9.81)}{5}} = 22.1472 \text{ rad/sec}$$

$$\tau_n = \frac{2\ \pi}{\omega_n} = 0.2837 \text{ sec}$$

Thus total time of vibration $= 6\ \tau_n = 1.7022$ sec.