Introduction to Machine Design

Lecture 4

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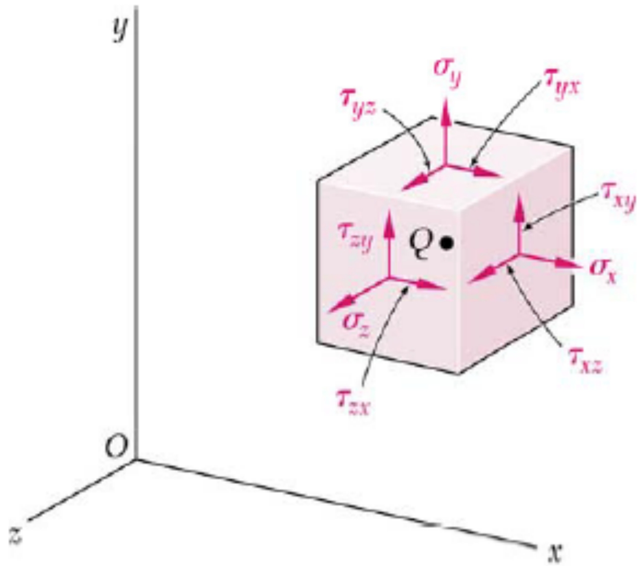
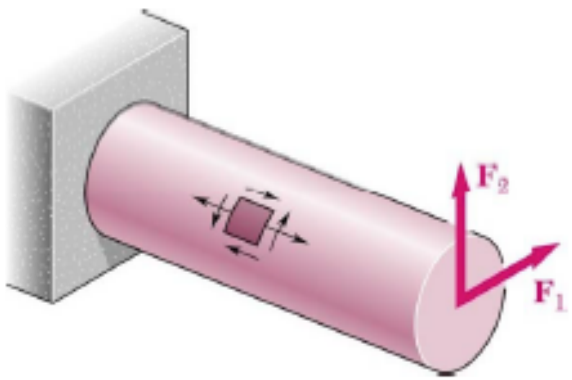
Transformation of Stress

- Recall the general state of stress at a point can be written in terms of 6 components: σ_x, σ_y, σ_z, $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$
- This general "stress state" is independent of the coordinate system used.
- The **components** of the stress state in the different directions **do** depend on the coordinate system.

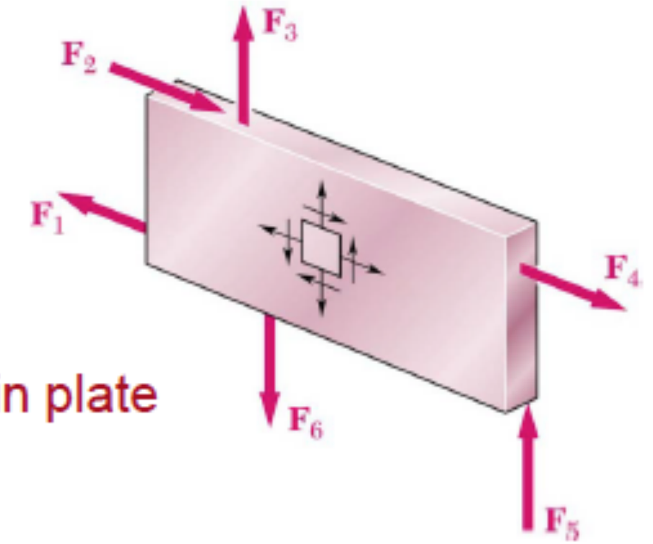


Transformation of Stress cont'd

- Consider a state of plane stress: $\sigma_z=\tau_{xz}=\tau_{yz}=0$
- Where does this occur?



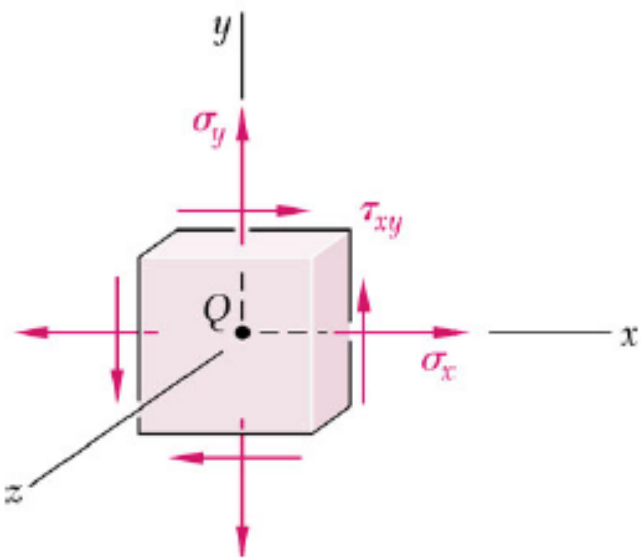
Outer surface



Thin plate

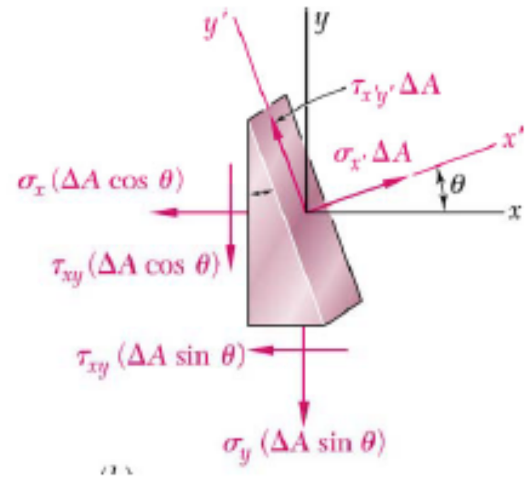
Transformation of Stress cont'd

- What do we want to calculate?
  - Principle stresses (σ maximum and σ minimum)
  - Principle planes of stresses (orientation at which they occur)
- Slice cube at an angle θ to the x axis (new coordinates x', y').
- Define forces in terms of angle and stresses.



Transformation of Stress cont'd

- Sum forces in x' direction.



$$\sigma_x'\Delta A=\sigma_x(\Delta A\cos\theta)\cos\theta+\sigma_y(\Delta A\sin\theta)\sin\theta$$
$$+\tau_{xy}(\Delta A\sin\theta)\cos\theta+\tau_{xy}(\Delta A\cos\theta)\sin\theta$$

$$\sigma_x'=\sigma_x\cos^2\theta+\sigma_y\sin^2\theta+2\tau_{xy}\sin\theta\cos\theta$$

- Sum forces in y' direction.

$$\tau_{xy}'\Delta A=-\sigma_x(\Delta A\cos\theta)\sin\theta+\sigma_y(\Delta A\sin\theta)\cos\theta-\tau_{xy}(\Delta A\sin\theta)\sin\theta+\tau_{xy}(\Delta A\cos\theta)\cos\theta$$

$$\tau_{xy}'=(\sigma_y-\sigma_x)\cos\theta\sin\theta+\tau_{xy}\left(\cos^2\theta-\sin^2\theta\right)$$

Transformation of Stress cont'd

$$\sigma_x' = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\boxed{\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)}$$

To get σ_y', evaluate σ_x' at $\theta + 90^o$.

$$\boxed{\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta)}$$

$$\tau_{xy}' = (\sigma_y - \sigma_x)\cos\theta\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\boxed{\tau_{xy}' = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)}$$

<u>Trig identities</u>

$$2\sin\theta\cos\theta = \sin(2\theta) \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\cos^2\theta - \sin^2\theta = \cos(2\theta) \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

Transformation of Stress cont'd

$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) - \tau_{xy}\sin(2\theta)$$

$$\tau_{xy}' = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

Now, let's perform some algebra:

$$\left(\sigma_x' - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2{}' =$$

Constants (we can find these stresses).

$$\left(\sigma_x' - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2{}' = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

Variables

Principle and Max Shearing Stress

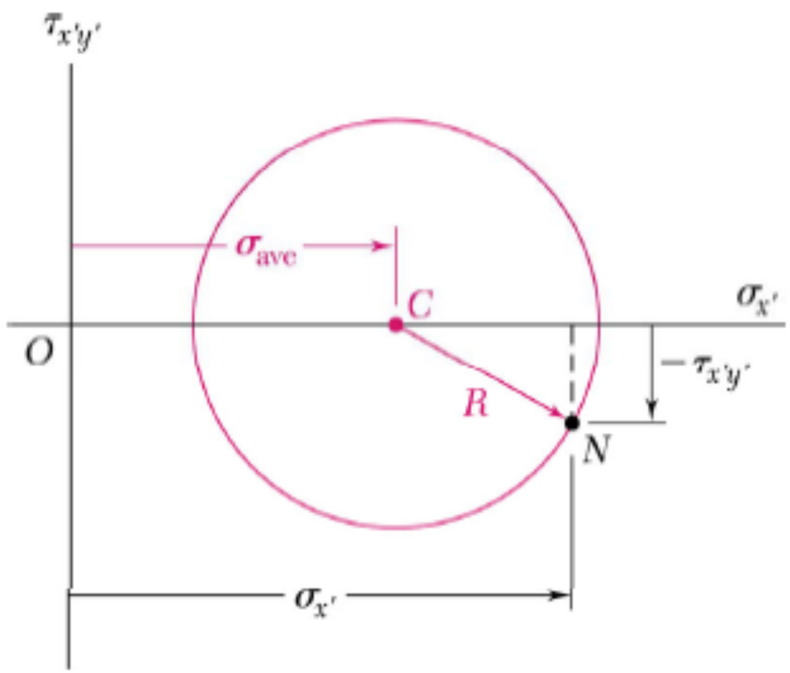
- Define

$$\sigma_{ave} = \frac{1}{2}(\sigma_x + \sigma_y) \qquad R^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

- Plug into previous equation

$$\left(\sigma_x' - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2{}' = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

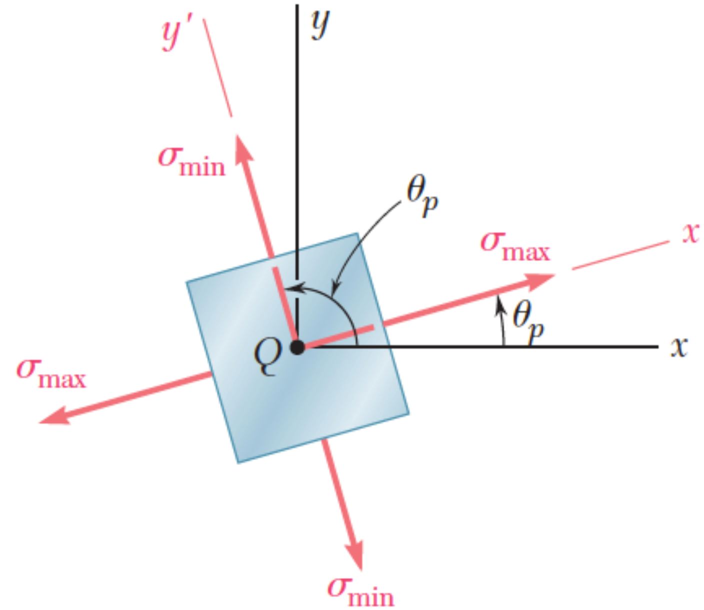
$$\boxed{(\sigma_{x1} - \sigma_{ave})^2 + \tau_{xy1}^2 = R^2}$$



- Which is the equation of a circle with center at $(\sigma_{ave}, 0)$ and radius R.

Summary

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



Principal Stresses

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Example 1

For the state of plane stress shown in Fig., determine (a) the principal planes, (b) the principal stresses, (c) the maximum shearing stress.

a. Principal Planes. Following the usual sign convention, the stress components are

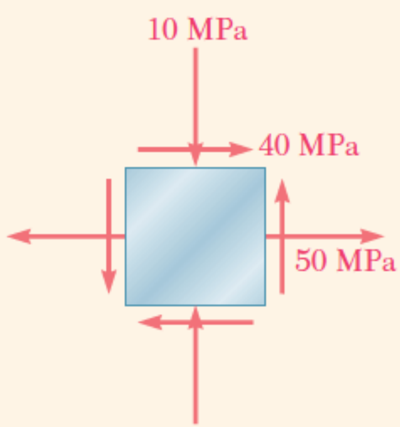
$$\sigma_x = +50 \text{ MPa} \qquad \sigma_y = -10 \text{ MPa} \qquad \tau_{xy} = +40 \text{ MPa}$$

Substituting into Eq. (7.12),

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+40)}{50 - (-10)} = \frac{80}{60}$$

$$2\theta_p = 53.1° \qquad \text{and} \qquad 180° + 53.1° = 233.1°$$

$$\theta_p = 26.6° \qquad \text{and} \qquad 116.6°$$



(a)

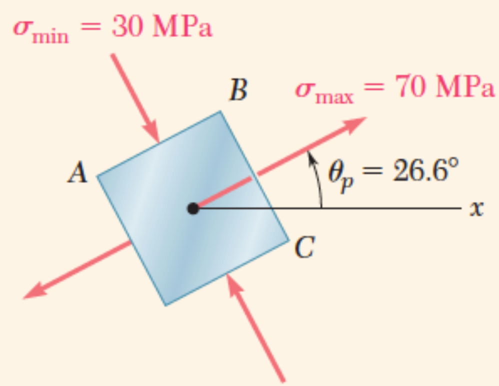
b. Principal Stresses. Equation (7.14) yields

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 20 \pm \sqrt{(30)^2 + (40)^2}$$

$$\sigma_{\max} = 20 + 50 = 70 \text{ MPa}$$

$$\sigma_{\min} = 20 - 50 = -30 \text{ MPa}$$



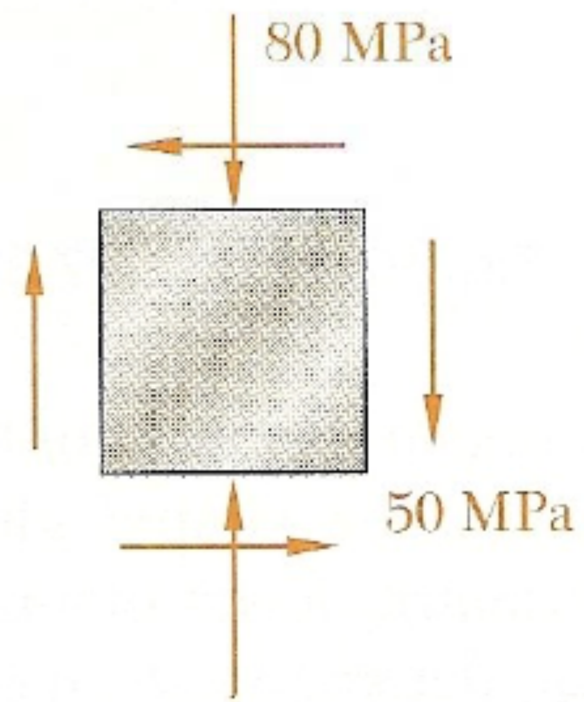
(b)

c. Maximum Shearing Stress. Equation (7.16) yields

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(30)^2 + (40)^2} = 50\,\text{MPa}$$

Example Problem 1

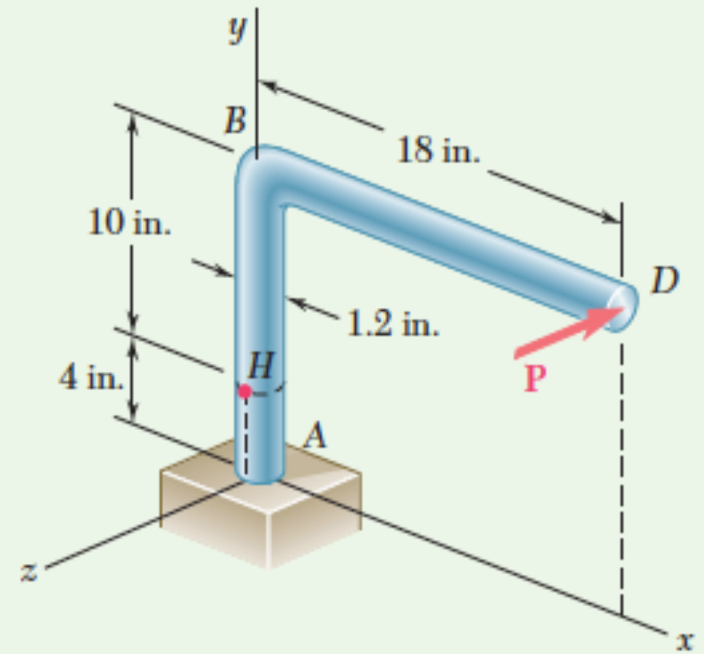
For the given state of stress, determine the normal and shearing stresses after the element shown has been rotated (a) 25° clockwise and (b) 10° counterclockwise.



Example Problem 2

Sample Problem 7.1 in Beer's Book

A single horizontal force **P** with a magnitude of 150 lb is applied to end *D* of lever *ABD*. Knowing that portion *AB* of the lever has a diameter of 1.2 in., determine (*a*) the normal and shearing stresses located at point *H* and having sides parallel to the *x* and *y* axes, (*b*) the principal planes and principal stresses at point *H*.



STRATEGY: You can begin by determining the forces and couples acting on the section containing the point of interest, and then use them to calculate the normal and shearing stresses acting at that point. These stresses can then be transformed to obtain the principal stresses and their orientation.

MODELING and ANALYSIS:

Force-Couple System. We replace the force **P** by an equivalent force-couple system at the center C of the transverse section containing point H (Fig.1):

$$P = 150 \text{ lb} \qquad T = (150 \text{ lb})(18 \text{ in.}) = 2.7 \text{ kip·in.}$$

$$M_x = (150 \text{ lb})(10 \text{ in.}) = 1.5 \text{ kip·in.}$$

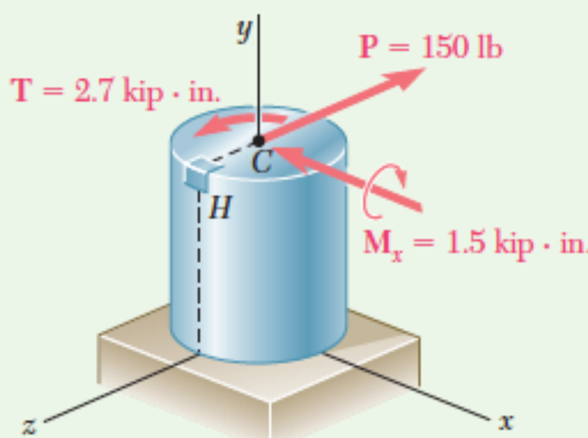


Fig. 1 Equivalent force-couple system acting on transverse section containing point *H*.

a. Stresses σ_x, σ_y, τ_{xy} at Point H. Using the sign convention shown in Fig. 7.2, the sense and the sign of each stress component are found by carefully examining the force-couple system at point *C* (Fig. 1):

$$\sigma_x = 0 \quad \sigma_y = +\frac{Mc}{I} = +\frac{(1.5 \text{ kip·in.})(0.6 \text{ in.})}{\frac{1}{4}\pi\,(0.6 \text{ in.})^4} \qquad \sigma_y = +8.84 \text{ ksi} \quad \blacktriangleleft$$

$$\tau_{xy} = +\frac{Tc}{J} = +\frac{(2.7 \text{ kip·in.})(0.6 \text{ in.})}{\frac{1}{2}\pi\,(0.6 \text{ in.})^4} \qquad \tau_{xy} = +7.96 \text{ ksi} \quad \blacktriangleleft$$

We note that the shearing force **P** does not cause any shearing stress at point *H*. The general plane stress element (Fig. 2) is completed to reflect these stress results (Fig. 3).

b. Principal Planes and Principal Stresses. Substituting the values of the stress components into Eq. (7.12), the orientation of the principal planes is

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(7.96)}{0 - 8.84} = -1.80$$

$$2\theta_p = -61.0^\circ \quad \text{and} \quad 180^\circ - 61.0^\circ = +119^\circ$$

$$\theta_p = -30.5^\circ \quad \text{and} \quad +59.5^\circ$$

Substituting into Eq. (7.14), the magnitudes of the principal stresses are

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{0 + 8.84}{2} \pm \sqrt{\left(\frac{0 - 8.84}{2}\right)^2 + (7.96)^2} = +4.42 \pm 9.10$$

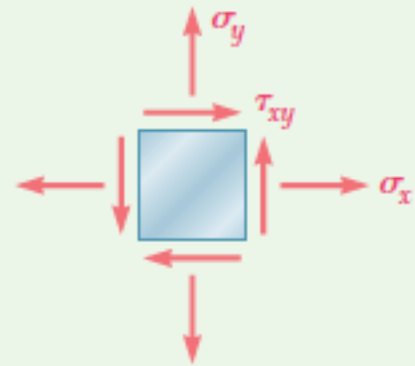


Fig. 2 General plane stress element (showing positive directions).

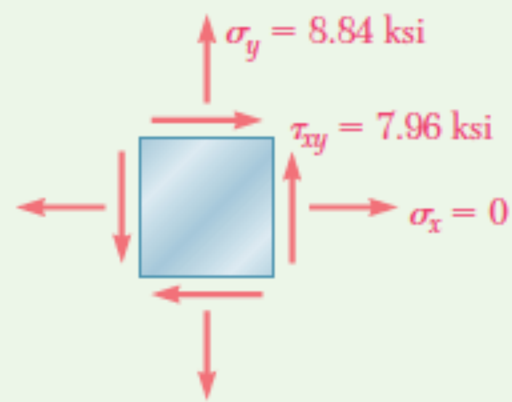


Fig. 3 Stress element at point *H*.

$$\sigma_{max} = +13.52 \text{ ksi}$$

$$\sigma_{min} = -4.68 \text{ ksi}$$

Considering face ab of the element shown, $\theta_p = -30.5°$ in Eq. (7.5) and $\sigma_{x'} = -4.68$ ksi. The principal stresses are as shown in Fig. 4.

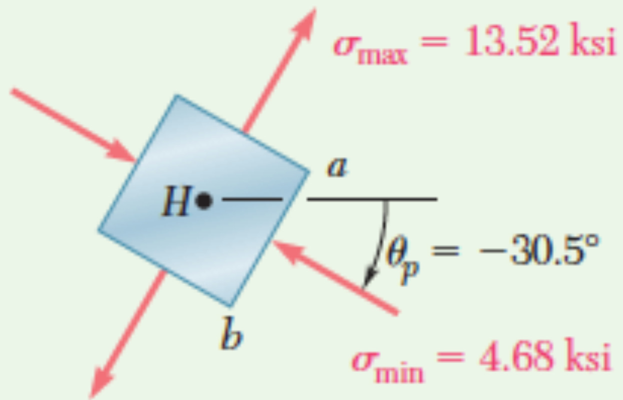


Fig. 4 Stress element at point *H* oriented in principal directions.

THEORIES OF FAILURE

Yield Criteria for Ductile Materials

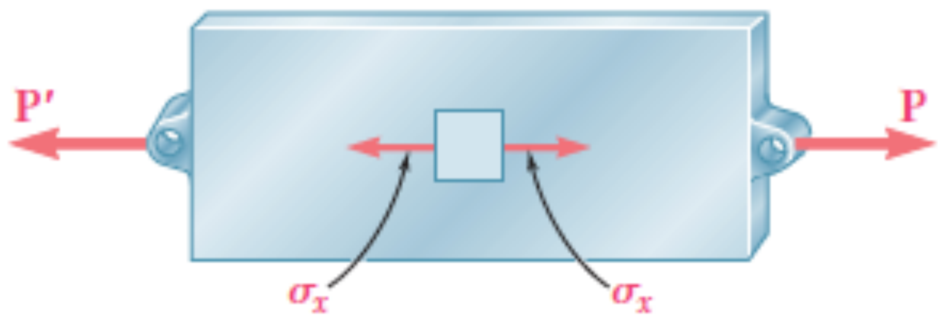


Fig. 7.29 Structural element under uniaxial stress.

$$\sigma_x < \sigma_Y.$$

THEORIES OF FAILURE

Yield Criteria for Ductile Materials

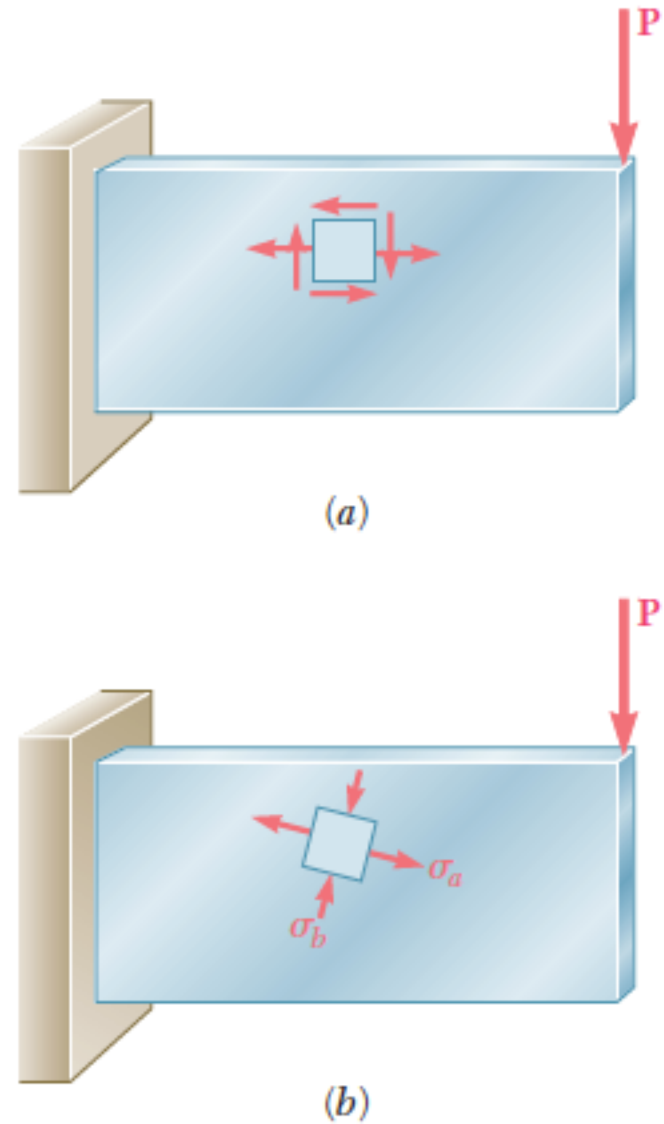


Fig. 7.30 Structural element in a state of plane stress. (*a*) Stress element referred to coordinate axes. (*b*) Stress element referred to principal axes.

principal stresses σ_a and σ_b $< \sigma_Y$.

2.1.1 Maximum Shear Stress Theory for Ductile Materials

The maximum shear stress theory predicts the yielding begins whenever the maximum shear stress in any element equals or exceeds the maximum shear stress for the simple tension test specimen of the same material when that specimen begins to yield.

The maximum shear stress theory predicts the yielding when:

$$\tau_{max} \le \frac{S_{s_y}}{n} \tag{2.1}$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}, \quad S_{s_y} = 0.577\, S_y$$

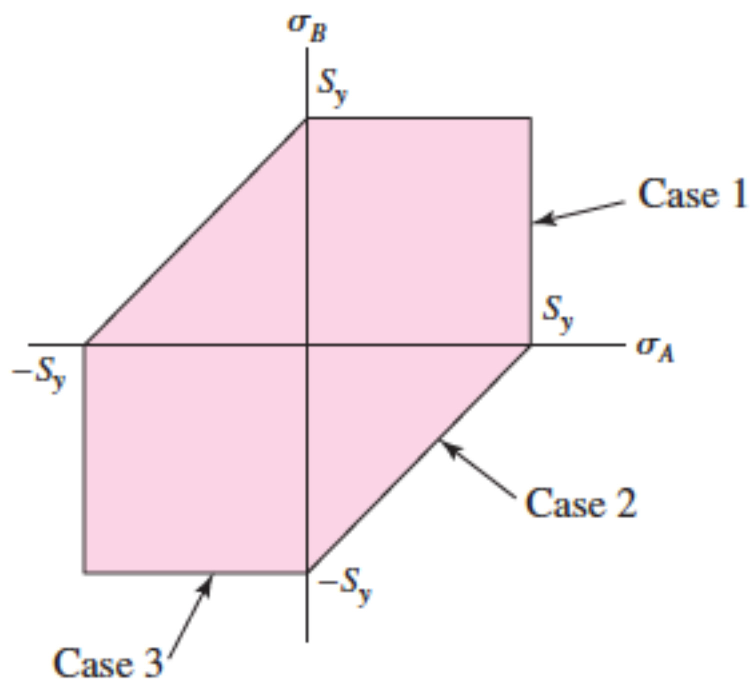
For one dimensional problem the theory will becomes:

$$\tau_{\max} = \sqrt{\left(\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2\right)} \le \frac{S_{s_y}}{n} \tag{2.2}$$

Maximum Shear Stress Theory for Ductile Materials

Figure 5–7

The maximum-shear-stress (MSS) theory for plane stress, where σ_A and σ_B are the two nonzero principal stresses.



Distortion Energy Theory for Ductile Materials

The distortion energy theory predicts the yielding occurs when the distortion strain per unit volume reaches or exceeds the distortion energy strain per unit volume for yield in a simple tension or compression of the same material.

The distortion energy theory predicts the yielding when:

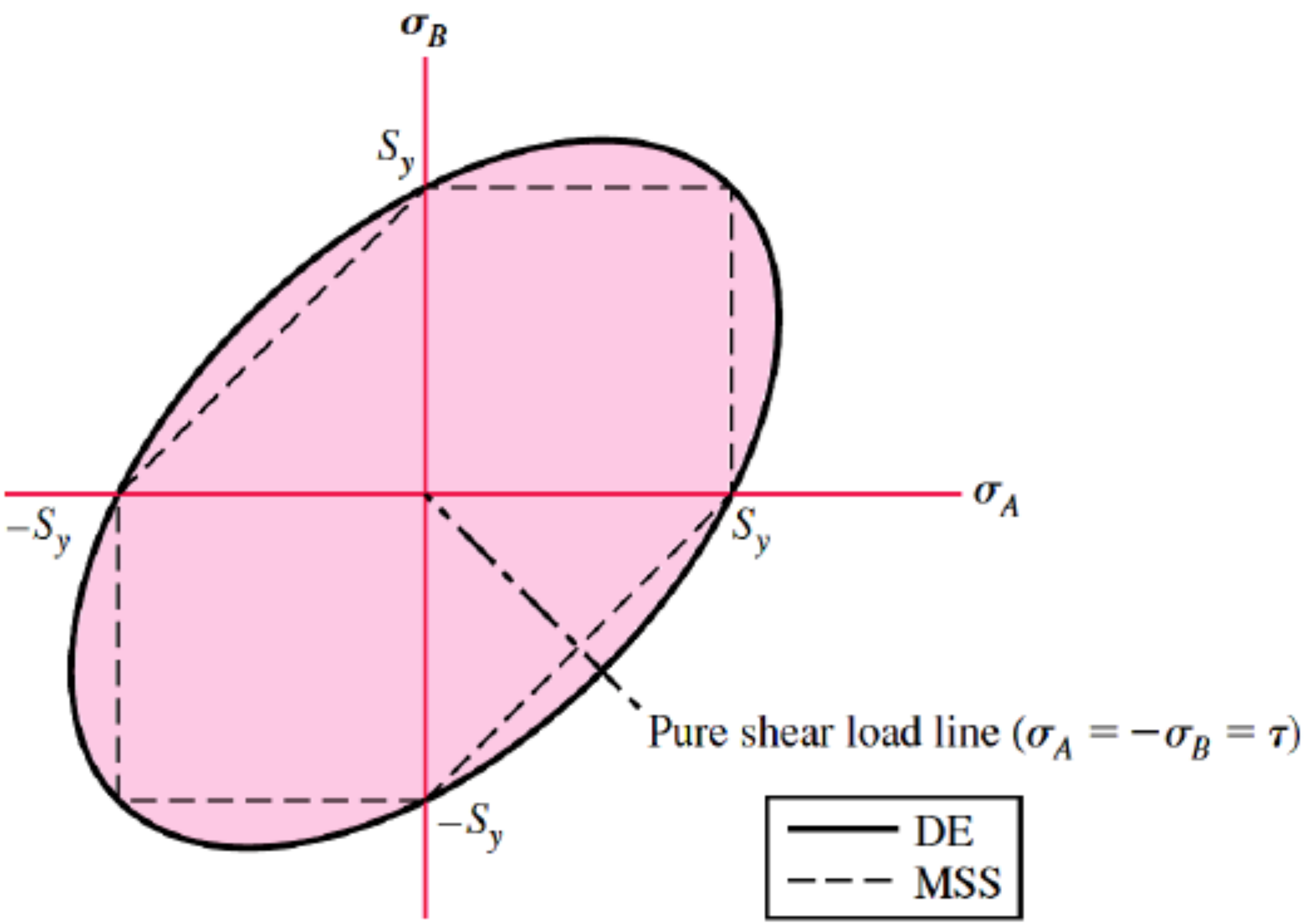
$$\grave{\sigma} \leq \frac{S_y}{n} \tag{2.3}$$

$$\grave{\sigma} = \sqrt{(\sigma_1)^2 - (\sigma_1\sigma_2) + (\sigma_2)^2}$$

For one dimensional problem the theory will becomes:

$$\grave{\sigma} = \sqrt{\left(\sigma_x^2 + 3\tau_{xy}^2\right)} \leq \frac{S_y}{n} \tag{2.4}$$

Distortion Energy Theory for Ductile Materials



Maximum Normal Stress for Ductile Materials

$$\sigma_{max} \leq \frac{S_y}{n} \tag{3.5}$$

σ_{max} = The maximum stress between σ_1 and σ_2 without take the stress sign in your account.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right)}$$

For one dimensional problem the theory will becomes:

$$\sigma_{max} = \left(\frac{\sigma_x}{2}\right) + \sqrt{\left(\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2\right)} \leq \frac{S_y}{n} \tag{3.6}$$

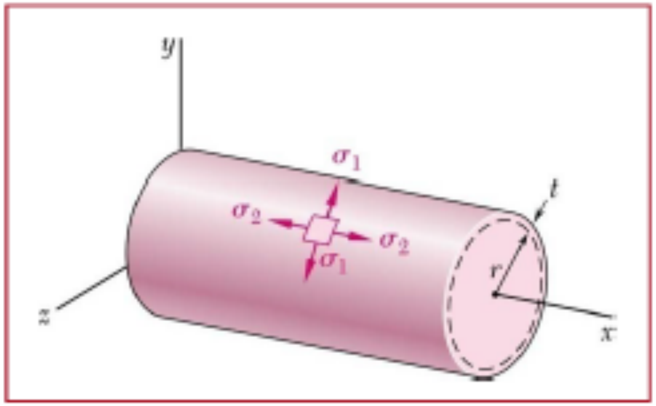
Thin-Walled Pressure Vessels

Cylindrical vessel with capped ends

Spherical vessel

- Assumptions
  - Constant gage pressure, p = internal pressure – external pressure
  - Thickness much less than radius ($t << r$, $t / r < 0.1$)
  - Internal radius = r
  - Point of calculation far away from ends (St. Venant's principle)

Cylindrical Pressure Vessel

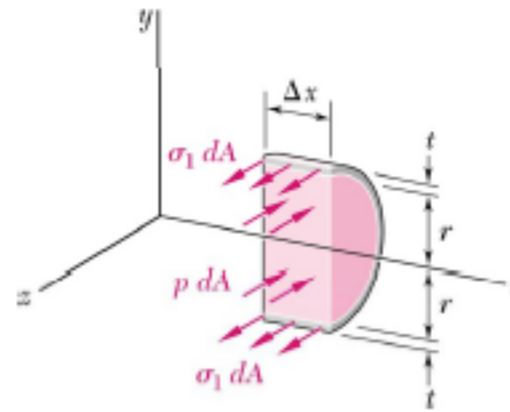


Circumferential (Hoop) Stress: σ_1

Sum forces in the vertical direction.

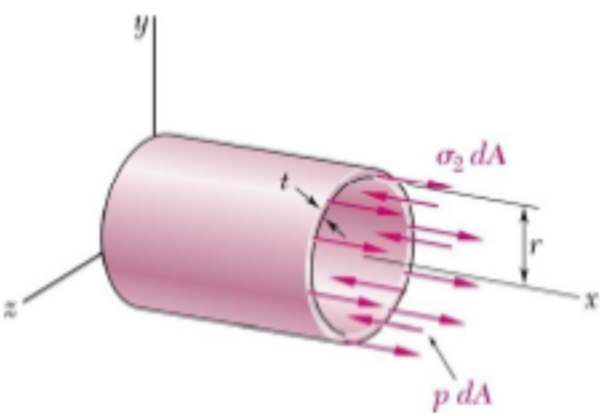
$$2\sigma_1(t\Delta x) - p(2r\Delta x) = 0$$

$$\boxed{\sigma_1 = \frac{pr}{t}}$$



Longitudinal stress: σ_2

Sum forces in the horizontal direction:



$$\sigma_2(2\pi r t) - p(\pi r^2) = 0$$

$$\boxed{\sigma_2 = \frac{pr}{2t}}$$

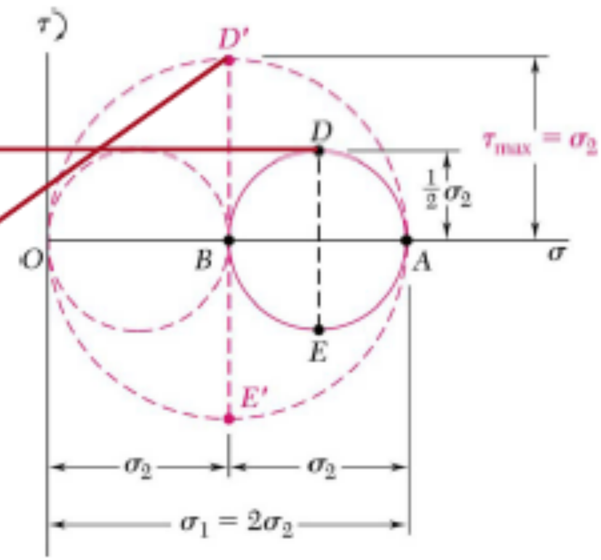
Cylindrical Pressure Vessel cont'd

- There is also a radial component of stress because the gage pressure must be balanced by a stress perpendicular to the surface.
  - $\sigma_r = p$
  - However $\sigma_r \ll \sigma_1$ and σ_2 , so we assume that $\sigma_r = 0$ and consider this a case of plane stress.
- Mohr's circle for a cylindrical pressure vessel:
  - Maximum shear stress (in-plane)

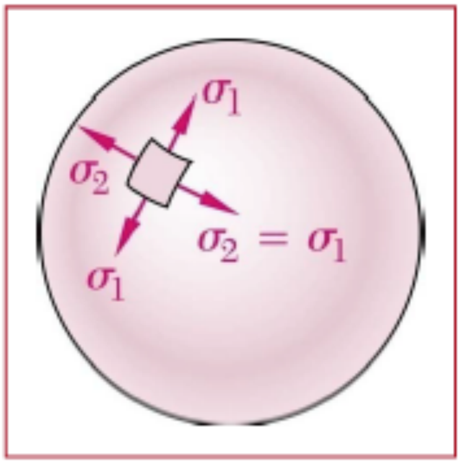
$$\boxed{\tau_{\max} = \frac{\sigma_2}{2} = \frac{pr}{4t}}$$

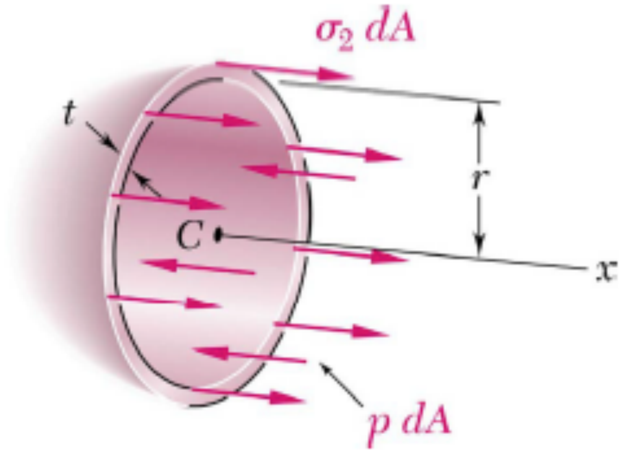
  - Maximum shear stress (out-of-plane)

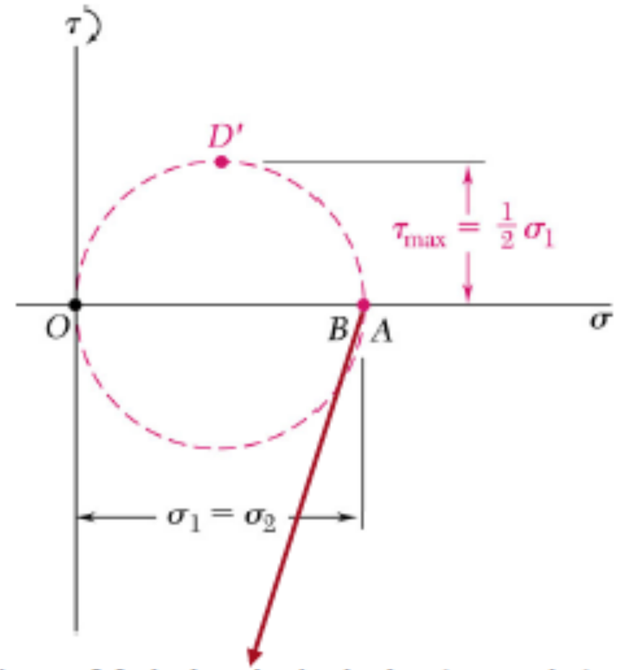
$$\boxed{\tau_{\max} = \sigma_2 = \frac{pr}{2t}}$$



Spherical Pressure Vessel







Sum forces in the horizontal direction:

$$\sigma_2(t2\pi r) - p(\pi r^2) = 0$$

$$\boxed{\sigma_1 = \sigma_2 = \frac{pr}{2t}}$$

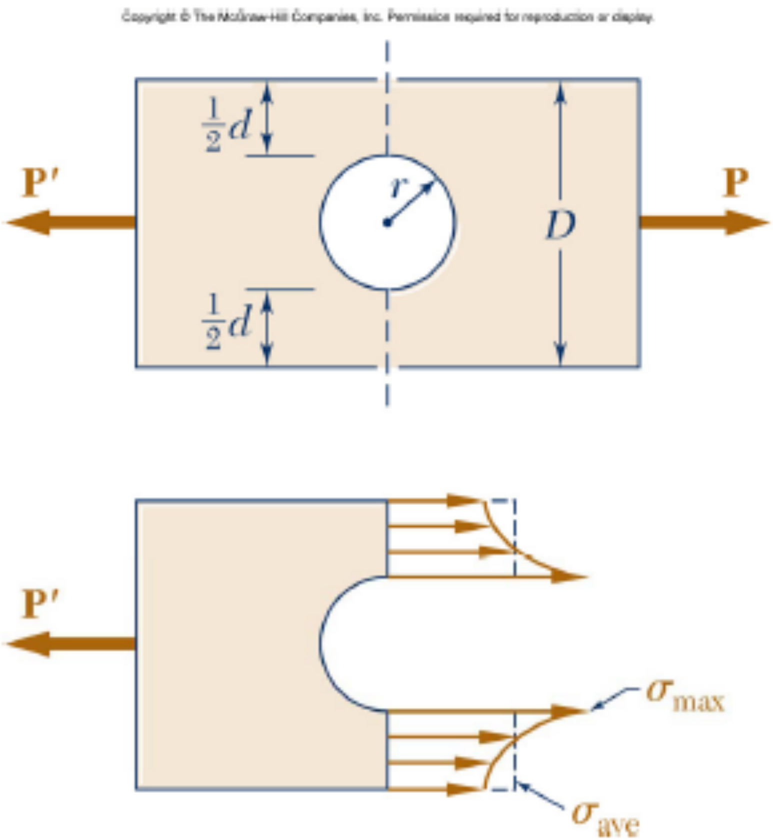
In-plane Mohr's circle is just a point

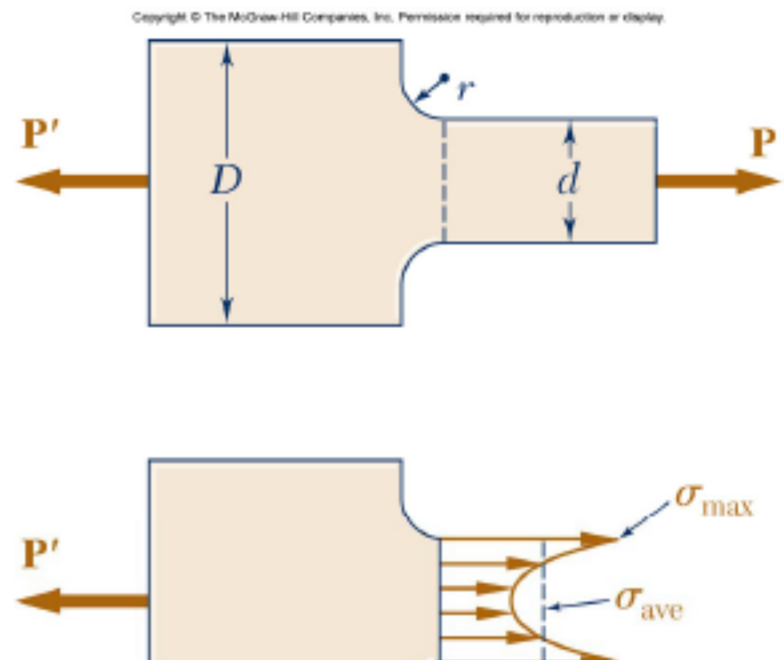
$$\boxed{\tau_{max} = \frac{\sigma_1}{2} = \frac{pr}{4t}}$$

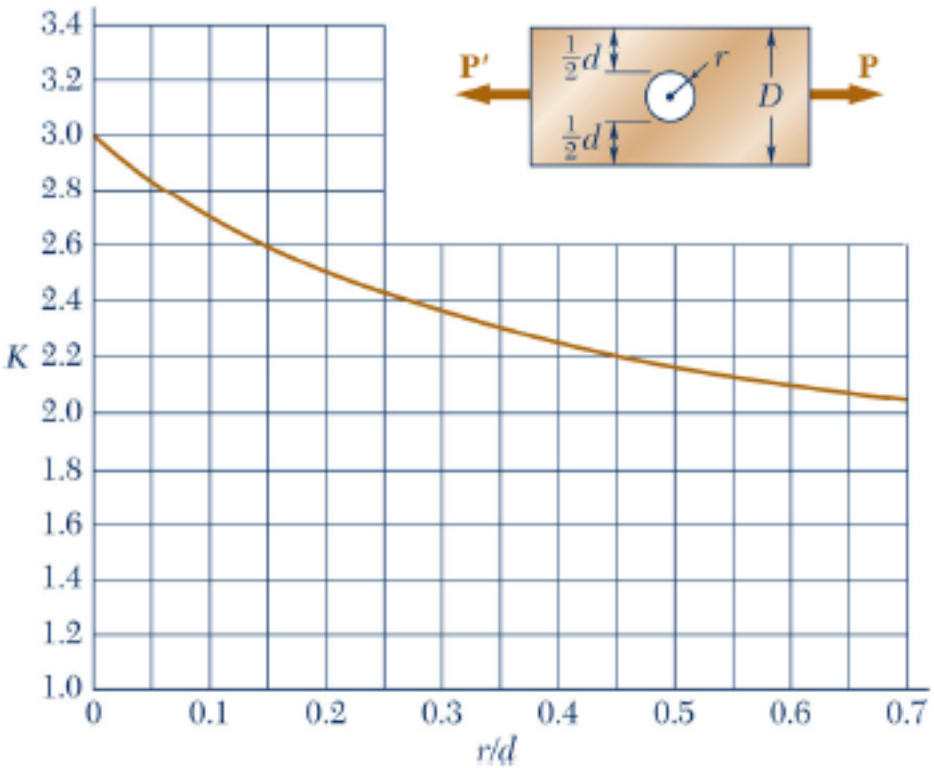
Stress Concentration

- The stresses near the points of application of concentrated loads can reach values much larger than the average value of the stress in the member.

- Stress concentration factor, $K = \sigma_{max}/\sigma_{ave}$






P′
P
$\frac{1}{2}d$
$\frac{1}{2}d$
r
D
K
r/d
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0
0.1
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0.3
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0.5
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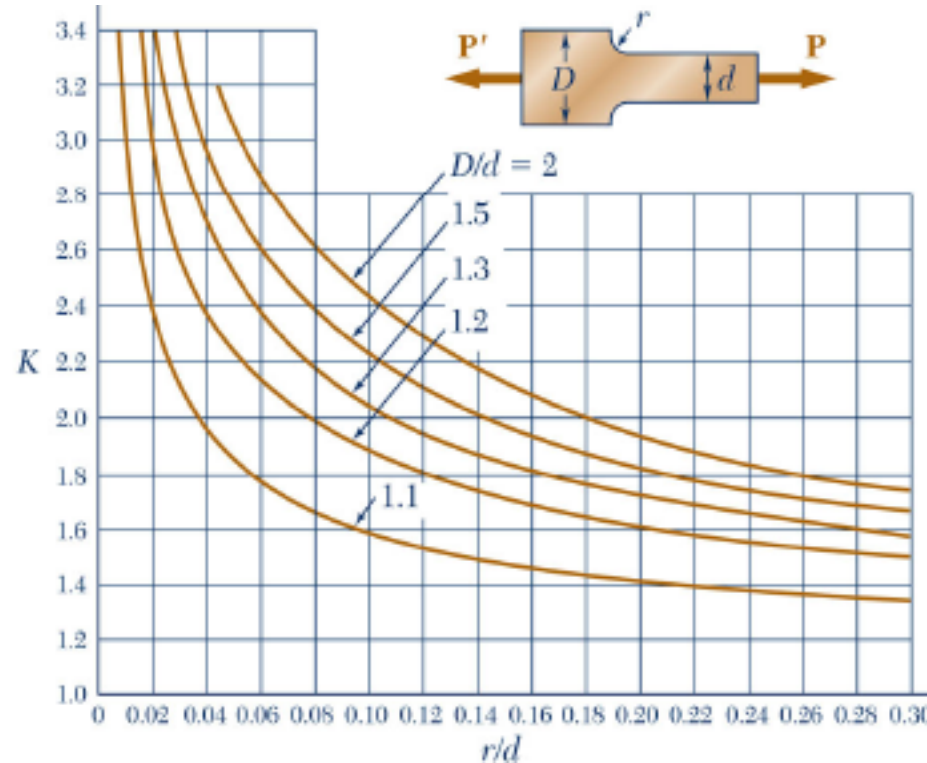
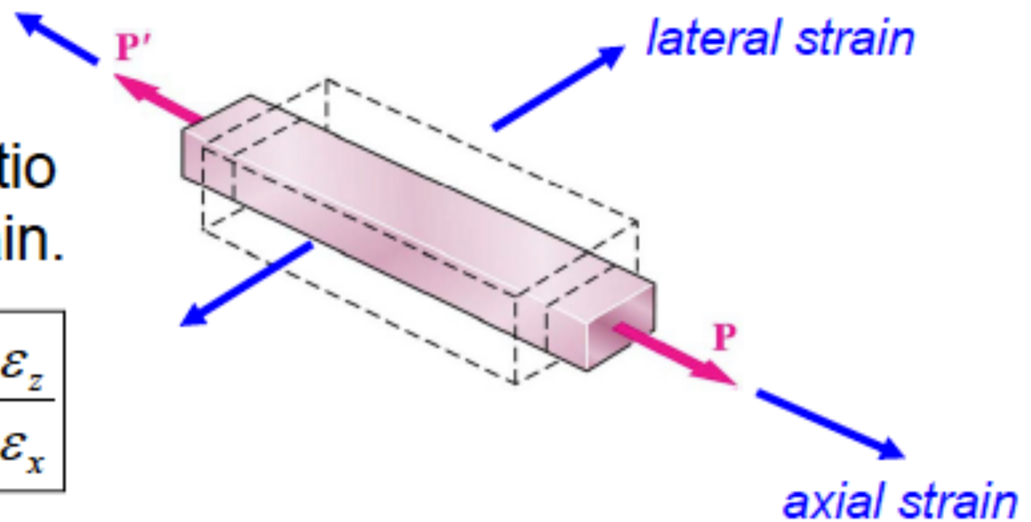
0.7


P′
P
D
d
r
D/d = 2
1.5
1.3
1.2
1.1
K
r/d
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
0.20
0.22
0.24
0.26
0.28
0.30

Poisson's Ratio

- When an axial force is applied to a bar, the bar not only elongates but also shortens in the other two orthogonal directions.

- Poisson's ratio (ν) is the ratio of lateral strain to axial strain.



$$\upsilon = -\frac{\textit{lateral strain}}{\textit{axial strain}} = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

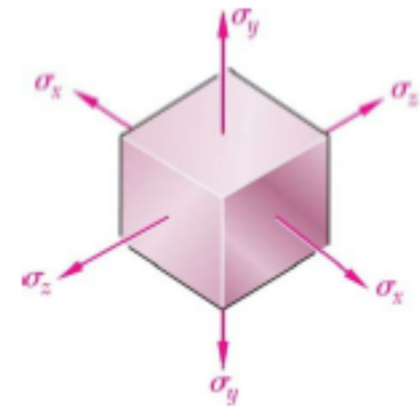
Minus sign needed to obtain a positive value – all engineering materials have opposite signs for axial and lateral strains

- ν is a material specific property and is dimensionless.

Generalized Hooke's Law

- Let's generalize Hooke's Law ($\sigma = E\varepsilon$).
  - Assumptions: linear elastic material, small deformations

$$\boxed{\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \qquad \varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \qquad \varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}}$$



- So, for the case of a homogenous isotropic bar that is axially loaded along the x-axis (σ_y=0 and σ_z=0), we get

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \varepsilon_y = \varepsilon_z = -\frac{\nu\sigma_x}{E}$$

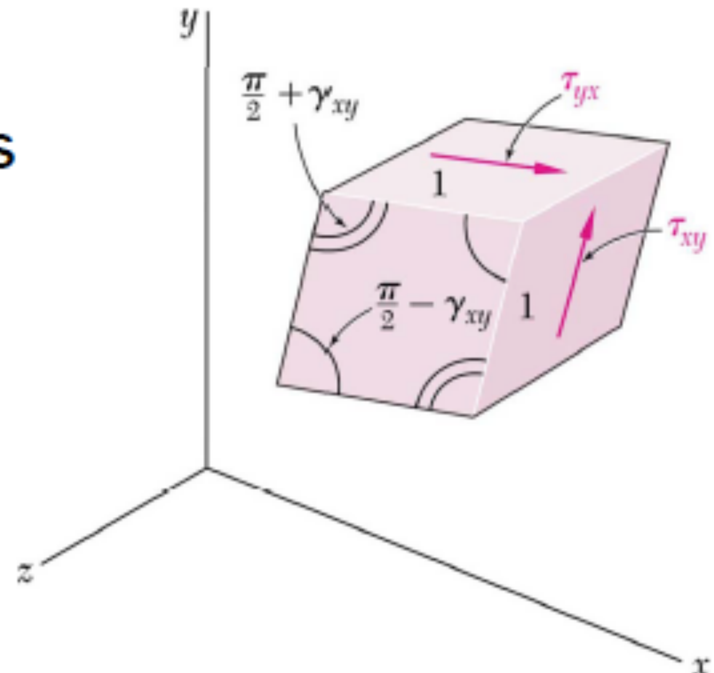
Even though the stress in the y and z axes are zero, the strain is not!

Shear Strain

- Recall that
  - Normal stresses produce a change in volume of the element
  - Shear stresses produce a change in shape of the element

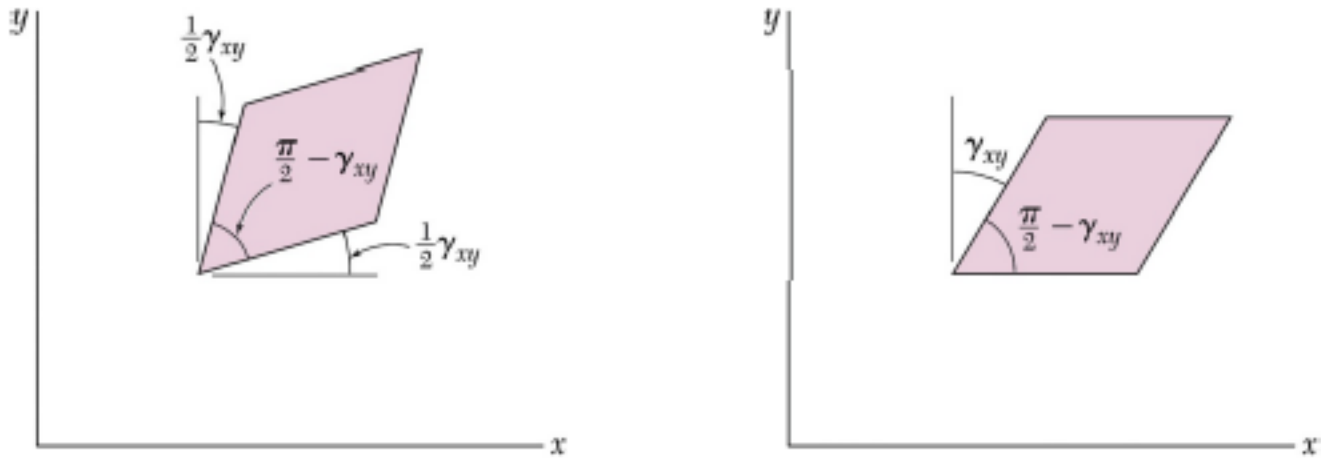
- Shear strain (γ) is an angle measured in degrees or radians (dimensionless)

- Sign convention is the same as for shear stress (τ)



Shear Strain *cont'd*

- There are two equivalent ways to visualize shear strain.

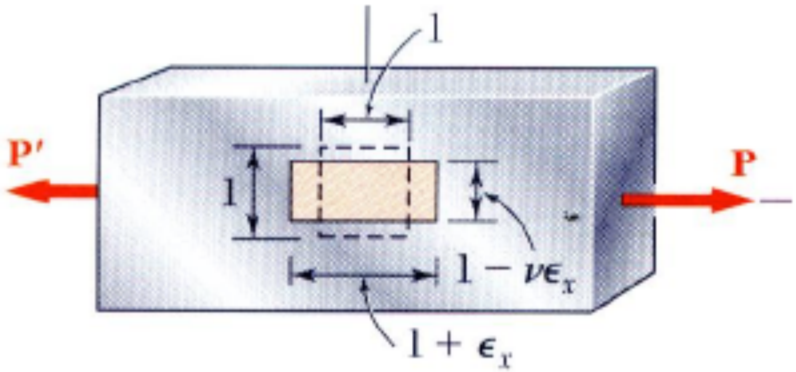


- Hooke's Law for shear stress is defined as

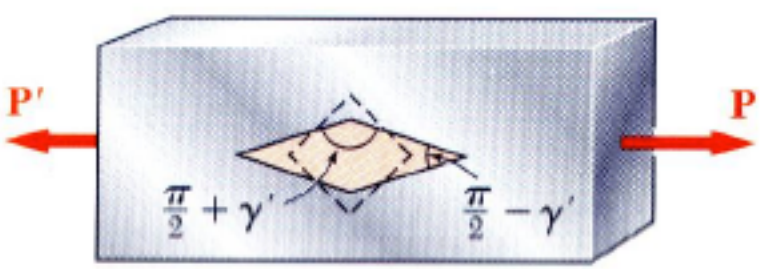
$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{xz} = G\gamma_{xz} \qquad \tau_{yz} = G\gamma_{yz}$$

  - G = shear modulus (or modulus of rigidity)
  - G is a material specific property with the same units as E (psi or Pa).

Relation Among E, ν, and G



(a)

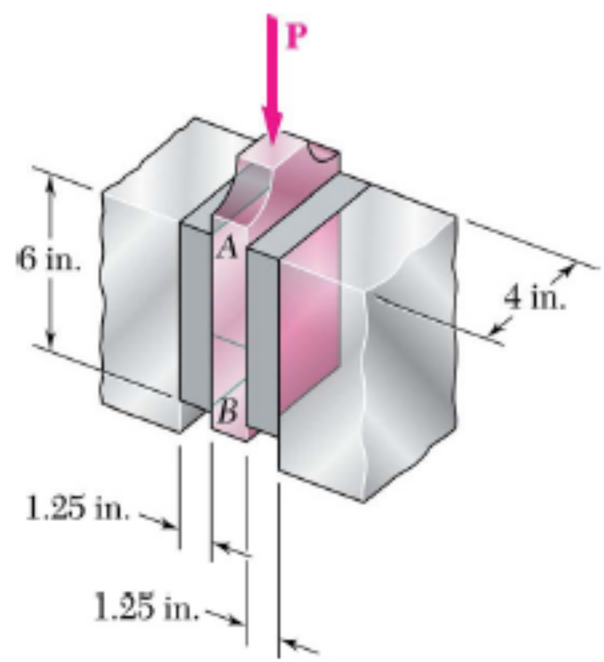


- An axially loaded slender bar will elongate in the axial direction and contract in the transverse directions.
- An initially cubic element oriented as in top figure will deform into a rectangular parallelepiped. The axial load produces a normal strain.
- If the cubic element is oriented as in the bottom figure, it will deform into a rhombus. Axial load also results in a shear strain.
- Components of normal and shear strain are related,

$$\frac{E}{2G} = (1 + \nu)$$

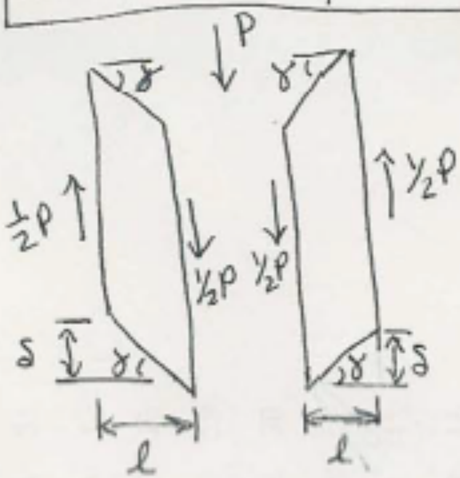
Example Problem 1

- A vibration isolation unit consists of two blocks of hard rubber bonded to a plate AB and to rigid supports as shown. Knowing that a force of magnitude P = 6 kips causes a deflection of δ=0.0625 in. of plate AB, determine the modulus of rigidity of the rubber used.



Example Problem 1 Solution

Lect. 6: Example Problem 1



$P = 6 \text{ Kips} \Rightarrow V = \frac{1}{2}(6 \text{ Kips}) = 3 \text{ kips}$

$\delta = 0.0625 \text{ in} \quad l = 1.25 \text{ in}$

$A = (6 \text{ in})(4 \text{ in}) = 24 \text{ in}^2$ (cross section area over which P acts)

$$\tau = G\gamma \Rightarrow G = \frac{\tau}{\gamma}$$

$$\tau = \frac{V}{A} = \frac{3 \text{ Kips}}{24 \text{ in}^2} = \frac{3000 \text{ lb}}{24 \text{ in}^2} = 125 \text{ psi}$$

$$\gamma = \frac{\delta}{l} = \frac{0.0625 \text{ in}}{1.25 \text{ in}} = 0.05$$

$$G = \frac{125 \text{ psi}}{0.05} = 2500 \text{ psi}$$

$$\boxed{G = 2.5 \text{ ksi}}$$