Finite Element Analysis
Lecture 4

Dr./ Ahmed Nagib

March 26, 2016

Axial members, beams and frames

Axial members

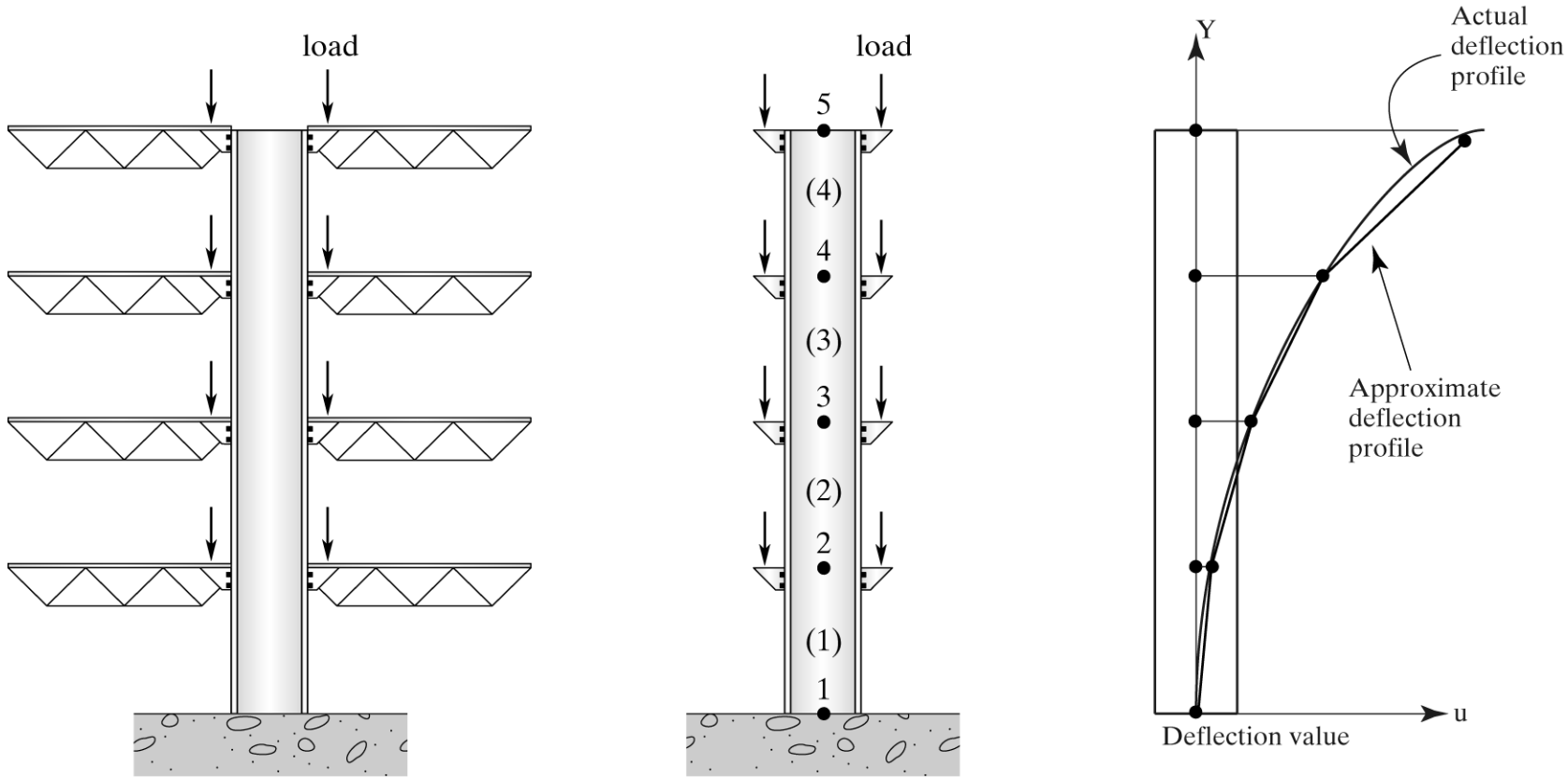


Figure 4-1

Deflection of a steel column subject to floor loading.

The linear deflection distribution for a typical element may be expressed as

$$u^{(e)} = c_1 + c_2 Y \tag{4.1}$$

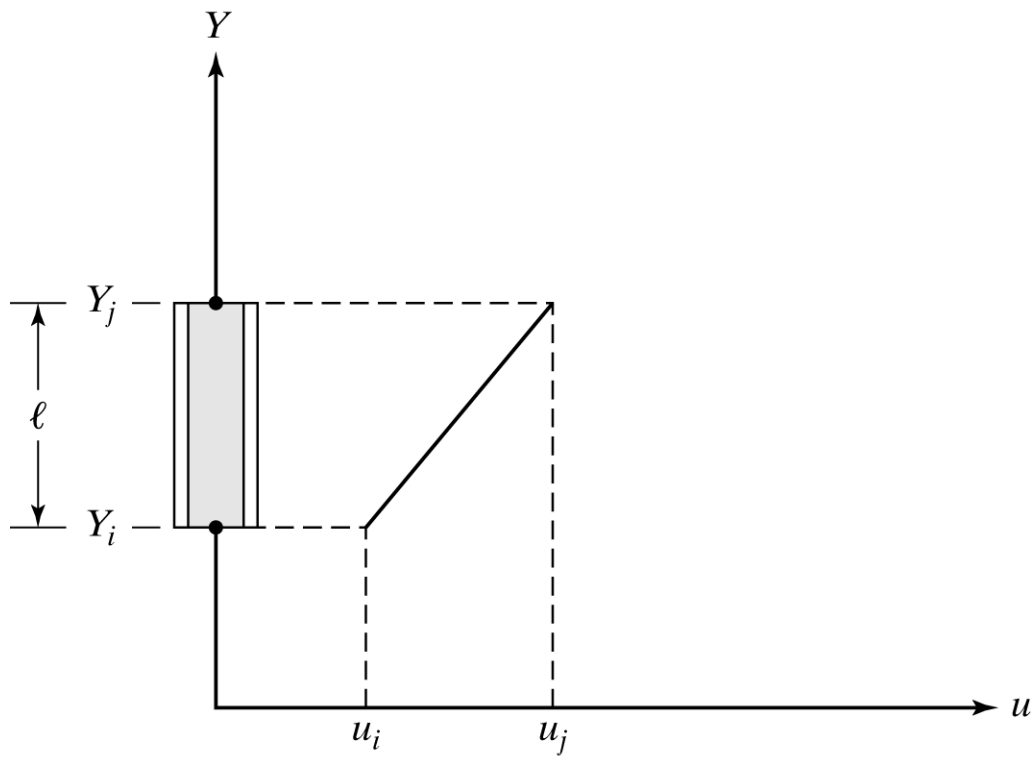


Figure 4-2
Linear approximation of deflection variation for an element.

$$u^{(e)} = c_1 + c_2 Y \tag{4.1}$$

In order to solve for the unknown coefficients c_1 and c_2, we make use of the element's end deflection values which are given by the nodal deflections u_i and u_j, according to the conditions

$$\begin{aligned} u &= u_i \quad \text{at} \quad Y = Y_i \\ u &= u_j \quad \text{at} \quad Y = Y_j \end{aligned} \tag{4.2}$$

Substitution of nodal values into Eq. (4.1) results in two equations and two unknowns:

$$\begin{aligned} u_i &= c_1 + c_2 Y_i \\ u_j &= c_1 + c_2 Y_j \end{aligned} \tag{4.3}$$

Solving for the unknowns c_1 and c_2, we get

$$c_1 = \frac{u_i Y_j - u_j Y_i}{Y_j - Y_i} \tag{4.4}$$

$$c_2 = \frac{u_j - u_i}{Y_j - Y_i} \tag{4.5}$$

The element's deflection distribution in terms of its nodal values is

$$u^{(e)} = \frac{u_i Y_j - u_j Y_i}{Y_j - Y_i} + \frac{u_j - u_i}{Y_j - Y_i} Y \tag{4.6}$$

Grouping the u_i terms together and the u_j terms together, Eq. (4.6) becomes

$$u^{(e)} = \left(\frac{Y_j - Y}{Y_j - Y_i}\right) u_i + \left(\frac{Y - Y_i}{Y_j - Y_i}\right) u_j \tag{4.7}$$

We now define the *shape functions*, S_i and S_j using the terms in parentheses appearing before u_i and u_j, according to the equations

$$S_i = \frac{Y_j - Y}{Y_j - Y_i} = \frac{Y_j - Y}{\ell} \tag{4.8}$$

$$S_j = \frac{Y - Y_i}{Y_j - Y_i} = \frac{Y - Y_i}{\ell} \tag{4.9}$$

where ℓ is the length of the element. Thus, the deflection for an element in terms of the shape functions and the nodal deflection values can be written as

$$u^{(e)} = S_i u_i + S_j u_j \tag{4.10}$$

Equation (4.10) can also be expressed in matrix form as

$$u^{(e)} = [S_i \quad S_j] \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{4.11}$$

For the one-dimensional element shown in Figure 4.2, the relationship between a global coordinate Y and a local coordinate y is given by $Y = Y_i + y$. This relationship is shown in Figure 4.3. Substituting for Y in terms of the local coordinate y in Eqs. (4.8) and (4.9), we get

$$S_i = \frac{Y_j - Y}{\ell} = \frac{Y_j - (Y_i + y)}{\ell} = 1 - \frac{y}{\ell} \tag{4.12}$$

$$S_i = \frac{Y - Y_i}{\ell} = \frac{(Y_i + y) - Y_i}{\ell} = \frac{y}{\ell} \tag{4.13}$$

where the local coordinate y varies from 0 to ℓ; that is, $0 \leq y \leq \ell$.

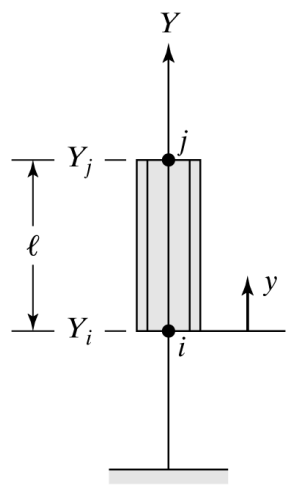


Figure 4-3
The relationship between a global coordinate Y and a local coordinate y.

Example 4.1, Moaveni

Consider a four-story building with steel columns. One column is subjected to the loading shown in Figure 4.4. Under axial loading assumption and using linear elements, the vertical displacements of the column at various floor-column connection points were determined to be

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = -\begin{Bmatrix} 0 \\ 0.03283 \\ 0.05784 \\ 0.07504 \\ 0.08442 \end{Bmatrix} \text{in.}$$

The modulus of elasticity of $E = 29 \times 10^6$ lb/in^2 and area of $A = 39.7$ in^2 were used in the calculations. A detailed analysis of this problem is given in the next section. For now, given the nodal displacement values, we are interested in determining the deflections of points A and B.

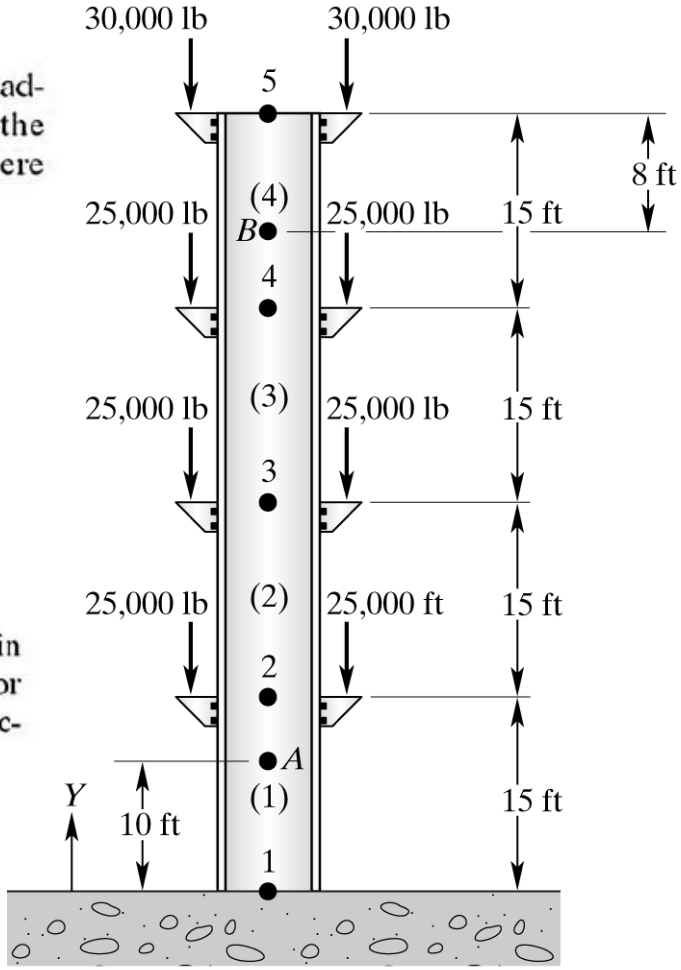


Figure 4-4

The column in Example 4.1.

In this section, we use the minimum total potential energy formulation to generate the stiffness and load matrices for members under axial loading. Previously, we showed that under axial loading, we can approximate the exact deflection of the column shown in Figure 4.1 by a series of linear functions. Moreover, as discussed in Section 1.6, applied external loads cause a body to deform. During the deformation, the work done by the external forces is stored in the material in the form of elastic energy, called strain energy. For a member (element) under axial loading, the strain energy $\Lambda^{(e)}$ is given by

$$\Lambda^{(e)} = \int_V \frac{\sigma\varepsilon}{2}\, dV = \int_V \frac{E\varepsilon^2}{2}\, dV \tag{4.14}$$

The total potential energy Π for a body consisting of n elements and m nodes is the difference between the total strain energy and the work done by the external forces:

$$\Pi = \sum_{1}^{n} \Lambda^{(e)} - \sum_{1}^{m} F_i u_i \tag{4.15}$$

The minimum total potential energy principle states that for a stable system, the displacement at the equilibrium position occurs such that the value of the system's total potential energy is a minimum. That is,

$$\frac{\partial \Pi}{\partial u_i} = \frac{\partial}{\partial u_i} \sum_{e=1}^{n} \Lambda^{(e)} - \frac{\partial}{\partial u_i} \sum_{i=1}^{m} F_i u_i = 0 \quad \text{for } i = 1, 2, 3, \ldots, m \tag{4.16}$$

where i takes on different values of node numbers. Recall that the deflection for an arbitrary element with nodes i and j in terms of local shape functions is given by

$$u^{(e)} = S_i u_i + S_j u_j \tag{4.17}$$

where $S_i = 1 - \frac{y}{\ell}$ and $S_j = \frac{y}{\ell}$ and y is the element's local coordinate, with its origin at node i. The strain in each member can be computed using the relation $\varepsilon = \frac{du}{dy}$ as

$$\varepsilon = \frac{du}{dy} = \frac{d}{dy}[S_i u_i + S_j u_j] = \frac{d}{dy}\left[\left(1 - \frac{y}{\ell}\right)u_i + \frac{y}{\ell} u_j\right] = \frac{-u_i + u_j}{\ell} \tag{4.18}$$

Incorporating Eq. (4.18) into Eq. (4.14) yields the strain energy for an arbitrary element (e):

$$\Lambda^{(e)} = \int_V \frac{E\varepsilon^2}{2} dV = \frac{AE}{2\ell}(u_j^2 + u_i^2 - 2u_j u_i) \tag{4.19}$$

Minimizing the strain energy with respect to u_i and u_j leads to

$$\frac{\partial \Lambda^{(e)}}{\partial u_i} = \frac{AE}{\ell}(u_i - u_j) \tag{4.20}$$

$$\frac{\partial \Lambda^{(e)}}{\partial u_j} = \frac{AE}{\ell}(u_j - u_i)$$

or, in matrix form,

$$\begin{Bmatrix} \dfrac{\partial \Lambda^{(e)}}{\partial u_i} \\ \dfrac{\partial \Lambda^{(e)}}{\partial u_j} \end{Bmatrix} = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{4.21}$$

where $k = \dfrac{(AE)}{\ell}$. Minimizing the work done by external forces, the second term on the right-hand side of Eq. (4.16) results in the load matrix

$$\{\mathrm{F}\}^{(e)} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} \tag{4.22}$$

Computing individual elemental stiffness and load matrices and connecting them leads to global stiffness and load matrices. This step is demonstrated by the next example.

Example 4.2, A column Problem, Moaveni

Consider a four-story building with steel columns. One column is subjected to the loading shown in Figure 4.5. Assuming axial loading, determine (a) vertical displacements of the column at various floor-column connection points and (b) the stresses in each portion of the column. $E = 29 \times 10^6$ lb/in^2, $A = 39.7$ in^2.

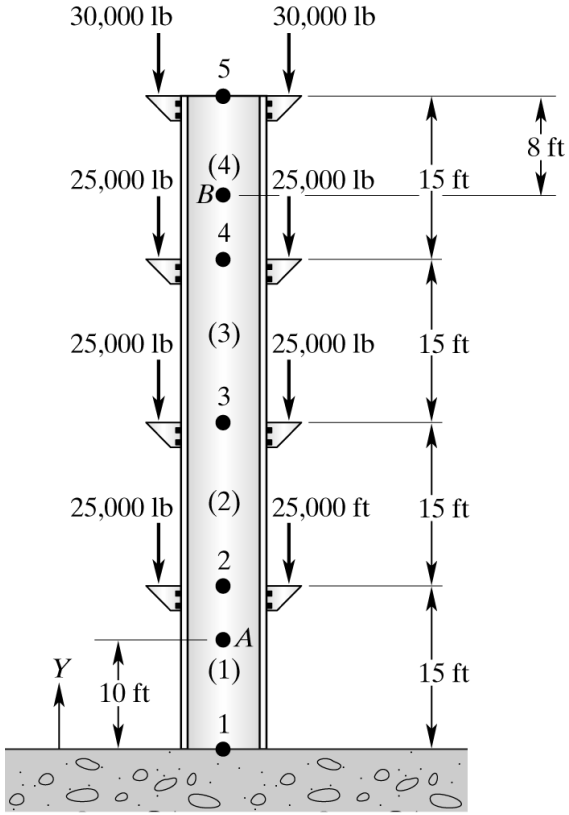


Figure 4-4

The column in Example 4.1.

Because all elements have the same length, cross-sectional area, and physical properties, the elemental stiffness for elements (1),(2),(3), and (4) is given by

$$[\mathbf{K}]^{(e)} = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{39.7 \times 29 \times 10^6}{15 \times 12}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 6.396 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\mathbf{K}]^{(1)} = [\mathbf{K}]^{(2)} = [\mathbf{K}]^{(3)} = [\mathbf{K}]^{(4)} = 6.396 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{\text{lb}}{\text{in}}$$

The global stiffness matrix is obtained by assembling the elemental matrices:

$$[\mathbf{K}]^{(G)} = 6.396 \times 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1+1 & -1 & 0 & 0 \\ 0 & -1 & 1+1 & -1 & 0 \\ 0 & 0 & -1 & 1+1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

The global load matrix is obtained from

$$\{\mathbf{F}\}^{(G)} = \left\{\frac{\partial F_i u_i}{\partial u_i}\right\}_{i=1,5} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix} = -\begin{Bmatrix} 0 \\ 50000 \\ 50000 \\ 50000 \\ 60000 \end{Bmatrix} \text{lb}$$

Note all applied forces act in the negative Y direction. Application of the boundary condition, $u_1 = 0$, and loads results in

$$6.396 \times 10^6 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = -\begin{Bmatrix} 0 \\ 50000 \\ 50000 \\ 50000 \\ 60000 \end{Bmatrix}$$

Solving for displacements, we have

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = -\begin{Bmatrix} 0 \\ 0.03283 \\ 0.05784 \\ 0.07504 \\ 0.08442 \end{Bmatrix} \text{ in}$$

The axial stresses in each element are determined from

$$\sigma^{(1)} = \frac{E(u_j - u_i)}{\ell} = \frac{29 \times 10^6(-0.03283 - 0)}{15 \times 12} = -5289 \text{ lb/in}^2$$

$$\sigma^{(2)} = \frac{29 \times 10^6(-0.05784 - (-0.03283))}{15 \times 12} = -4029 \text{ lb/in}^2$$

$$\sigma^{(3)} = \frac{29 \times 10^6(-0.07504 - (-0.05784))}{15 \times 12} = -2771 \text{ lb/in}^2$$

$$\sigma^{(4)} = \frac{29 \times 10^6(-0.08442 - (-0.07504))}{15 \times 12} = -1511 \text{ lb/in}^2$$

Beams

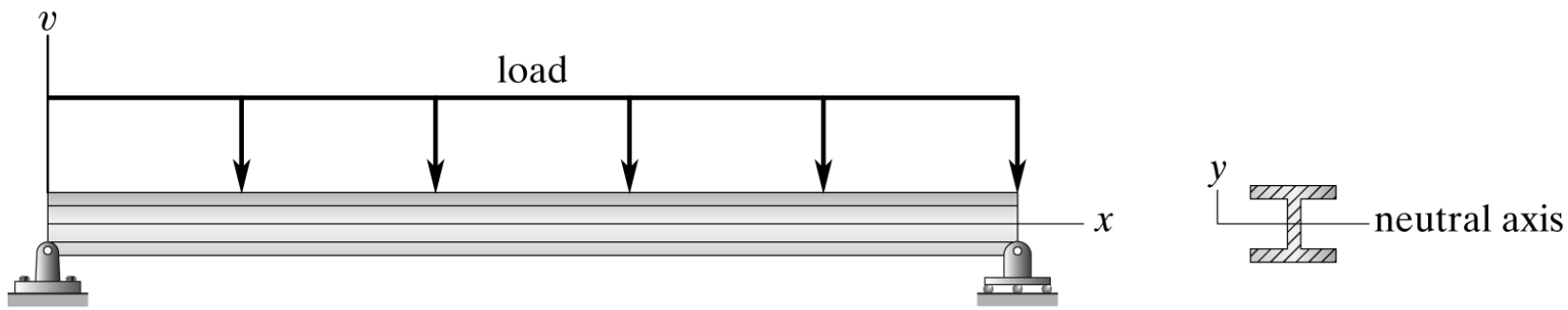
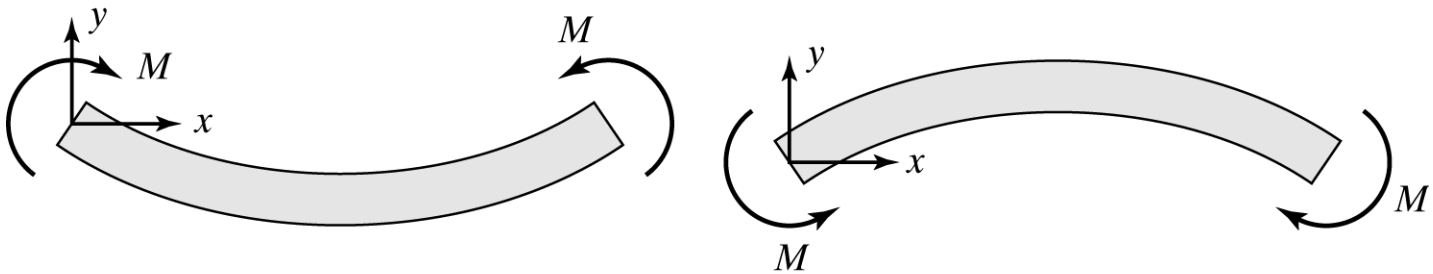


Figure 4-6

A beam subjected to a distributed load.



Positive bending and
Positive curvature

Negative moment
and negative curvature

Figure 4-7

The positive and negative bending moments and curvature sign convention.

$$\sigma = -\frac{My}{I} \tag{4.23}$$

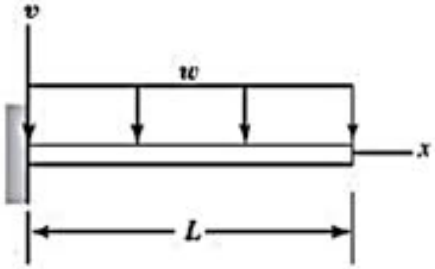
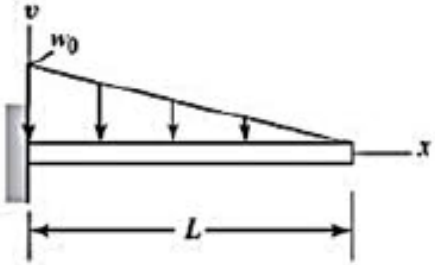
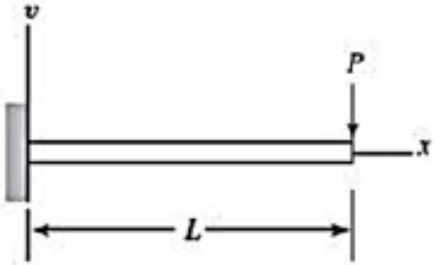
where y locates a point in the cross section of the beam and represents the lateral distance from the neutral axis to that point. The deflection of the neutral axis v is also related to the internal bending moment $M(x)$, the transverse shear $V(x)$, and the load $w(x)$ according to the equations

$$EI\frac{d^2v}{dx^2} = M(x) \tag{4.24}$$

$$EI\frac{d^3v}{dx^3} = \frac{dM(x)}{dx} = V(x) \tag{4.25}$$

$$EI\frac{d^4v}{dx^4} = \frac{dV(x)}{dx} = w(x) \tag{4.26}$$

TABLE 4.1 Deflections and slopes of beams under some typical loads and supports

Beam Support and Load	Equation of Elastic Curve	Maximum Deflection	Slope
	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$	$v_{max} = \frac{-wL^4}{8EI}$	$\theta_{max} = \frac{-wL^3}{6EI}$
	$v = \frac{-w_0x^2}{120LEI}(-x^3 + 5Lx^2 - 10L^2x + 10L^3)$	$v_{max} = \frac{-w_0L^4}{30EI}$	$\theta_{max} = \frac{-w_0L^3}{24EI}$
	$v = \frac{-Px^2}{6EI}(3L - x)$	$v_{max} = \frac{-PL^3}{3EI}$	$\theta_{max} = \frac{-PL^2}{2EI}$

4.3 FINITE ELEMENT FORMULATION OF BEAMS

Before we proceed with finite element formulation of beams, we should define what we mean by a beam element. A simple beam element consists of two nodes. At each node, there are two degrees of freedom, a vertical displacement, and a rotation angle (slope), as shown in Figure 4.8.

There are four nodal values associated with a beam element. Therefore, we will use a third-order polynomial with four unknown coefficients to represent the displacement field. Moreover, we want the first derivatives of the shape functions to be continuous. The resulting shape functions are commonly referred to as Hermite shape functions. As you will see, they differ in some ways from the linear shape functions you have already studied. We start with the third-order polynomial

$$v = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \tag{4.27}$$

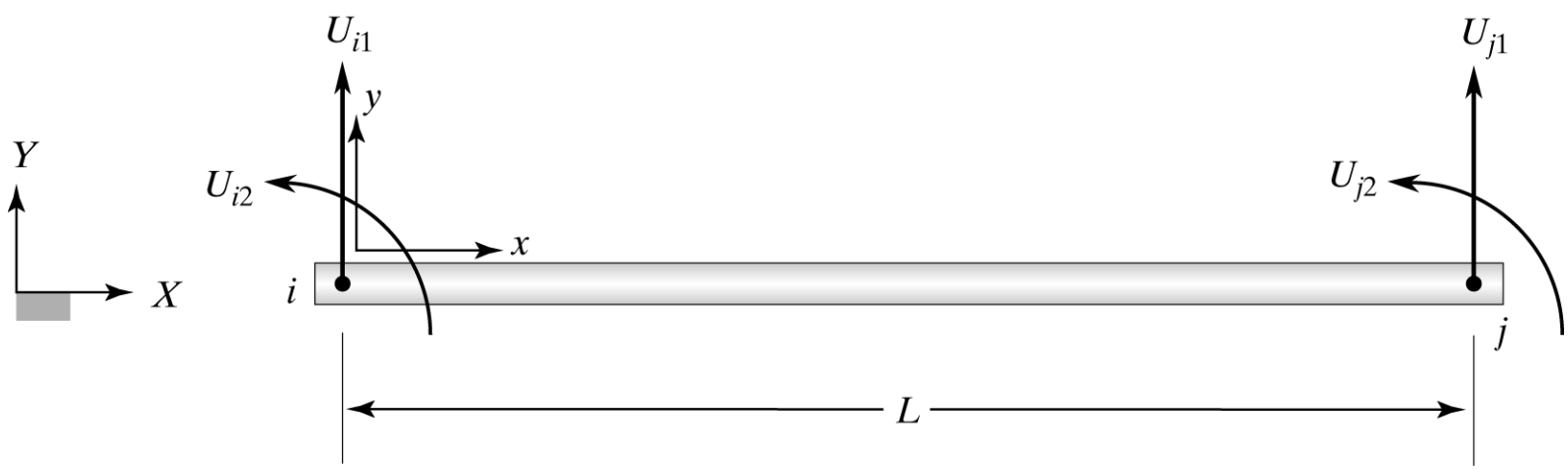


Figure 4-8

A beam element.

$$v = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \tag{4.27}$$

The element's end conditions are given by the following nodal values:

For node i: The vertical displacement at $x = 0$ $v = c_1 = U_{i1}$

For node i: The slope at $x = 0$ $\left.\dfrac{dv}{dx}\right|_{x=0} = c_2 = U_{i2}$

For node j: The vertical displacement at $x = L$ $v = c_1 + c_2 L + c_3 L^2 + c_4 L^3 = U_{j1}$

For node j: The slope at $x = L$ $\left.\dfrac{dv}{dx}\right|_{x=L} = c_2 + 2c_3 L + 3c_4 L^2 = U_{j2}$

We now have four equations with four unknowns. Solving for c_1, c_2, c_3, and c_4; substituting into Eq. (4.27); and regrouping the $U_{i1}, U_{i2}, U_{j1}, U_{j2}$ terms results in the equation

$$v = S_{i1}U_{i1} + S_{i2}U_{i2} + S_{j1}U_{j1} + S_{j2}U_{j2} \tag{4.28}$$

where the shape functions are given by

$$S_{i1} = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \tag{4.29}$$

$$S_{i2} = x - \frac{2x^2}{L} + \frac{x^3}{L^2} \tag{4.30}$$

$$S_{j1} = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \tag{4.31}$$

$$S_{j2} = -\frac{x^2}{L} + \frac{x^3}{L^2} \tag{4.32}$$

Evaluating Eq. (4.42) leads to the expression

$$\frac{\partial \Lambda^{(e)}}{\partial U_k} = EI \int_0^L [\mathbf{D}]^T [\mathbf{D}] dx \{\mathbf{U}\} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} U_{i1} \\ U_{i2} \\ U_{j1} \\ U_{j2} \end{Bmatrix}$$

The stiffness matrix for a beam element with two degrees of freedom at each node—the vertical displacement and rotation—is

$$[\mathbf{K}]^{(e)} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{4.43}$$

Load Matrix

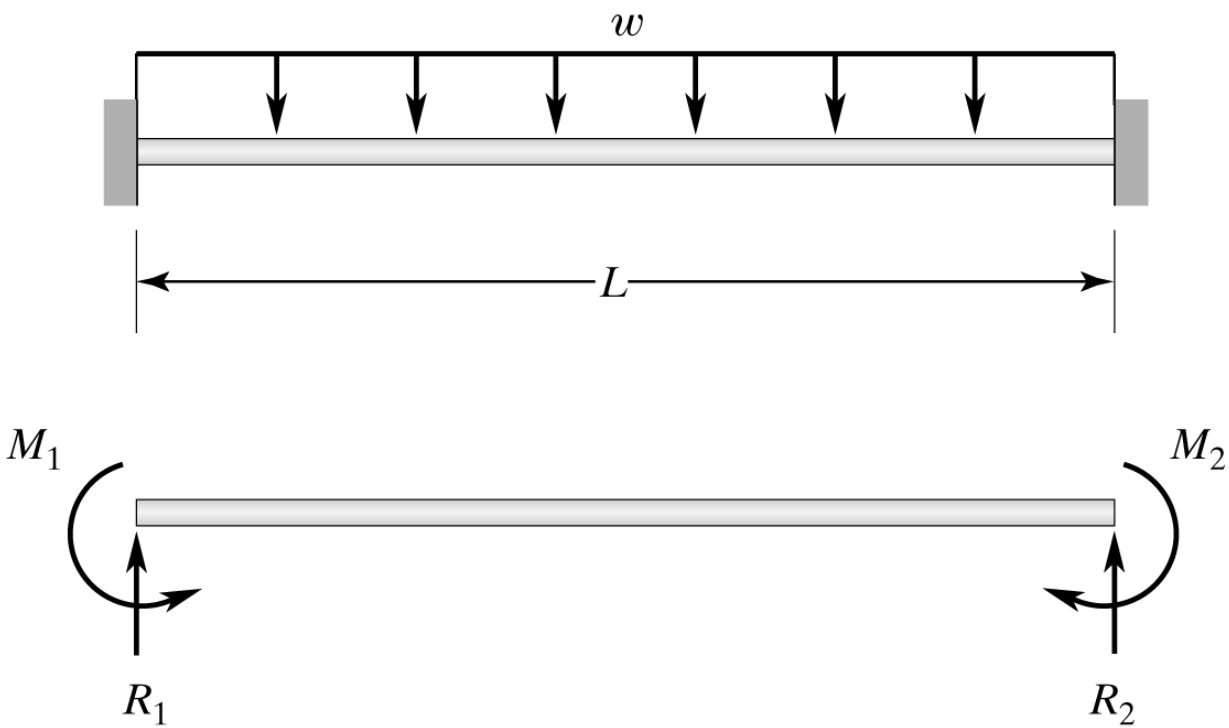


Figure 4-9

A beam element subjected to a uniform distributed load.

Figure 4-10

Reaction results for a beam subjected to a uniformly distributed load.

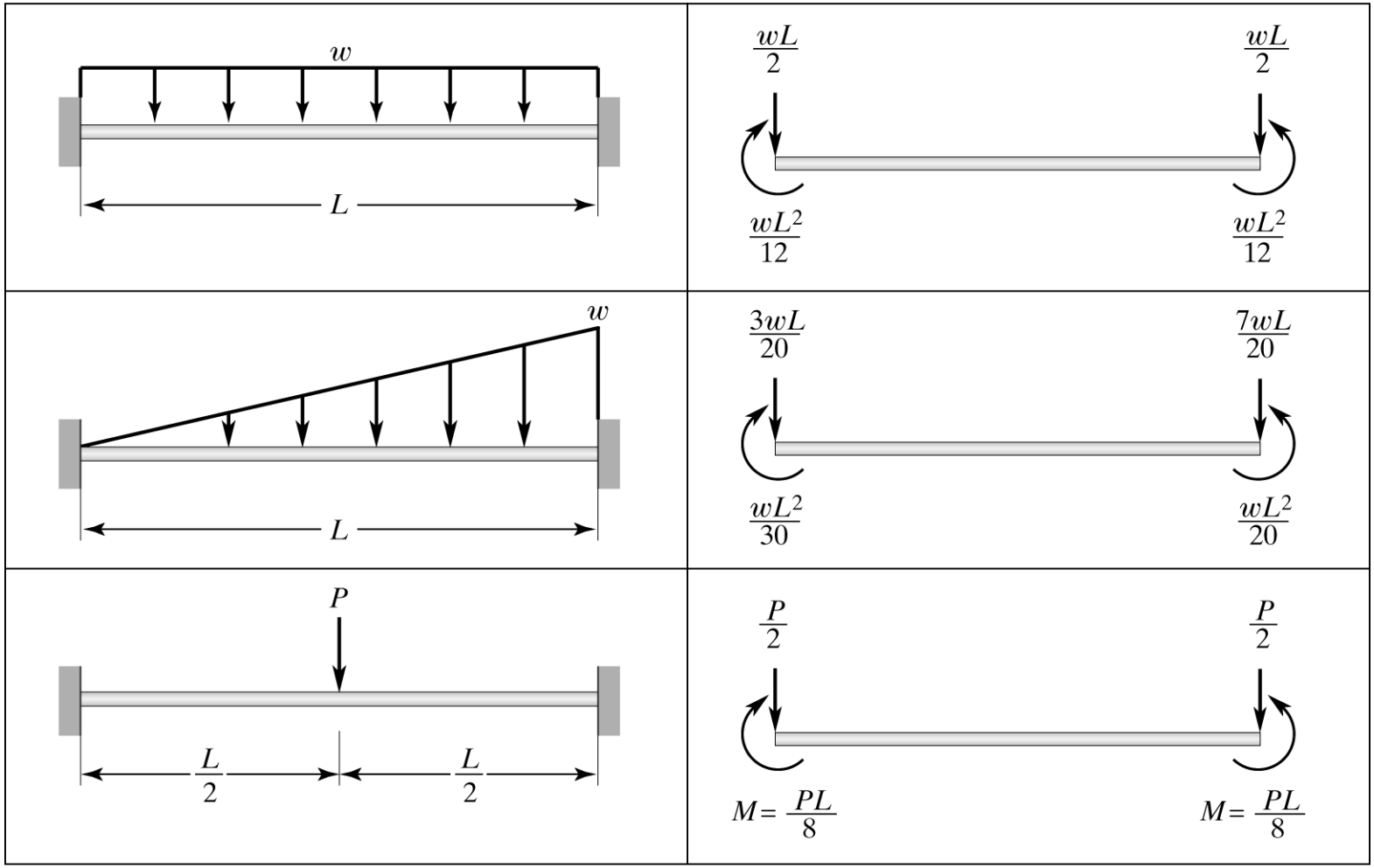
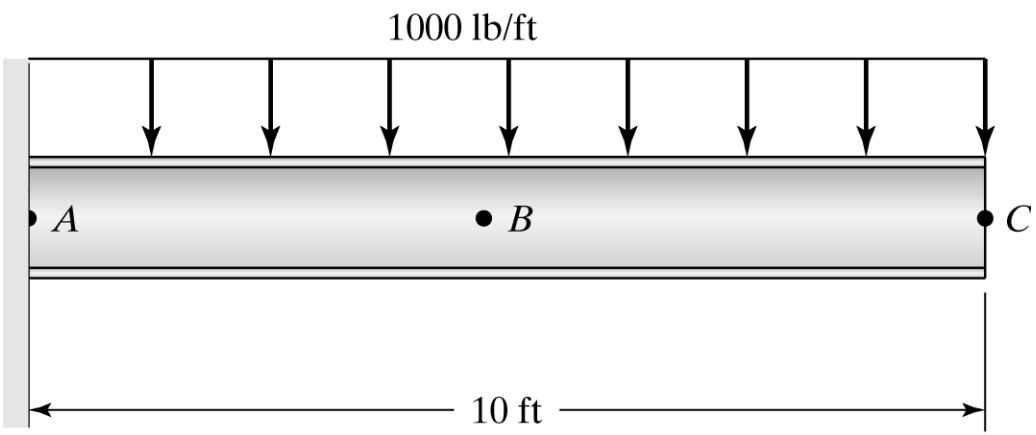


Table 4-2

Equivalent nodal loading of beams.

Example 4.3, Moaveni

Let us consider the cantilevered balcony beam of Example 4.3 again and solve it using a single beam element. Recall that the beam is a wide-flange W18 × 35, with a cross-sectional area of 10.3 in^2 and a depth of 17.7 in. The second moment of area is 510 in^4. The beam is subjected to a uniformly distributed load of 1000 lb/ft. The modulus of elasticity of the beam $E = 29 \times 10^6$ lb/in^2. We are interested in determining the deflection of the beam at the midpoint B and the endpoint C. Also, we will compute the maximum slope that will occur at point C.



Example 4-3 (Revisited)

Because we are using a single element to model this problem, the elemental stiffness and load matrices are the same as the global matrices.

$$[K]^{(e)} = [K]^{(G)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad \{F\}^{(e)} = \{F\}^{(G)} = \begin{Bmatrix} -\frac{wL}{2} \\ -\frac{wL^2}{12} \\ -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{Bmatrix} = \begin{Bmatrix} -\frac{wL}{2} \\ -\frac{wL^2}{12} \\ -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

Applying the boundary conditions $U_{11} = 0$ and $U_{12} = 0$ at node 1, we have

$$\frac{EI}{L^3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix}$$

And simplifying, we get

$$\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} U_{21} \\ U_{22} \end{Bmatrix} = \frac{L^3}{EI}\begin{Bmatrix} -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix}$$

$$\begin{bmatrix} 12 & -6(10\text{ ft}) \\ -6(10\text{ ft}) & 4(10\text{ ft})^2 \end{bmatrix}\begin{Bmatrix} U_{21} \\ U_{22} \end{Bmatrix} = \frac{(10\text{ ft})^3}{(29 \times 10^6\text{ lb/in}^2)(510\text{ in}^4)\left(\dfrac{1\text{ ft}}{12\text{ in.}}\right)^2}\begin{Bmatrix} -\dfrac{1000(10)}{2} \\ \dfrac{(1000)(10)^2}{12} \end{Bmatrix}$$

The deflection and the slope at endpoint C is

$$U_{21} = -0.01217 \text{ ft} = -0.146 \text{ in} \quad \text{and} \quad U_{22} = -0.00163 \text{ rad}$$

To determine the deflection at point B, we use the deflection equation for the beam element and evaluate the shape functions at $x = \frac{L}{2}$.

$$\begin{aligned} v &= S_{11}U_{11} + S_{12}U_{12} + S_{21}U_{21} + S_{22}U_{22} \\ &= S_{11}(0) + S_{12}(0) + S_{21}(-0.146) + S_{22}(-0.00163) \end{aligned}$$

Computing the values of the shape functions at point B, we have

$$S_{21} = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} = \frac{3}{L^2}\left(\frac{L}{2}\right)^2 - \frac{2}{L^3}\left(\frac{L}{2}\right)^3 = \frac{1}{2}$$

$$S_{22} = -\frac{x^2}{L} + \frac{x^3}{L^2} = -\frac{\left(\frac{L}{2}\right)^2}{L} + \frac{\left(\frac{L}{2}\right)^3}{L^2} = -\frac{L}{8}$$

$$v_B = \left(\frac{1}{2}\right)(-0.146 \text{ in}) + \left(-\frac{120 \text{ in.}}{8}\right)(-0.00163 \text{ rad}) = -0.048 \text{ in}$$