Finite Element Analysis
Lecture 5

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Example 4.4, Moaveni

The beam shown in Figure 4.11 is a wide-flange W310 $\times$ 52 with a cross-sectional area of 6650 mm^2 and depth of 317 mm. The second moment of the area is $118.6 \times 10^6\, mm^4$. The beam is subjected to a uniformly distributed load of 25,000 N/m. The modulus of elasticity of the beam is $E = 200$ GPa. Determine the vertical displacement at node 3 and the rotations at nodes 2 and 3. Also, compute the reaction forces and moment at nodes 1 and 2.

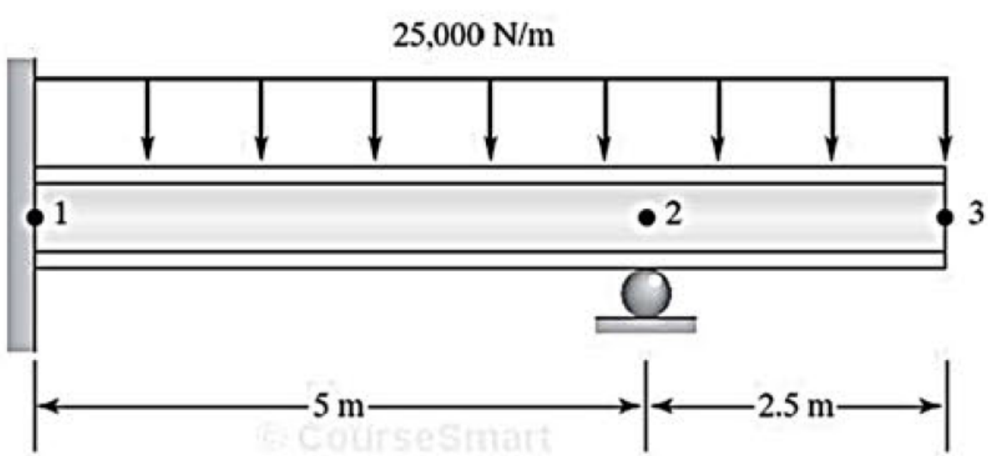


FIGURE 4.11 A schematic of the beam in Example 4.4

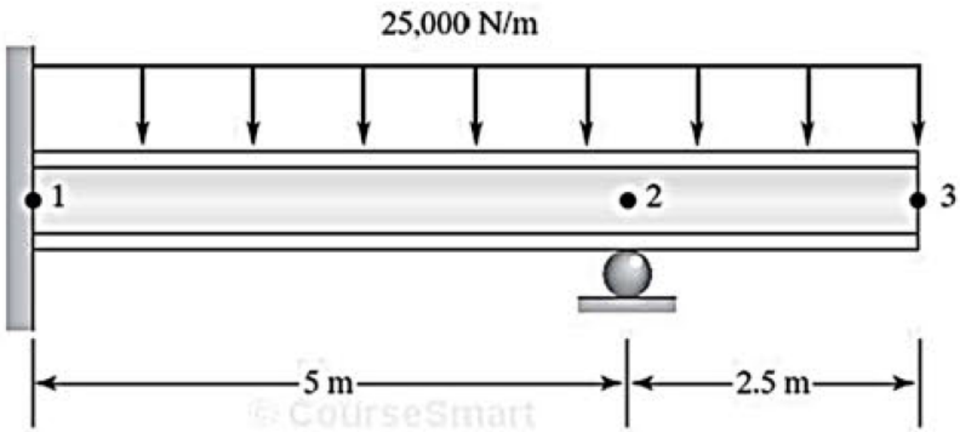


FIGURE 4.11 A schematic of the beam in Example 4.4

Note that this problem is statically indeterminate. We will use two elements to represent this problem. The stiffness matrices of the elements are computed from Eq. (4.43):

$$[\mathbf{K}]^{(e)} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Substituting appropriate values for element (1), we have

$$[\mathbf{K}]^{(1)} = \frac{200 \times 10^9 \times 1.186 \times 10^{-4}}{5^3}\begin{bmatrix} 12 & 6(5) & -12 & 6(5) \\ 6(5) & 4(5)^2 & -6(5) & 2(5)^2 \\ -12 & -6(5) & 12 & -6(5) \\ 6(5) & 2(5)^2 & -6(5) & 4(5)^2 \end{bmatrix}$$

For convenience, the nodal degrees of freedom are shown alongside the stiffness matrices. For element (1), we have

$$[\mathbf{K}]^{(1)} = \begin{array}{c} \begin{bmatrix} 2277120 & 5692800 & -2277120 & 5692800 \\ 5692800 & 18976000 & -5692800 & 9488000 \\ -2277120 & -5692800 & 2277120 & -5692800 \\ 5692800 & 9488000 & -5692800 & 18976000 \end{bmatrix} \end{array} \begin{array}{l} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{array}$$

Computing the stiffness matrix for element (2), we have

$$[\mathbf{K}]^{(2)} = \frac{200 \times 10^9 \times 1.186 \times 10^{-4}}{(2.5)^3}\begin{bmatrix} 12 & 6(2.5) & -12 & 6(2.5) \\ 6(2.5) & 4(2.5)^2 & -6(2.5) & 2(2.5)^2 \\ -12 & -6(2.5) & 12 & -6(2.5) \\ 6(2.5) & 2(2.5)^2 & -6(2.5) & 4(2.5)^2 \end{bmatrix}$$

Showing the nodal degrees of freedom alongside the stiffness matrix for element (2), we have

$$[\mathbf{K}]^{(2)} = \begin{array}{c} \begin{bmatrix} 18216960 & 22771200 & -18216960 & 22771200 \\ 22771200 & 37952000 & -22771200 & 18976000 \\ -18216960 & -22771200 & 18216960 & -22771200 \\ 22771200 & 18976000 & -22771200 & 37952000 \end{bmatrix} \end{array} \begin{array}{l} U_{21} \\ U_{22} \\ U_{31} \\ U_{32} \end{array}$$

Assembling $[\mathbf{K}]^{(1)}$ and $[\mathbf{K}]^{(2)}$ to obtain the global stiffness matrix yields

$$[\mathbf{K}]^{(G)} = \begin{bmatrix} 2277120 & 5692800 & -2277120 & 5692800 & 0 & 0 \\ 5692800 & 18976000 & -5692800 & 9488000 & 0 & 0 \\ -2277120 & -5692800 & 20494080 & 17078400 & -18216960 & 22771200 \\ 5692800 & 9488000 & 17078400 & 56928000 & -22771200 & 18976000 \\ 0 & 0 & -18216960 & -22771200 & 18216960 & -22771200 \\ 0 & 0 & 22771200 & 18976000 & -22771200 & 37952000 \end{bmatrix}$$

Referring to Table 4.2, we can compute the load matrix for elements (1) and (2). The respective load matrices are

$$\{\mathbf{F}\}^{(1)} = \begin{Bmatrix} -\dfrac{wL}{2} \\ -\dfrac{wL^2}{12} \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{25 \times 10^3 \times 5}{2} \\ -\dfrac{25 \times 10^3 \times 5^2}{12} \\ -\dfrac{25 \times 10^3 \times 5}{2} \\ \dfrac{25 \times 10^3 \times 5^2}{12} \end{Bmatrix} = \begin{Bmatrix} -62500 \\ -52083 \\ -62500 \\ 52083 \end{Bmatrix}$$

$$\{\mathbf{F}\}^{(2)} = \begin{Bmatrix} -\dfrac{wL}{2} \\ -\dfrac{wL^2}{12} \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{25 \times 10^3 \times 2.5}{2} \\ -\dfrac{25 \times 10^3 \times 2.5^2}{12} \\ -\dfrac{25 \times 10^3 \times 2.5}{2} \\ \dfrac{25 \times 10^3 \times 2.5^2}{12} \end{Bmatrix} = \begin{Bmatrix} -31250 \\ -13021 \\ -31250 \\ 13021 \end{Bmatrix}$$

Combining the two load matrices to obtain the global load matrix, we obtain

$$\{\mathbf{F}\}^{(G)} = \begin{Bmatrix} -62500 \\ -52083 \\ -62500 - 31250 \\ 52083 - 13021 \\ -31250 \\ 13021 \end{Bmatrix} = \begin{Bmatrix} -62500 \\ -52083 \\ -93750 \\ 39062 \\ -31250 \\ 13021 \end{Bmatrix}$$

Applying the boundary conditions $U_{11} = U_{12} = 0$ at node 1 and the boundary condition $U_{21} = 0$ at node 2, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 5692800 & 9488000 & 17078400 & 56928000 & -22771200 & 18976000 \\ 0 & 0 & -18216960 & -22771200 & 18216960 & -22771200 \\ 0 & 0 & 22771200 & 18976000 & -22771200 & 37952000 \end{bmatrix} \begin{Bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \\ U_{31} \\ U_{32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 39062 \\ -31250 \\ 13021 \end{Bmatrix}$$

Considering the applied boundary conditions, we reduce the global stiffness matrix and the load matrix to

$$\begin{bmatrix} 56928000 & -22771200 & 18976000 \\ -22771200 & 18216960 & -22771200 \\ 18976000 & -22771200 & 37952000 \end{bmatrix} \begin{Bmatrix} U_{22} \\ U_{31} \\ U_{32} \end{Bmatrix} = \begin{Bmatrix} 39062 \\ -31250 \\ 13021 \end{Bmatrix}$$

Solving the three equations simultaneously results in the unknown nodal values. The displacement result is

$$[\mathbf{U}]^T = [0 \quad 0 \quad 0 \quad -0.0013723(\text{rad}) \quad -0.0085772(\text{m}) \quad -0.004117(\text{rad})]$$

We can compute the nodal reaction forces and moments from the relationship

$$\{\mathbf{R}\} = [\mathbf{K}]\{\mathbf{U}\} - \{\mathbf{F}\} \tag{4.50}$$

where $\{\mathbf{R}\}$ is the reaction matrix. Substituting for the appropriate values in Eq. (4.50), we have

$$\begin{Bmatrix} R_1 \\ M_1 \\ R_2 \\ M_2 \\ R_3 \\ M_3 \end{Bmatrix} = \begin{bmatrix} 2277120 & 5692800 & -2277120 & 5692800 & 0 & 0 \\ 5692800 & 18976000 & -5692800 & 9488000 & 0 & 0 \\ -2277120 & -5692800 & 20494080 & 17078400 & -18216960 & 22771200 \\ 5692800 & 9488000 & 17078400 & 56928000 & -22771200 & 18976000 \\ 0 & 0 & -18216960 & -22771200 & 18216960 & -22771200 \\ 0 & 0 & 22771200 & 18976000 & -22771200 & 37952000 \end{bmatrix} \times$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0013723 \\ -0.0085772 \\ -0.0041170 \end{Bmatrix} - \begin{Bmatrix} -62500 \\ -52083 \\ -93750 \\ 39062 \\ -31250 \\ 13021 \end{Bmatrix}$$

Performing the matrix operation results in the following reaction forces and moments at each node:

$$\begin{Bmatrix} R_1 \\ M_1 \\ R_2 \\ M_2 \\ R_3 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} 54687(\mathrm{N}) \\ 39062(\mathrm{N \cdot m}) \\ 132814(\mathrm{N}) \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

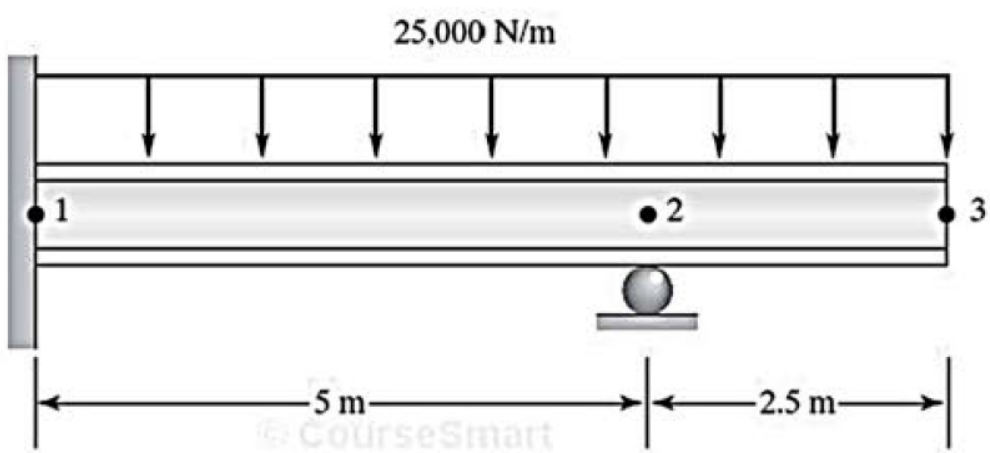
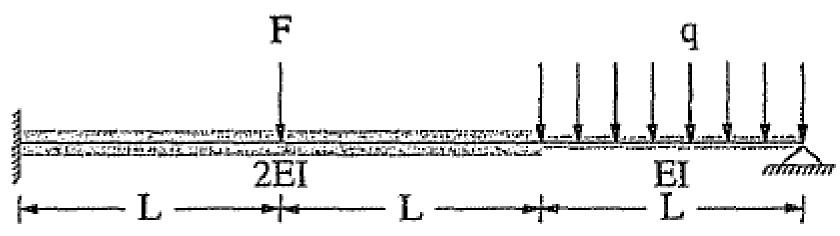


FIGURE 4.11 A schematic of the beam in Example 4.4

Beam Example – Bhatti Book P. 255

Example 4.7 Find the deflection, bending moment, and shear force distribution in the nonuniform beam shown in Figure 4.26. Use the following numerical data:

$L = 2$ m; $F = 18$ kN; $q = 10$ kN/m; $E = 210$ GPa; $I = 4 \times 10^{-4}$ m^4



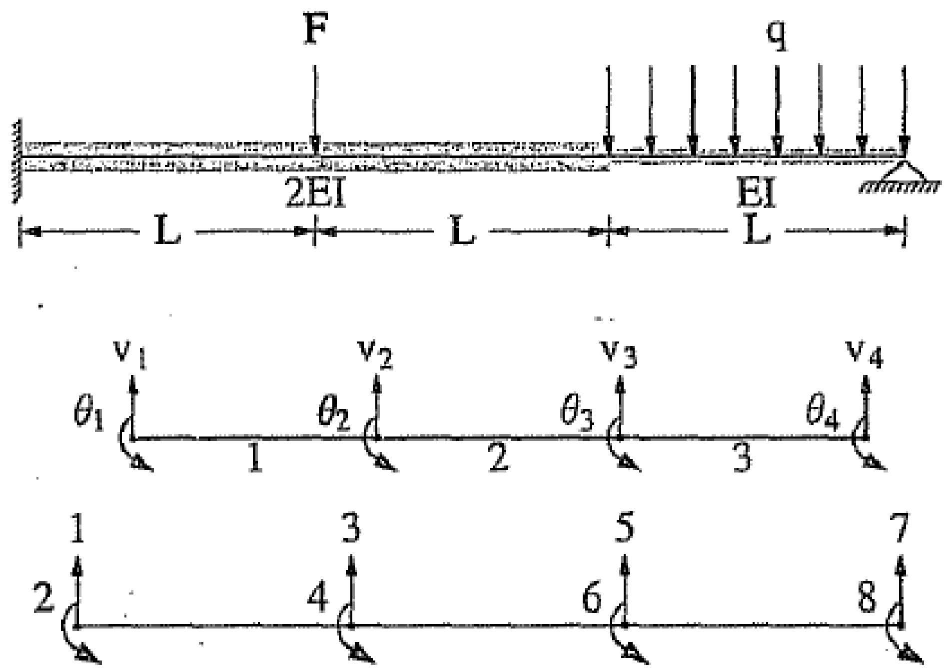
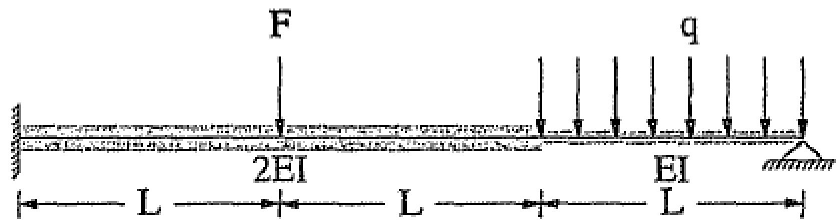


Figure 4.26. Nonuniform beam and three-element model

Specified nodal loads:

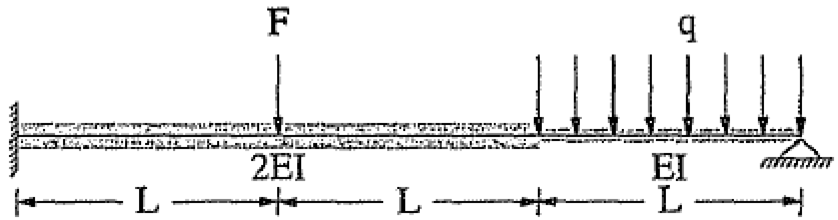
Node	dof	Value
2	v_2	−18



$$k = \frac{EI}{L^3}\begin{pmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{pmatrix}$$

Equations for element 1:

$$\begin{pmatrix} 252000. & 252000. & -252000. & 252000. \\ 252000. & 336000. & -252000. & 168000. \\ -252000. & -252000. & 252000. & -252000. \\ 252000. & 168000. & -252000. & 336000. \end{pmatrix}\begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



Equations for element 2:

$$\begin{pmatrix} 252000. & 252000. & -252000. & 252000. \\ 252000. & 336000. & -252000. & 168000. \\ -252000. & -252000. & 252000. & -252000. \\ 252000. & 168000. & -252000. & 336000. \end{pmatrix} \begin{pmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equations for element 3:

$$E = 210000000; \qquad I = \frac{1}{2500}; \qquad q = -10$$

$$\begin{pmatrix} 126000. & 126000. & -126000. & 126000. \\ 126000. & 168000. & -126000. & 84000. \\ -126000. & -126000. & 126000. & -126000. \\ 126000. & 84000. & -126000. & 168000. \end{pmatrix} \begin{pmatrix} v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} -10. \\ -3.33333 \\ -10. \\ 3.33333 \end{pmatrix}$$

Adding element equations into appropriate locations, we have

$$\begin{pmatrix} 252000. & 252000. & -252000. & 252000. & 0 & 0 & 0 & 0 \\ 252000. & 336000. & -252000. & 168000. & 0 & 0 & 0 & 0 \\ -252000. & -252000. & 504000. & 0 & -252000. & 252000. & 0 & 0 \\ 252000. & 168000. & 0 & 672000. & -252000. & 168000. & 0 & 0 \\ 0 & 0 & -252000. & -252000. & 378000. & -126000. & -126000. & 126000. \\ 0 & 0 & 252000. & 168000. & -126000. & 504000. & -126000. & 84000. \\ 0 & 0 & 0 & 0 & -126000. & -126000. & 126000. & -126000. \\ 0 & 0 & 0 & 0 & 126000. & 84000. & -126000. & 168000. \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -18 \\ 0 \\ -10. \\ -3.33333 \\ -10. \\ 3.33333 \end{pmatrix}$$

Essential boundary conditions:

Node	dof	Value
1	v_1	0
	θ_1	0
4	v_4	0

Remove {1, 2, 7} rows and columns:

$$\begin{pmatrix} 504000. & 0 & -252000. & 252000. & 0 \\ 0 & 672000. & -252000. & 168000. & 0 \\ -252000. & -252000. & 378000. & -126000. & 126000. \\ 252000. & 168000. & -126000. & 504000. & 84000. \\ 0 & 0 & 126000. & 84000. & 168000. \end{pmatrix} \begin{pmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} -18. \\ 0 \\ -10. \\ -3.33333 \\ 3.33333 \end{pmatrix}$$

Solving the final system of global equations, we get

$$\{v_2 = -0.000213152, \theta_2 = -0.000131378, v_3 = -0.00034127,$$
$$\theta_3 = 0.0000136054, \theta_4 = 0.000268991\}$$

Thus the assumed solution is

$$v(s) = \begin{pmatrix} \frac{2s^3}{L^3} - \frac{3s^2}{L^2} + 1 & \frac{s^3}{L^2} - \frac{2s^2}{L} + s & \frac{3s^2}{L^2} - \frac{2s^3}{L^3} & \frac{s^3}{L^2} - \frac{s^2}{L} \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix} \equiv \boldsymbol{N}^T \boldsymbol{d}$$

$$N_1 = 1 - \frac{3s^2}{L^2} + \frac{2s^3}{L^3}; \qquad N_2 = s - \frac{2s^2}{L} + \frac{s^3}{L^2}; \qquad N_3 = \frac{3s^2}{L^2} - \frac{2s^3}{L^3}; \qquad N_4 = -\frac{s^2}{L} + \frac{s^3}{L^2}$$

Solution for element 1:

Left node: $x_1 = 0$; Right node: $x_2 = 2$.

Local coordinate: $s = x - x_1 = x$; Length, $L = 2$.

Using these s and L values, the interpolation functions in terms of x are

$$N^T = \{0.25x^3 - 0.75x^2 + 1, 0.25x^3 - 1.x^2 + x, 0.75x^2 - 0.25x^3, 0.25x^3 - 0.5x^2\}$$

Nodal values: $\boldsymbol{d}^T = \{0, 0, -0.000213152, -0.000131378\}$

$v(x) = N^T\boldsymbol{d} = 0.0000204436x^3 - 0.0000941752x^2$

$M(x) = EI\, d^2v/dx^2 = 20.6071x - 31.6429$

$V(x) = dM/dx = 20.6071$

Solution for element 2:

$v(x) = N^T\boldsymbol{d} = 2.58645 \times 10^{-6}x^3 + 0.0000129677x^2 - 0.000214286x + 0.000142857$

$M(x) = EI\, d^2v/dx^2 = 2.60714x + 4.35714$

$V(x) = dM/dx = 2.60714$

Solution for element 3:

$v(x) = N^T\boldsymbol{d} = -0.0000146684x^3 + 0.000283872x^2 - 0.00155329x + 0.00226871$

$M(x) = EI\, d^2v/dx^2 = 47.6905 - 7.39286x$

$V(x) = dM/dx = -7.39286$

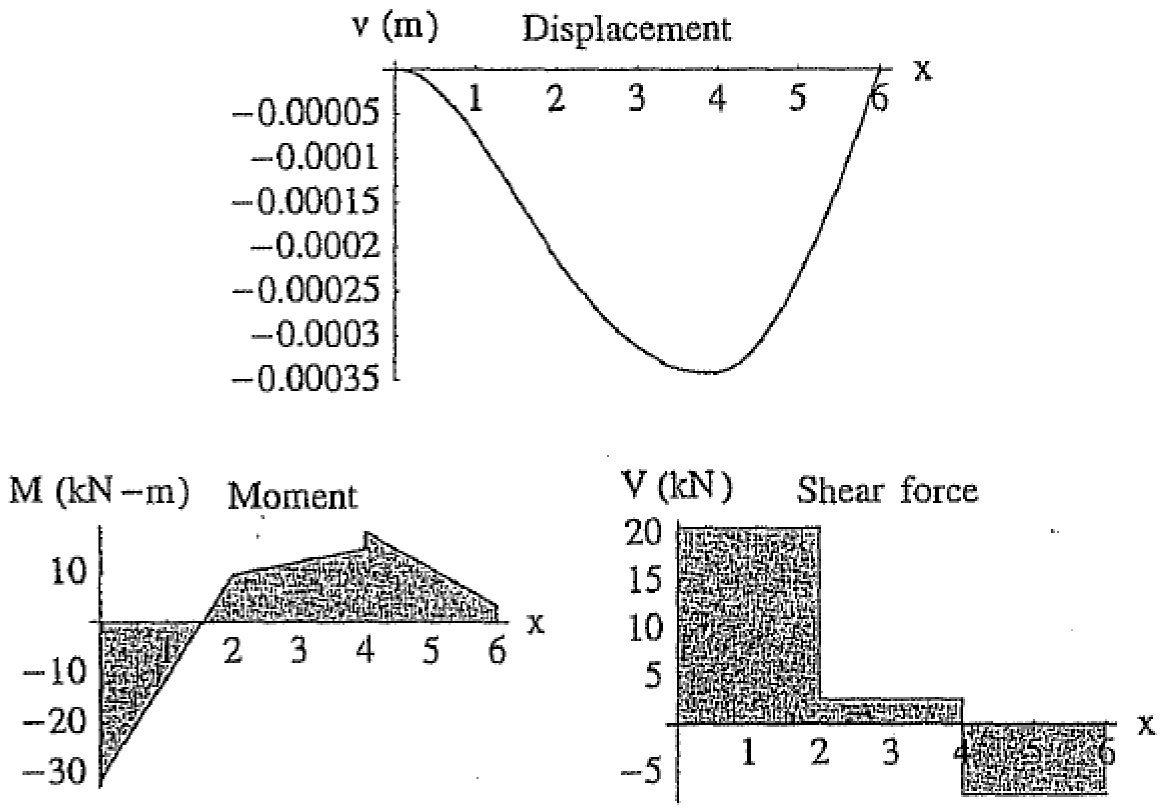


Figure 4.27. Three-element solution of nonuniform beam

The deflection, bending moment, and shear force diagrams are shown in Figure 4.27. The exact solution in the distributed load segment should show a linear shear diagram and a quadratic bending moment diagram. However, since the element is based on a cubic interpolation, it gives a constant shear and a linear moment diagram, and therefore the solution for moments and shears is not very good.

To get accurate results, we could use more elements in the segment with the distributed load. Using five elements in the distributed load segment, we get the moment and shear force diagrams shown in Figure 4.28 that are clearly a lot closer to the expected solution. However, for uniform beams it is not necessary to use more than one element per span. A simple superposition procedure is presented in the following section that gives exact solution without having to use many elements in the loaded span. That procedure obviously is preferred for practical applications.

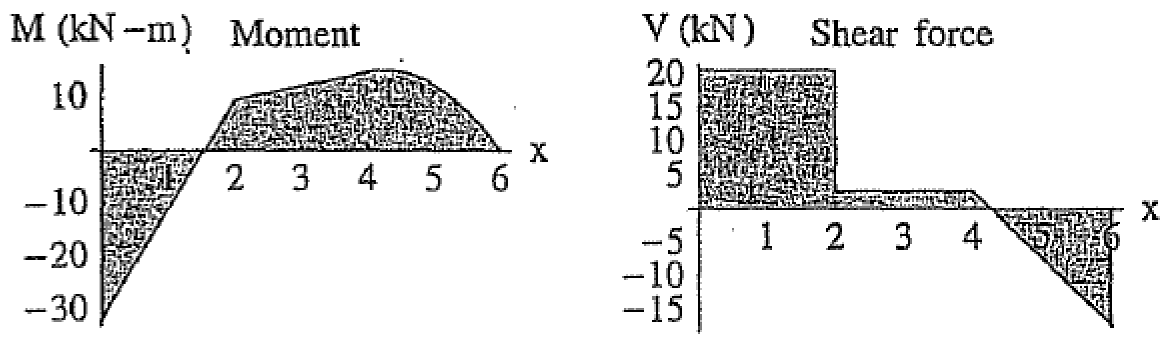


Figure 4.28. Solution with five elements in the distributed load segment

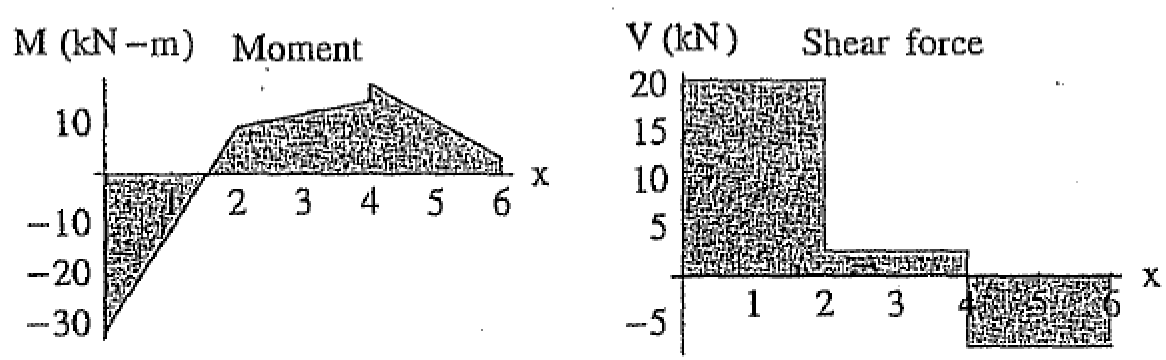


Figure 4.27. Three-element solution of nonuniform beam

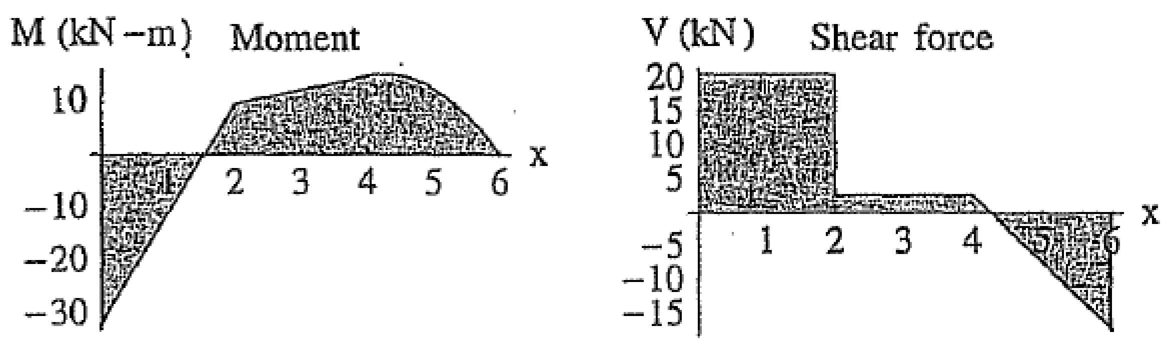
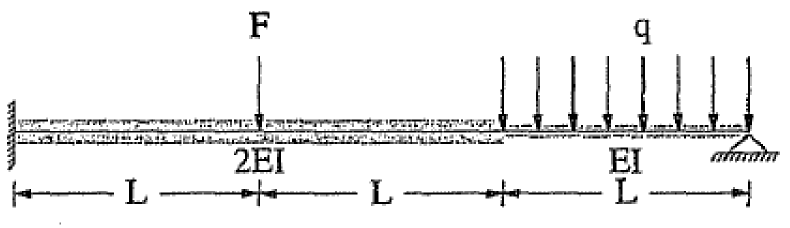
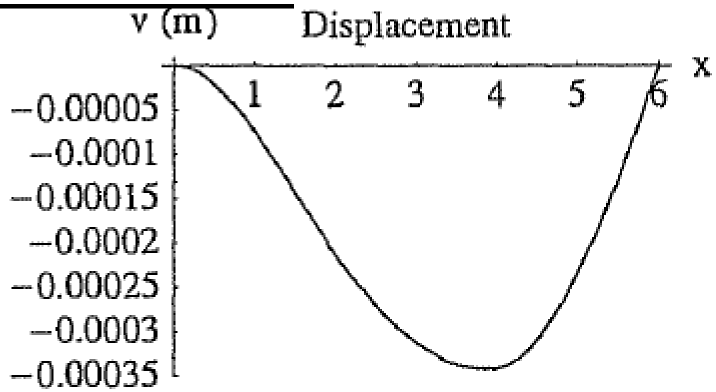
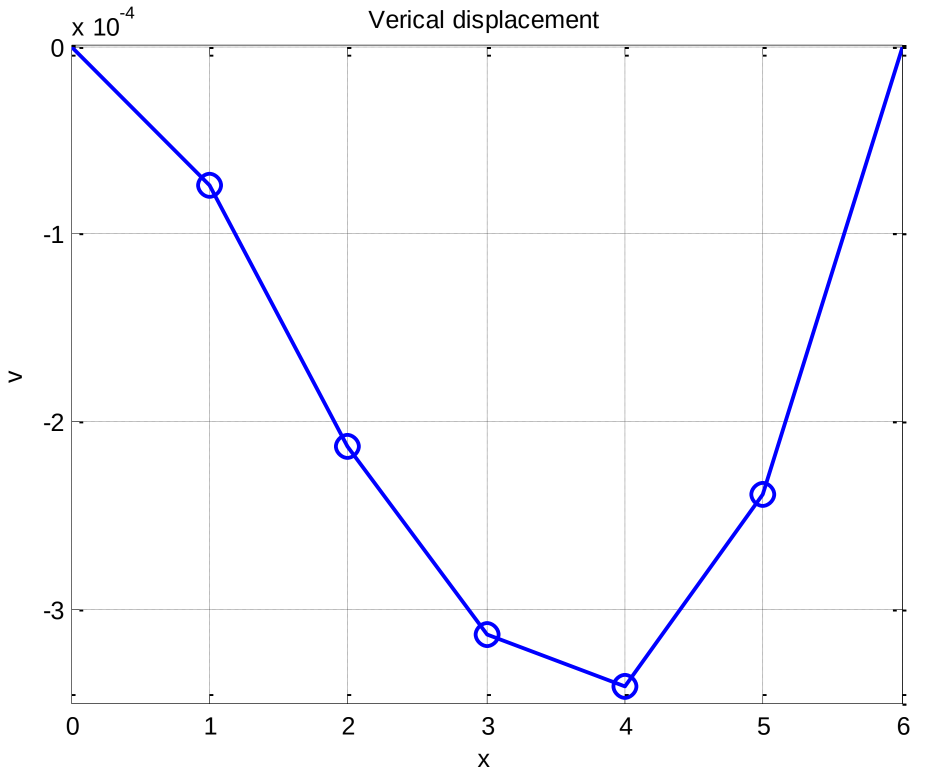


Figure 4.28. Solution with five elements in the distributed load segment

Matlab Results, 3 points in each element







Matlab Results, 3 points in each element

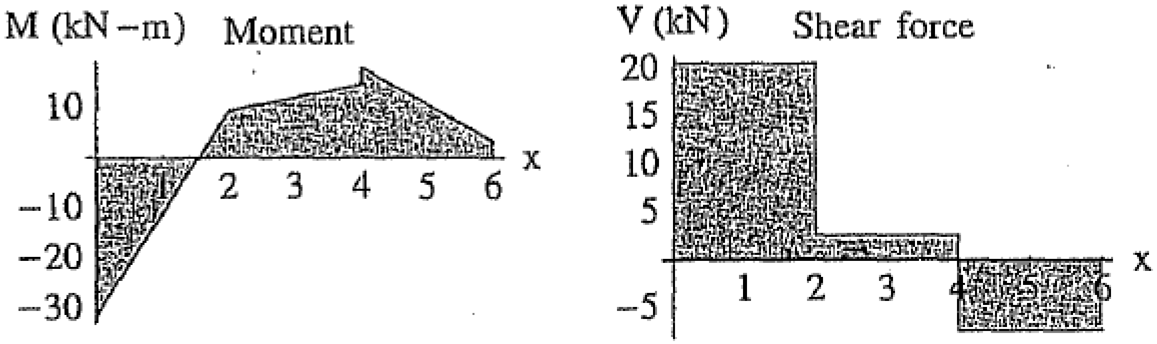
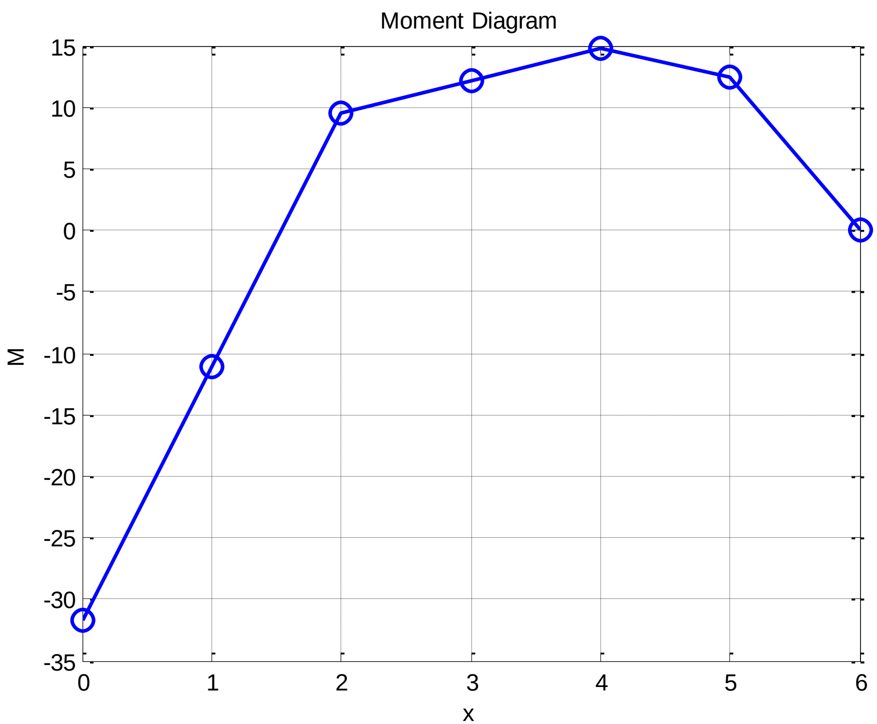
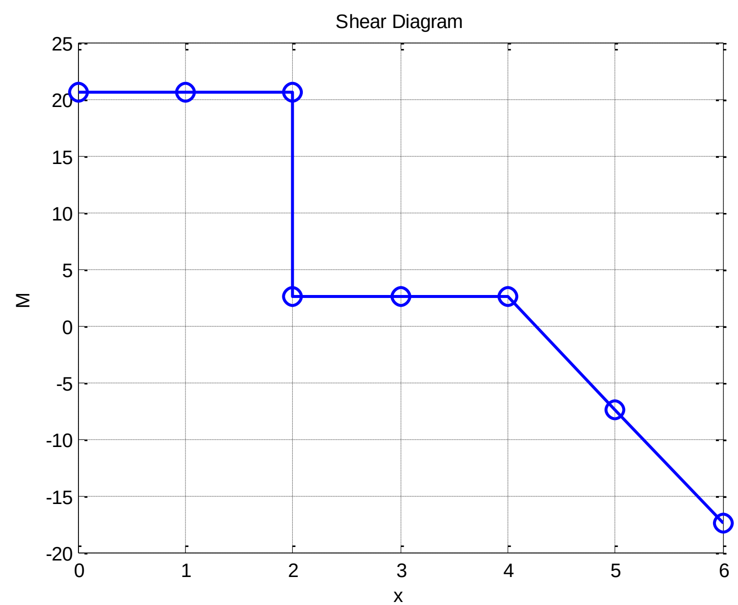


Figure 4.27. Three-element solution of nonuniform beam





Matlab Results, 3 points in each element

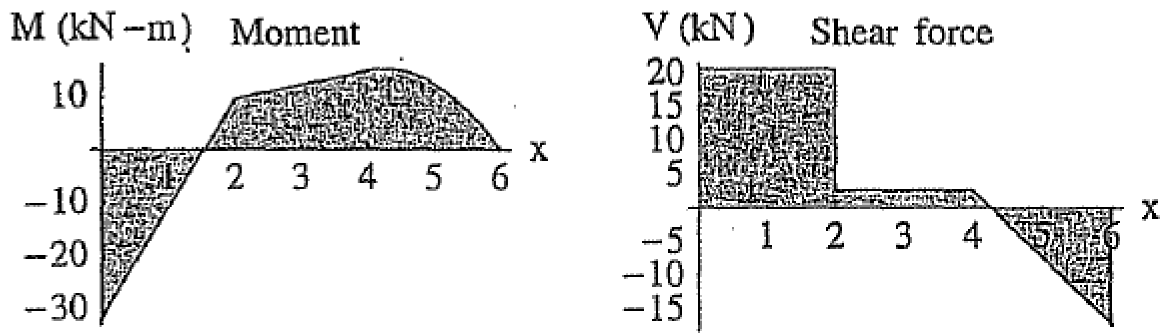
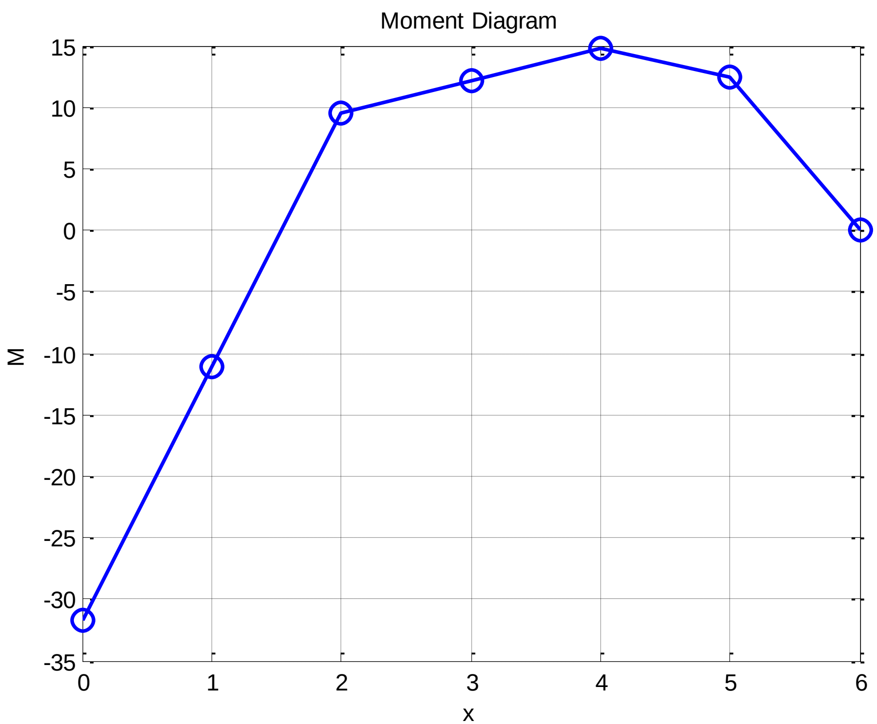
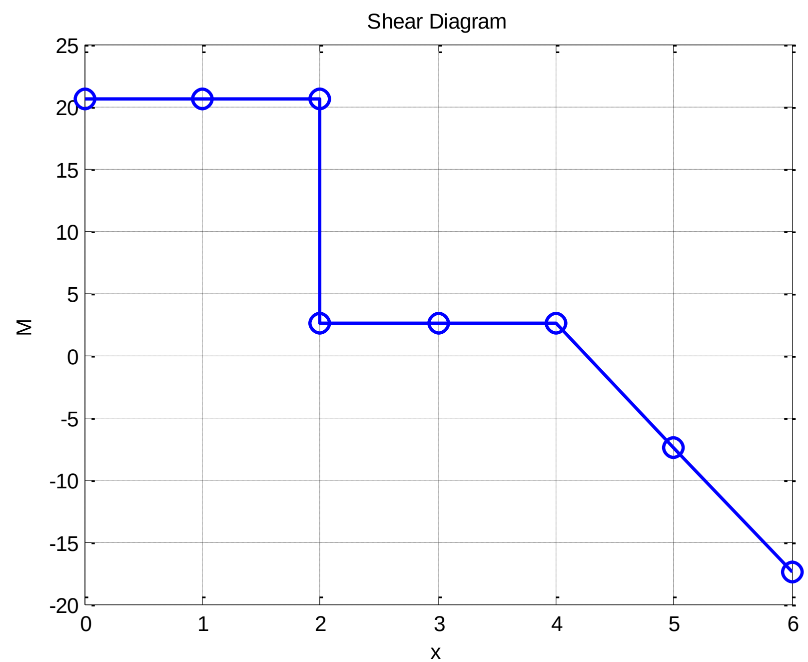
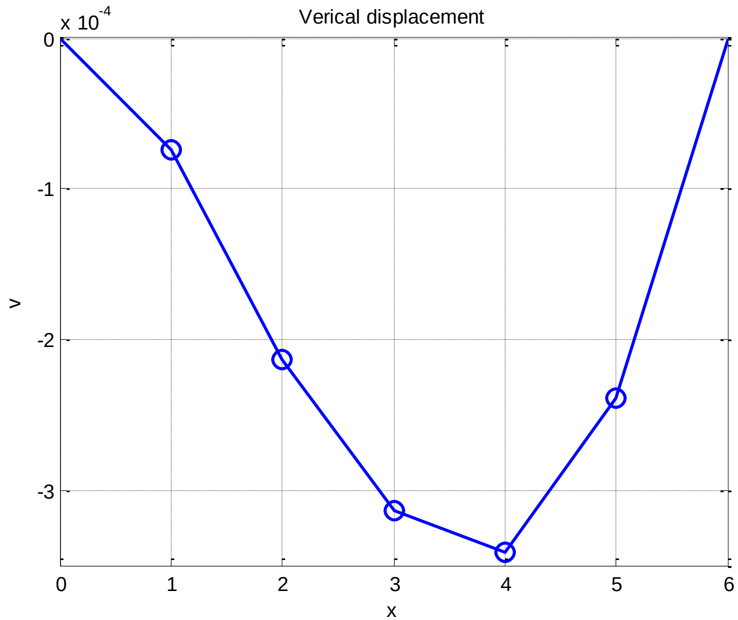


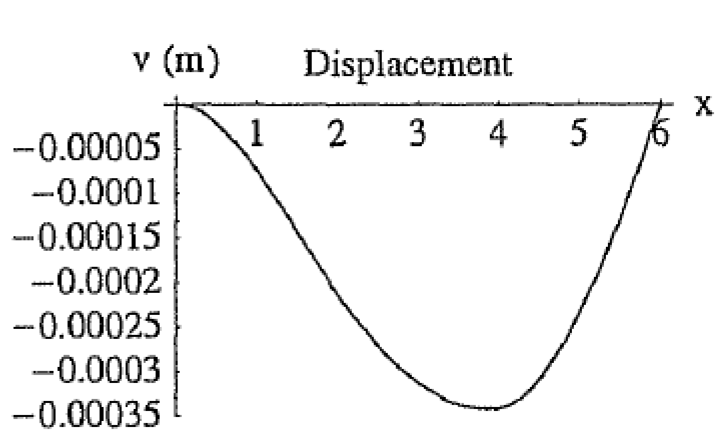
Figure 4.28. Solution with five elements in the distributed load segment



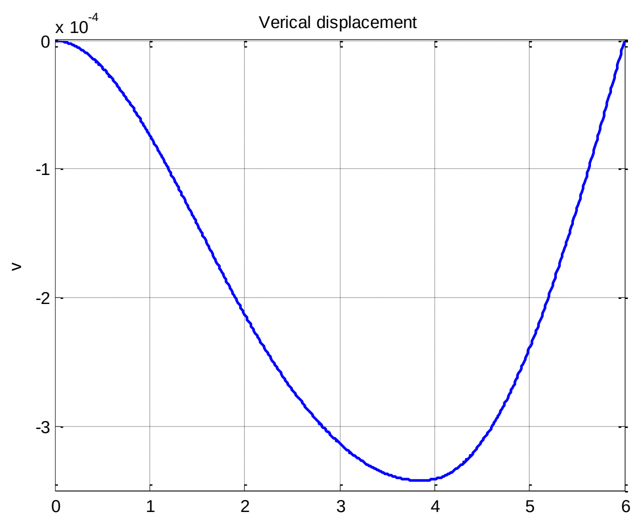


Matlab Results, 3 points in each element

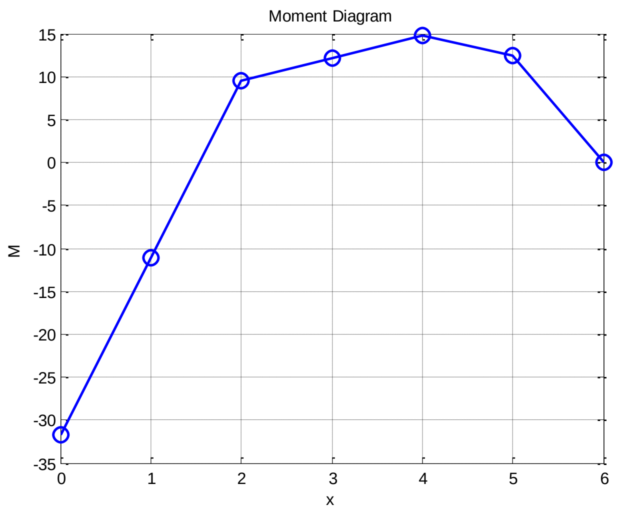


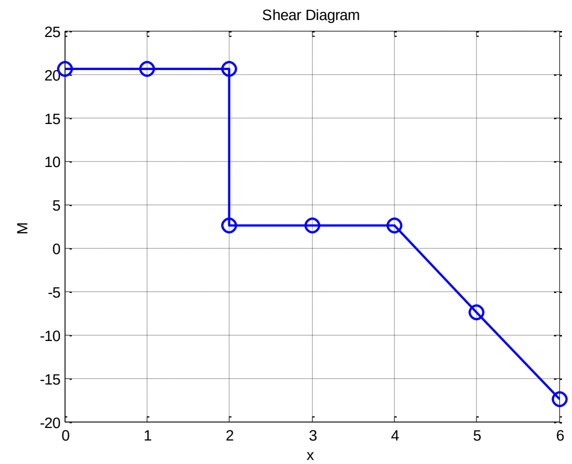


Matlab Results, 100 points in each element

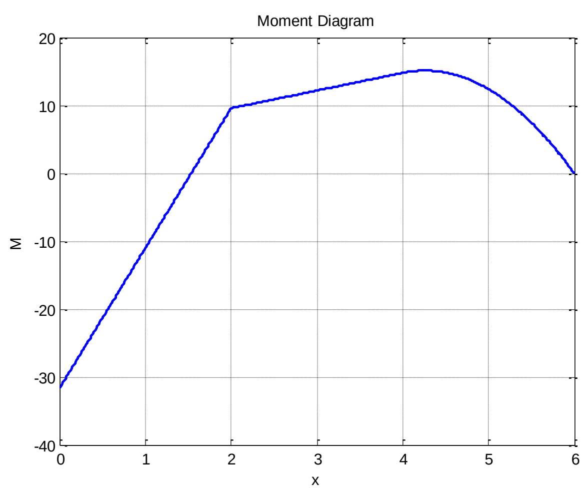


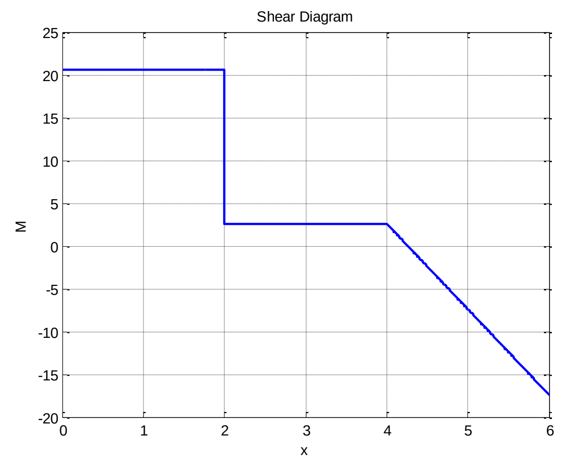
Matlab Results, 3 points in each element





Matlab Results, 100 points in each element





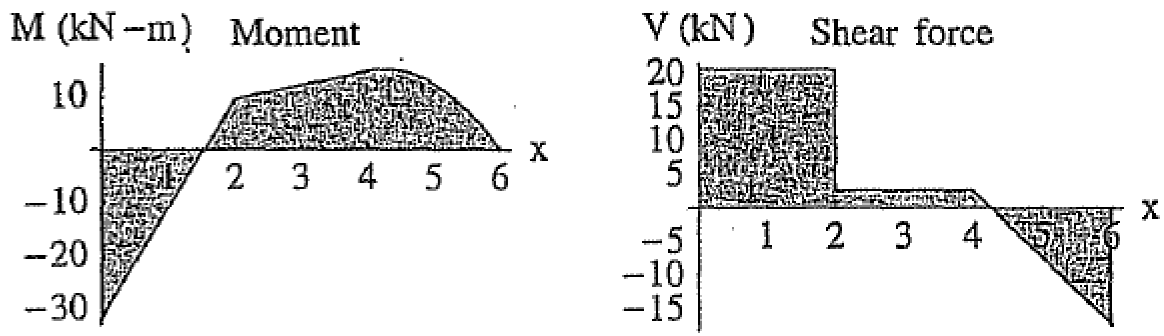
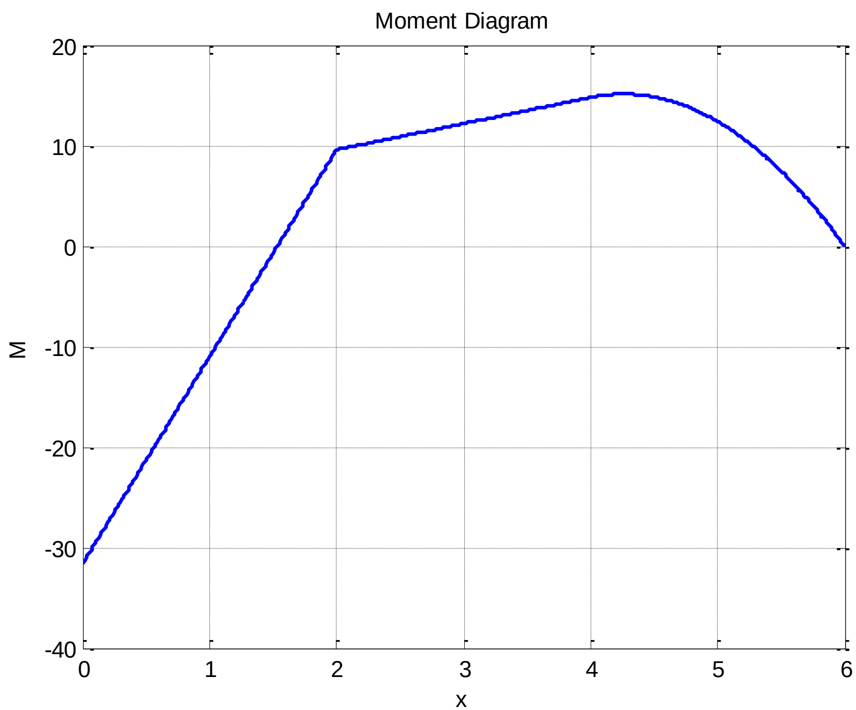
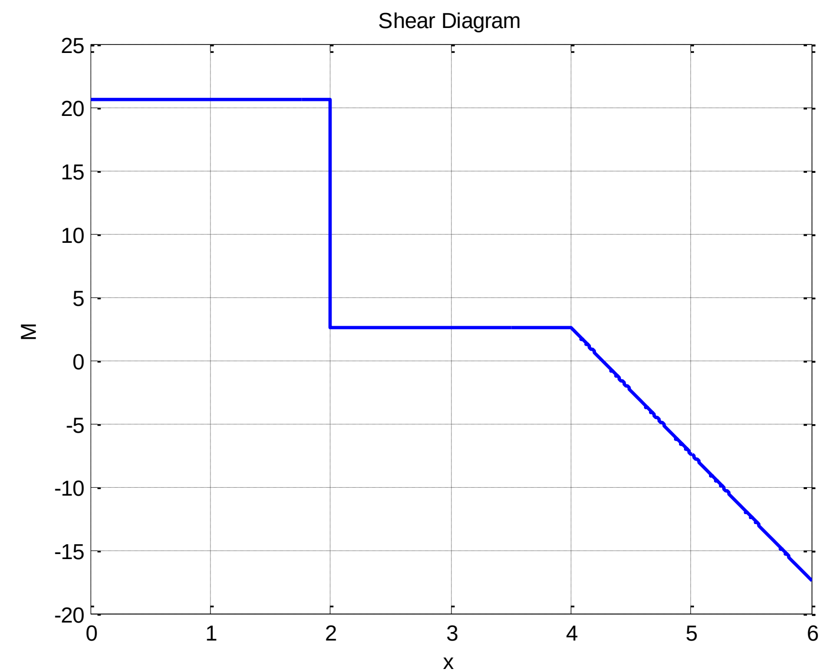


Figure 4.28. Solution with five elements in the distributed load segment





4.4 FINITE ELEMENT FORMULATION OF FRAMES

Frames represent structural members that may be rigidly connected with welded joints or bolted joints. For such structures, in addition to rotation and lateral displacement, we also need to be concerned about axial deformations. Here, we focus on plane frames. The frame element, shown in Figure 4.12, consists of two nodes. At each node, there are three degrees of freedom: a longitudinal displacement, a lateral displacement, and a rotation.

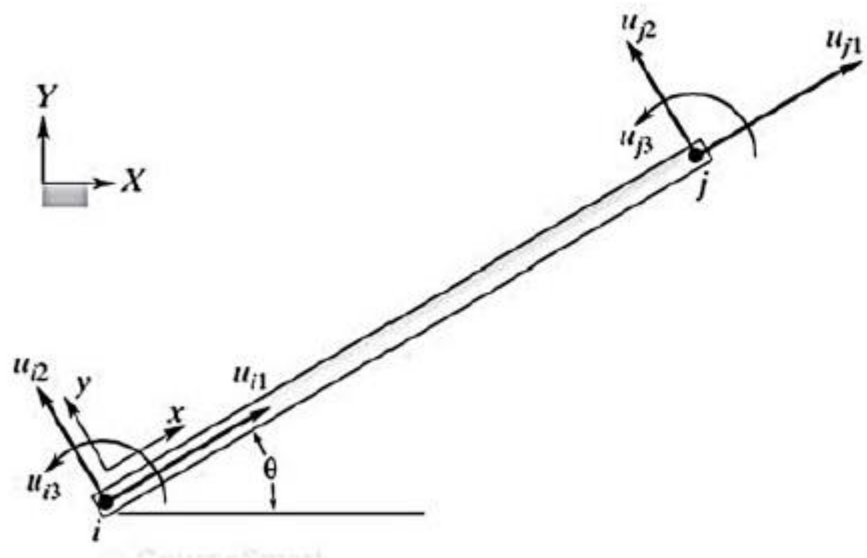


FIGURE 4.12 A frame element.

global coordinate system (X, Y) is shown in Figure 4.12. Because there are three degrees of freedom associated with each node, the stiffness matrix for the frame element will be a 6×6 matrix. The local degrees of freedom are related to the global degrees of freedom through the transformation matrix, according to the relationship

$$[\mathbf{u}] = [\mathbf{T}][\mathbf{U}] \tag{4.51}$$

where the transformation matrix is

$$[\mathrm{T}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.52}$$

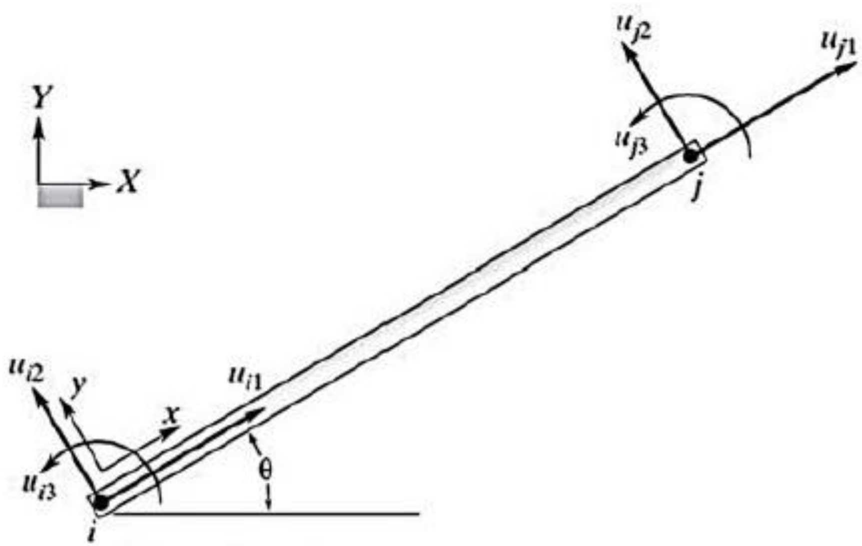


FIGURE 4.12 A frame element.

In the previous section, we developed the stiffness matrix attributed to bending for a beam element. This matrix accounts for lateral displacements and rotations at each node and is

$$[\mathbf{K}]_{xy}^{(e)} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccccc} u_{i1} & u_{i2} & u_{i3} & u_{j1} & u_{j2} & u_{j3} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{array}{c} \\ u_{i1} \\ u_{i2} \\ u_{i3} \\ u_{j1} \\ u_{j2} \\ u_{j3} \end{array} \tag{4.53}$$

To represent the contribution of each term to nodal degrees of freedom, the degrees of freedom are shown above and alongside the stiffness matrix in Eq. (4.53). In Section 4.1 we derived the stiffness matrix for members under axial loading as

$$
[\mathbf{K}]^{(e)}_{\text{axial}} = \begin{array}{c} \begin{array}{cccccc} u_{i1} & u_{i2} & u_{i3} & u_{j1} & u_{j2} & u_{j3} \end{array} \\ \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_{i1} \\ u_{i2} \\ u_{i3} \\ u_{j1} \\ u_{j2} \\ u_{j3} \end{array} \tag{4.54}
$$

Adding Eqs. (4.53) and (4.54) results in the stiffness matrix for a frame element with respect to local coordinate system x, y

$$[\mathbf{K}]_{xy}^{(e)} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{4.55}$$

Note that we need to represent Eq. (4.55) with respect to the global coordinate system. To perform this task, we must substitute for the local displacements in terms of the global displacements in the strain energy equation, using the transformation matrix and performing the minimization. (See Problem 4.13.) These steps result in the relationship

$$[\mathbf{K}]^{(e)} = [\mathbf{T}]^T[\mathbf{K}]_{xy}^{(e)}[\mathbf{T}] \tag{4.56}$$

Example 4.5, Moaveni

Consider the overhang frame shown in Figure 4.13. The frame is made of steel, with $E = 30 \times 10^6$ lb/in^2. The cross-sectional areas and the second moment of areas for the two members are shown in Figure 4.13. The frame is fixed as shown in the figure, and we are interested in determining the deformation of the frame under the given distributed load.

We model the problem using two elements. For element (1), the relationship between the local and the global coordinate systems is shown in Figure 4.14.

Similarly, the relationship between the coordinate systems for element (2) is shown in Figure 4.15.

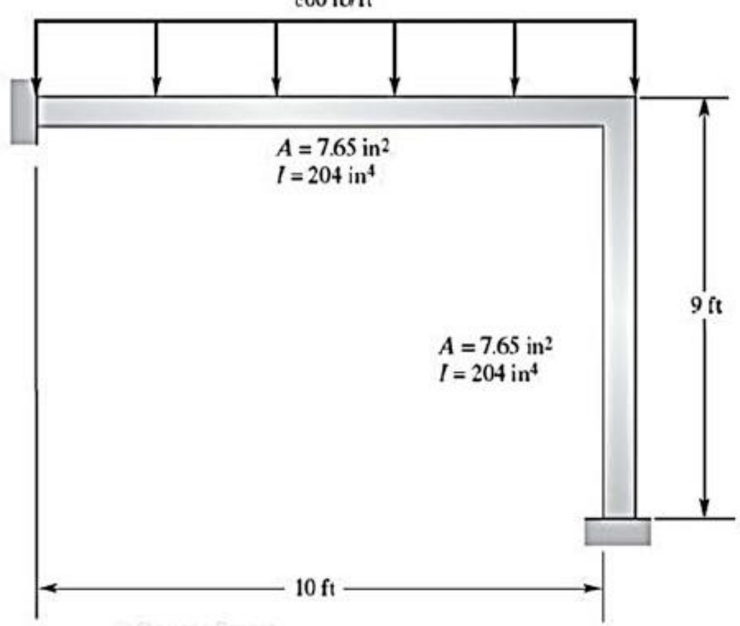


FIGURE 4.13 An overhang frame supporting a distributed load.

Note that for this problem, the boundary conditions are $U_{11} = U_{12} = U_{13} = U_{31} = U_{32} = U_{33} = 0$. For element (1), the local and the global frames of reference are aligned in the same direction; therefore, the stiffness matrix for element (1) can be computed from Eq. (4.55) resulting in

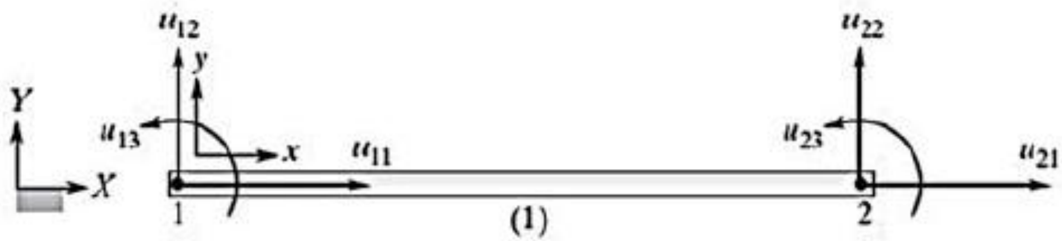


FIGURE 4.14 The configuration of element (1).

$$[\mathbf{K}]^{(1)} = 10^3 \begin{bmatrix} 1912.5 & 0 & 0 & -1912.5 & 0 & 0 \\ 0 & 42.5 & 2550 & 0 & -42.5 & 2550 \\ 0 & 2550 & 204000 & 0 & -2550 & 102000 \\ -1912.5 & 0 & 0 & 1912.5 & 0 & 0 \\ 0 & -42.5 & -2550 & 0 & 42.5 & -2550 \\ 0 & 2550 & 102000 & 0 & -2550 & 204000 \end{bmatrix}$$

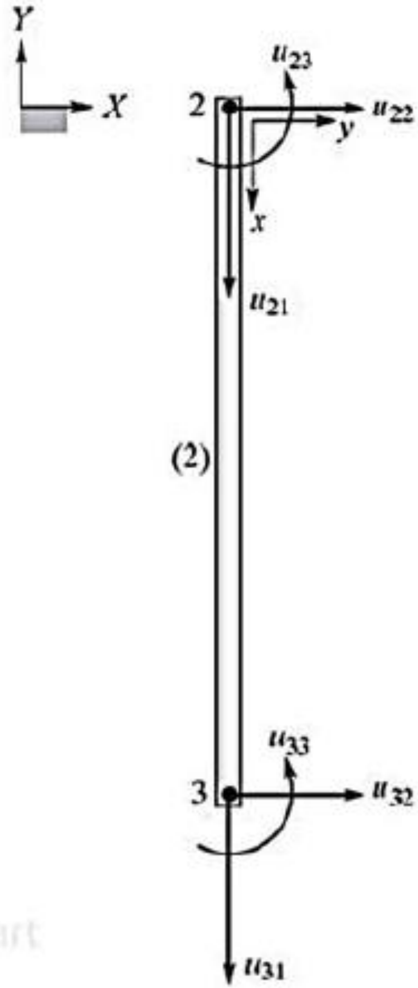


FIGURE 4.15 The configuration of element (2).

For element (2), the stiffness matrix represented with respect to the local coordinate system is

$$[\mathbf{K}]_{xy}^{(2)} = 10^3 \begin{bmatrix} 2125 & 0 & 0 & -2125 & 0 & 0 \\ 0 & 58.299 & 3148.148 & 0 & -58.299 & 3148.148 \\ 0 & 3148.148 & 226666 & 0 & -3148.148 & 113333 \\ -2125 & 0 & 0 & 2125 & 0 & 0 \\ 0 & -58.299 & -3148.148 & 0 & 58.299 & -3148.148 \\ 0 & 3148.148 & 113333 & 0 & -3148.148 & 226666 \end{bmatrix}$$

For element (2), the transformation matrix is

$$[\mathbf{T}] = \begin{bmatrix} \cos(270) & \sin(270) & 0 & 0 & 0 & 0 \\ -\sin(270) & \cos(270) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(270) & \sin(270) & 0 \\ 0 & 0 & 0 & -\sin(270) & \cos(270) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{T}] = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{K}]^{(e)} = [\mathbf{T}]^T[\mathbf{K}]^{(e)}_{xy}[\mathbf{T}] \tag{4.56}$$

Substituting for $[\mathbf{T}]^T$, $[\mathbf{K}]^{(2)}_{xy}$, and $[\mathbf{T}]$ into Eq. (4.56), we have

$$[\mathbf{K}]^{(2)} = 10^3 \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2125 & 0 & 0 & -2125 & 0 & 0 \\ 0 & 58.299 & 3148.148 & 0 & -58.299 & 3148.148 \\ 0 & 3148.148 & 226666 & 0 & -3148.148 & 113333 \\ -2125 & 0 & 0 & 2125 & 0 & 0 \\ 0 & -58.299 & -3148.148 & 0 & 58.299 & -3148.148 \\ 0 & 3148.148 & 113333 & 0 & -3148.148 & 226666 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and performing the matrix operation, we obtain

$$[\mathbf{K}]^{(2)} = 10^3 \begin{bmatrix} 58.299 & 0 & 3148.148 & -58.299 & 0 & 3148.148 \\ 0 & 2125 & 0 & 0 & -2125 & 0 \\ 3148.148 & 0 & 226666 & -3148.148 & 0 & 113333 \\ -58.299 & 0 & -3148.148 & 58.299 & 0 & -3148.1480 \\ 0 & -2125 & 0 & 0 & 2125 & 0 \\ 3148.148 & 0 & 113333 & -3148.148 & 0 & 226666 \end{bmatrix}$$

Constructing the global stiffness matrix by assembling $[\mathbf{K}]^{(1)}$ and $[\mathbf{K}]^{(2)}$, we have

$$[\mathbf{K}]^{(2)} = 10^3 \begin{bmatrix} 1912.5 & 0 & 0 & -1912.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 42.5 & 2550 & 0 & -42.5 & 2550 & 0 & 0 & 0 \\ 0 & 2550 & 204000 & 0 & -2550 & 102000 & 0 & 0 & 0 \\ -1912.5 & 0 & 0 & 1912.5 + 58.299 & 0 & 0 + 3148.148 & -58.299 & 0 & 3148.148 \\ 0 & -42.5 & -2550 & 0 & 42.5 + 2125 & -2550 & 0 & -2125 & 0 \\ 0 & 2550 & 102000 & 0 + 3148.148 & -2550 & 204000 + 226666 & -3148.148 & 0 & 113333 \\ 0 & 0 & 0 & -58.299 & 0 & -3148.148 & 58.299 & 0 & -3148.1480 \\ 0 & 0 & 0 & 0 & -2125 & 0 & 0 & 2125 & 0 \\ 0 & 0 & 0 & 3148.148 & 0 & 113333 & -3148.148 & 0 & 226666 \end{bmatrix}$$

The load matrix is

$$\{\mathbf{F}\}^{(1)} = \begin{Bmatrix} 0 \\ -\dfrac{wL}{2} \\ -\dfrac{wL^2}{12} \\ 0 \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\dfrac{800 \times 10}{2} \\ -\dfrac{800 \times 10^2 \times 12}{12} \\ 0 \\ -\dfrac{800 \times 10}{2} \\ \dfrac{800 \times 10^2 \times 12}{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -4000 \\ -80000 \\ 0 \\ -4000 \\ 80000 \end{Bmatrix}$$

In the load matrix, the force terms have the units of lb, whereas the moment terms have the units of lb · in. Application of the boundary conditions ($U_{11} = U_{12} = U_{13} = U_{31} = U_{32} = U_{33} = 0$) reduces the 9×9 global stiffness matrix to the following 3×3 matrix:

$$10^3 \begin{bmatrix} 1970.799 & 0 & 3148.148 \\ 0 & 2167.5 & -2550 \\ 3148.148 & -2550 & 430666 \end{bmatrix} \begin{Bmatrix} U_{21} \\ U_{22} \\ U_{23} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -4000 \\ 80000 \end{Bmatrix}$$

Solving these equations simultaneously results in the following displacement matrix:

$$[\mathbf{U}]^T = [0 \quad 0 \quad 0 \quad -0.0002845(\text{in}) \quad -0.0016359(\text{in}) \quad 0.00017815(\text{rad}) \quad 0 \quad 0 \quad 0]$$