

Finite Element Analysis

Lecture 6

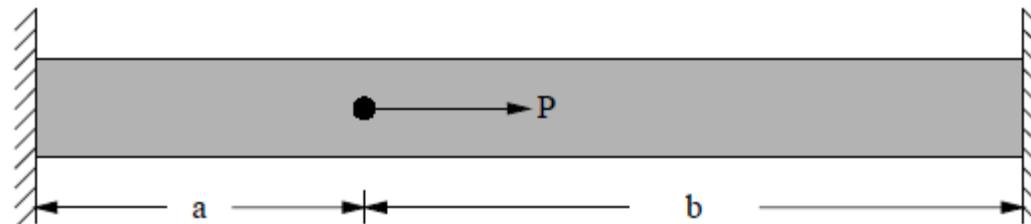
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Example

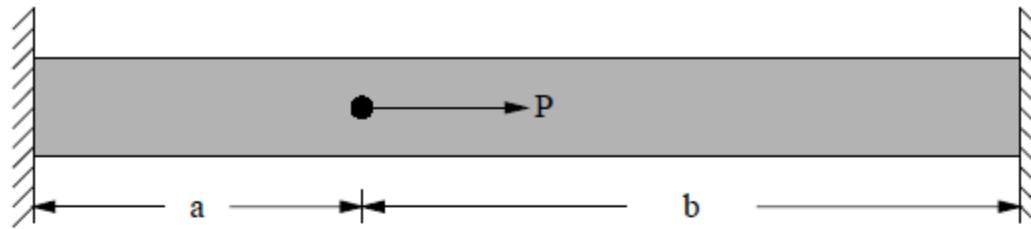
A bar of constant cross section A and modulus of elasticity E is attached at both ends to rigid supports and is loaded axially by force P as shown in the Figure 2.32. Find axial displacement and force distribution using only two linear axial deformation elements. Use the following numerical values.

$$a = 300 \text{ mm}; \quad b = 600 \text{ mm}; \quad E = 200 \text{ GPa}; \quad A = 200 \text{ mm}^2; \quad P = 10 \text{ kN}$$

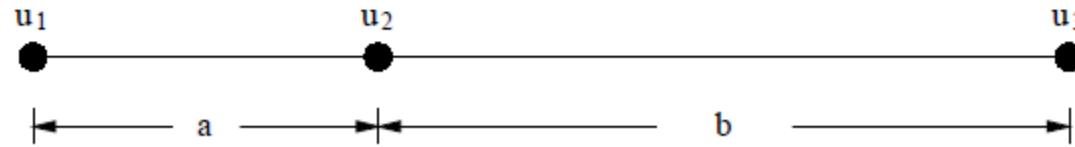


Axially loaded bar

Solution



Axially loaded bar



Enter E in MPa, load in N and dimensions in mm units. The displacements will be in mm, axial forces in N and the stresses in MPa.

The concentrated nodal loads will be added directly to the global equations after assembly. Thus the element equations are as follows.

$$\frac{EA}{x_2 - x_1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 1

Element nodes: $\{x_1 \rightarrow 0, x_2 \rightarrow 300\}$

$$\begin{pmatrix} \frac{400000}{3} & -\frac{400000}{3} \\ -\frac{400000}{3} & \frac{400000}{3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} \frac{400000}{3} & -\frac{400000}{3} & 0 \\ -\frac{400000}{3} & \frac{400000}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Element 2

Element nodes: $\{x_2 \rightarrow 300, x_3 \rightarrow 900\}$

$$\begin{pmatrix} \frac{200000}{3} & -\frac{200000}{3} \\ -\frac{200000}{3} & \frac{200000}{3} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Global equations after assembly of this element

$$\begin{pmatrix} \frac{400000}{3} & -\frac{400000}{3} & 0 \\ -\frac{400000}{3} & 200000 & -\frac{200000}{3} \\ 0 & -\frac{200000}{3} & \frac{200000}{3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Global equations before boundary conditions

$$\begin{pmatrix} \frac{400000}{3} & -\frac{400000}{3} & 0 \\ -\frac{400000}{3} & 200000 & -\frac{200000}{3} \\ 0 & -\frac{200000}{3} & \frac{200000}{3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Incorporating the concentrated load applied at node 2 the global equations are as follows.

$$\begin{pmatrix} \frac{400000}{3} & -\frac{400000}{3} & 0 \\ -\frac{400000}{3} & 200000 & -\frac{200000}{3} \\ 0 & -\frac{200000}{3} & \frac{200000}{3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10000 \\ 0 \end{pmatrix}$$

Essential boundary conditions

DOF	Value
u_1	0
u_3	0

Incorporating EBC the final system of equations is

$$(200000)(u_2) = (10000)$$

Solution for nodal unknowns

DOF	x	Solution
u_1	0	0
u_2	300	$\frac{1}{20}$
u_3	900	0

Solution over elements

Element 1

Nodes: $\{x_1 \rightarrow 0, x_2 \rightarrow 300\}$

Interpolation functions: $N^T = \left\{1 - \frac{x}{300}, \frac{x}{300}\right\}$

Nodal values: $d^T = \left\{0, \frac{1}{20}\right\}$

Solution: $u(x) = N^T d = \frac{x}{6000}$

Element 2

Nodes: $\{x_1 \rightarrow 300, x_2 \rightarrow 900\}$

Interpolation functions: $N^T = \left\{\frac{3}{2} - \frac{x}{600}, \frac{x}{600} - \frac{1}{2}\right\}$

Nodal values: $d^T = \left\{\frac{1}{20}, 0\right\}$

Solution: $u(x) = N^T d = \frac{3}{40} - \frac{x}{12000}$

Solution summary

	Range	Solution
1	$0 \leq x \leq 300$	$\frac{x}{6000}$
2	$300 \leq x \leq 900$	$\frac{3}{40} - \frac{x}{12000}$

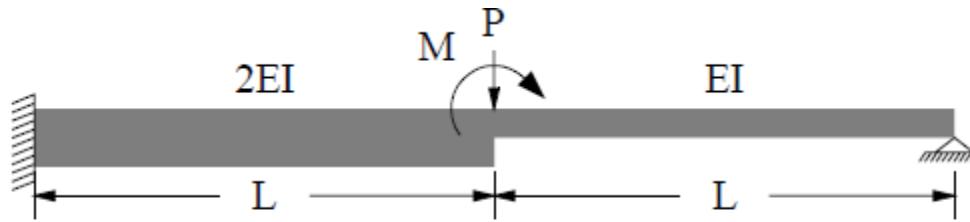
From these displacements we get the following axial strains, stresses and axial forces.

$$\epsilon_x = \frac{du}{dx}; \quad \sigma_x = E \epsilon_x; \quad F = A \sigma_x$$

	Range	ϵ	σ	F
1	$0 \leq x \leq 300$	$\frac{1}{6000}$	$\frac{100}{3}$	$\frac{20000}{3}$
2	$300 \leq x \leq 900$	$-\frac{1}{12000}$	$-\frac{50}{3}$	$-\frac{10000}{3}$

Example

Find displacements and draw shear force and bending moment diagrams for the beam shown in Figure 4.65. Assume $E = 210 \text{ GPa}$, $I = 4 \times 10^{-4} \text{ m}^4$, $L = 2 \text{ m}$, $P = 10 \text{ kN}$, $M = 20 \text{ kN}\cdot\text{m}$.



Beam with variable section

Element equations

1

2

3

Element stiffness matrix in global coordinates

$$\text{Element 1} \rightarrow \begin{pmatrix} 252000. & 252000. & -252000. & 252000. \\ 252000. & 336000. & -252000. & 168000. \\ -252000. & -252000. & 252000. & -252000. \\ 252000. & 168000. & -252000. & 336000. \end{pmatrix}$$

$$\text{Element 2} \rightarrow \begin{pmatrix} 126000. & 126000. & -126000. & 126000. \\ 126000. & 168000. & -126000. & 84000. \\ -126000. & -126000. & 126000. & -126000. \\ 126000. & 84000. & -126000. & 168000. \end{pmatrix}$$

Global stiffness matrix and equivalent distributed load vector

After Assembly of Element 1:

$$\begin{pmatrix} 252000 & 252000 & -252000 & 252000 & 0 & 0 \\ 252000 & 336000 & -252000 & 168000 & 0 & 0 \\ -252000 & -252000 & 252000 & -252000 & 0 & 0 \\ 252000 & 168000 & -252000 & 336000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

After Assembly of Element 2:

$$\begin{pmatrix} 252000 & 252000 & -252000 & 252000 & 0 & 0 \\ 252000 & 336000 & -252000 & 168000 & 0 & 0 \\ -252000 & -252000 & 378000 & -126000 & -126000 & 126000 \\ 252000 & 168000 & -126000 & 504000 & -126000 & 84000 \\ 0 & 0 & -126000 & -126000 & 126000 & -126000 \\ 0 & 0 & 126000 & 84000 & -126000 & 168000 \end{pmatrix}$$

Global load vector and solution

Add concentrated loads to the load vector and solve for nodal displacements

$$\{0., 0., -0.0000881834, -0.0000793651, 0., 0.00010582\}$$

Solution for element quantities

Quantities for Element 1:

$$v(s): 0.0000176367 s^3 - 0.00010582 s^2 + 0. s + 0.$$

$$bm(s): 4.44444 s - 8.88889$$

$$V(s): 2.22222$$

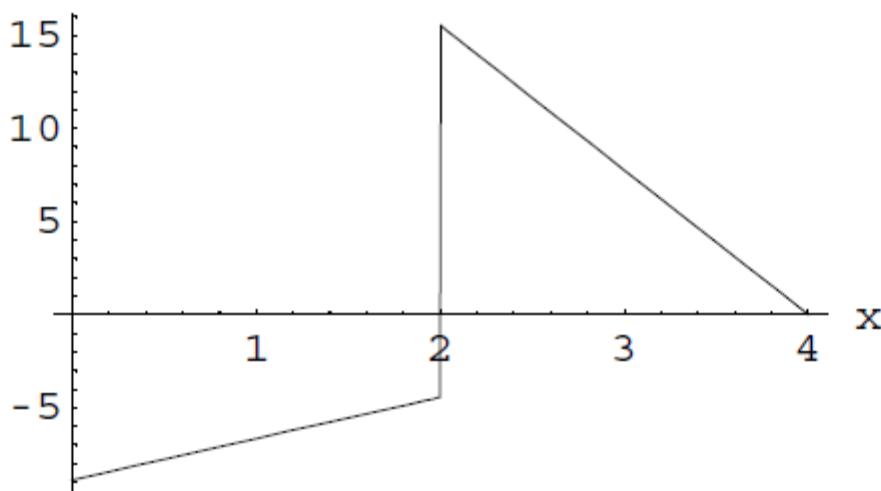
Quantities for Element 2:

$$v(s): -0.000123457 s^3 + 0.00037037 s^2 - 0.00015873 s - 0.0000881834$$

$$bm(s): 15.5556 - 15.5556 s$$

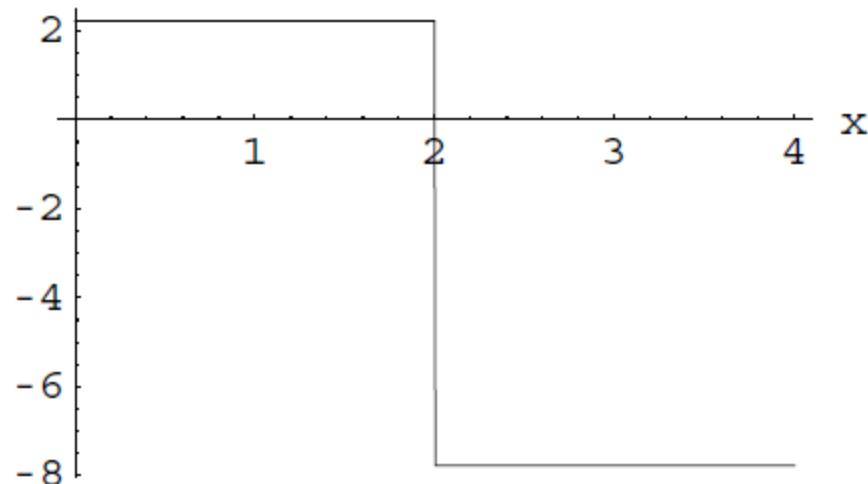
$$V(s): -7.77778$$

Moment

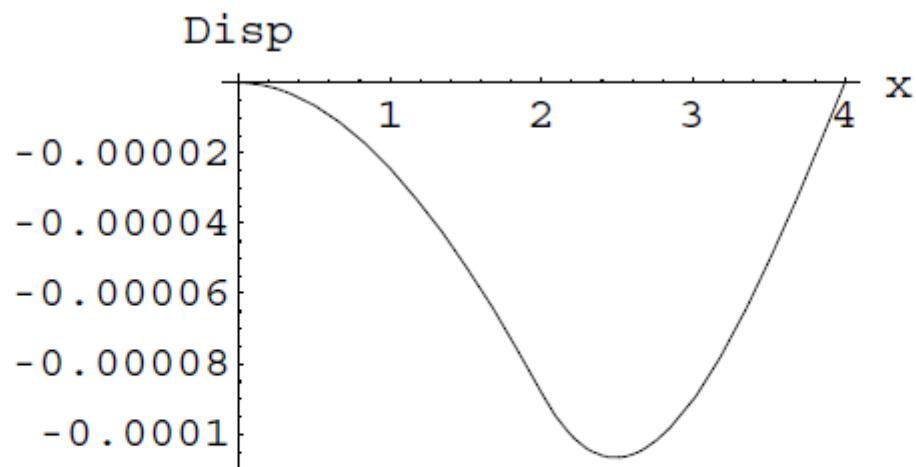


Which[$x \leq 2, -8.88889 + 2.22222x, x \leq 4, 31.1111 - 7.77778x]$]

Shear



Which[x \leq 2, 2.22222, x \leq 4, -7.77778]



Which $[x \leq 2, 0 + 0.x - 0.000026455x^2 + 2.20459 \times 10^{-6} x^3,$
 $x \leq 4, 0.000564374 - 0.000634921x + 0.000185185x^2 - 0.0000154321x^3]$