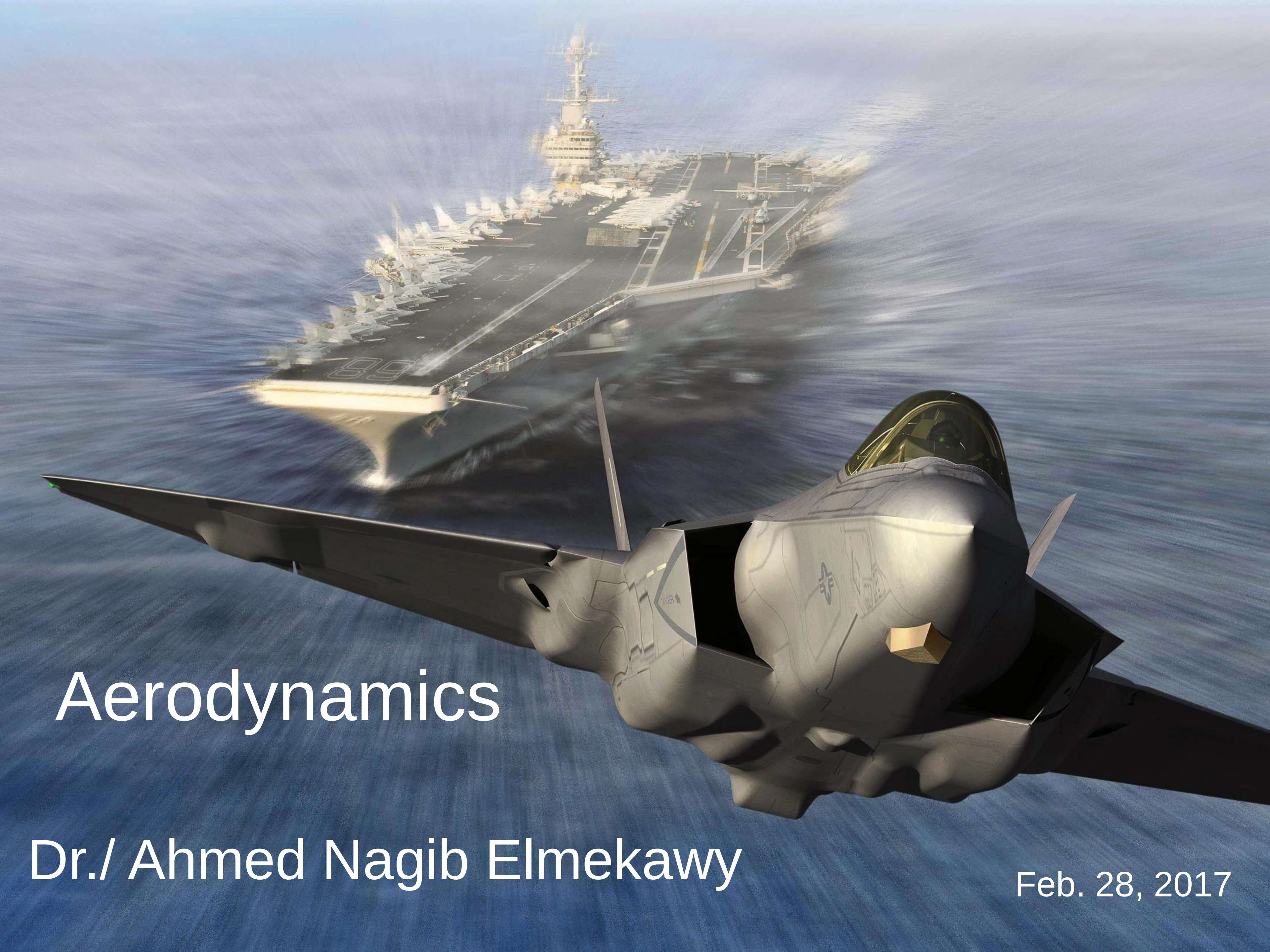

Aerodynamics

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Feb. 28, 2017

Review of Fluid Mechanics

Review: Fluid Kinematics -Acceleration Field

- Consider a fluid particle and Newton's second law,

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

- The acceleration of the particle is the time derivative of the particle's velocity.

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

- However, particle velocity at a point is the same as the fluid velocity,
- To take the time derivative of V, chain rule must be used because,

$$\vec{V}_{particle} = \vec{V}\left(x_{particle}(t), y_{particle}(t), z_{particle}(t)\right)$$

$$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x}\frac{dx_{particle}}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy_{particle}}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz_{particle}}{dt}$$

Acceleration Field

- Since $\dfrac{dx_{particle}}{dt} = u, \dfrac{dy_{particle}}{dt} = v, \dfrac{dz_{particle}}{dt} = w$

  then $$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

- In vector form, the acceleration can be written as

$$\vec{a}(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}$$

Where:

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

- First term is called the **local acceleration** and is nonzero only for unsteady flows.
- Second term is called the **advective acceleration** and accounts for the effect of the fluid particle moving to a new location in the flow, where the velocity is different.

Acceleration Components

The components of the acceleration are:

Vector equation: $$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

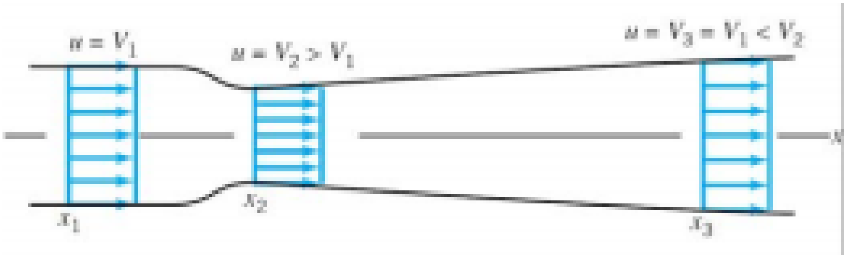
x- component $$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Y-component $$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

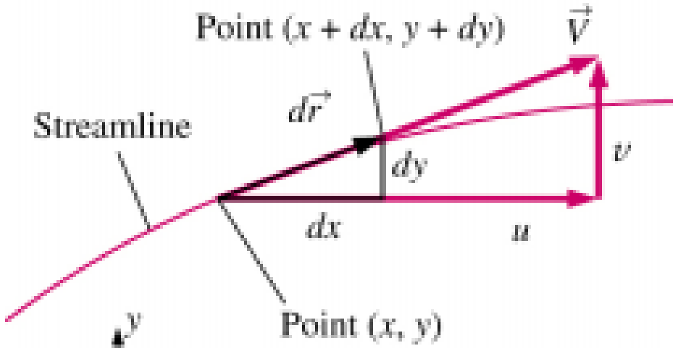
z-component $$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

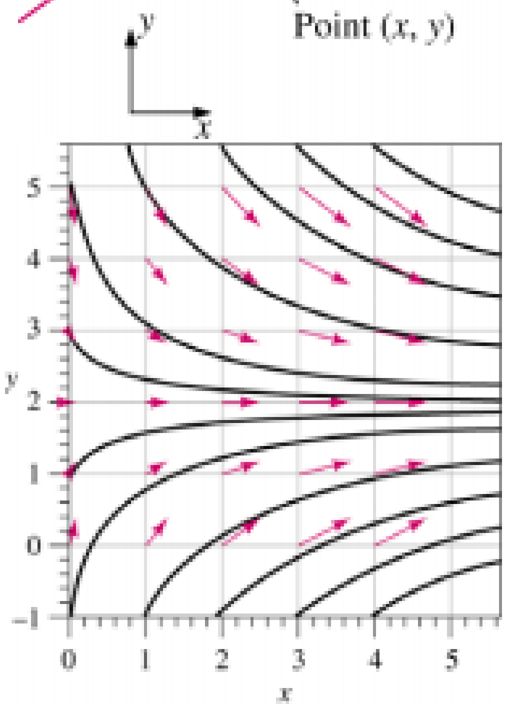
Question: Give examples of steady flows with acceleration

Incompressible Steady ideal flow in a variable-area duct



Streamlines





- A **Streamline** is a curve that is everywhere tangent to the *instantaneous* local velocity vector.
- Consider an arc length

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

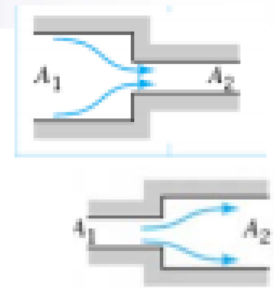
- $d\vec{r}$ must be parallel to the local velocity vector

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

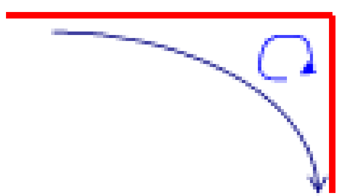
- Geometric arguments results in the equation for a streamline

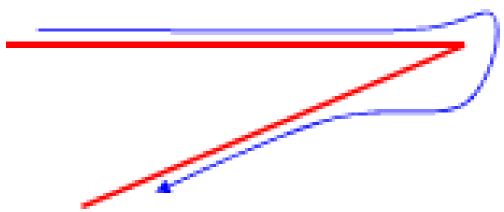
$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \longrightarrow \boxed{\frac{dy}{dx} = \frac{v}{u}}$$

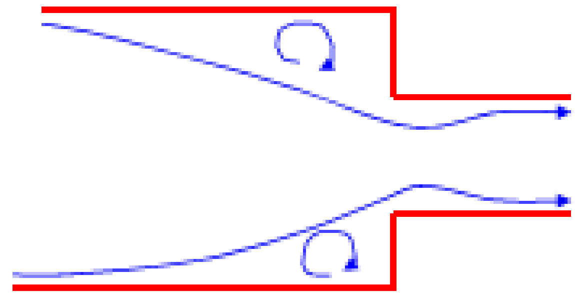
Properties of Streamlines

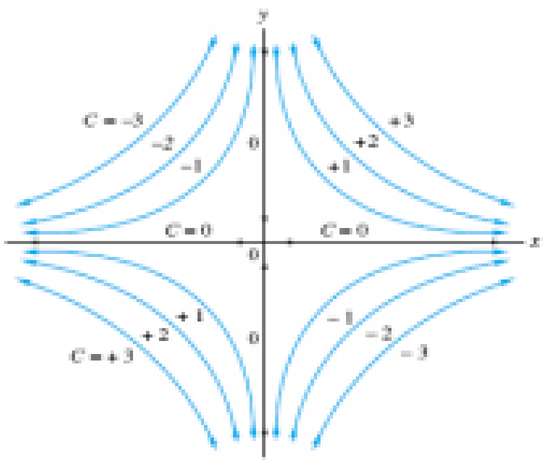


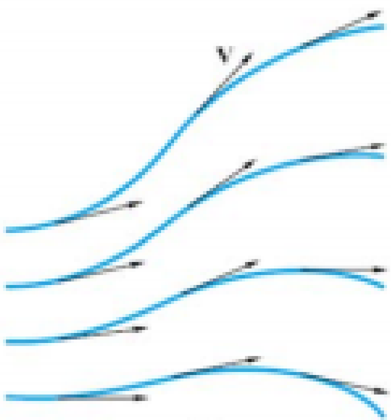
- A streamline can not have a sharp corner
- Two streamlines can not intersect or meet at a sharp corner
- No flow across a tube formed by streamlines (steam tube).



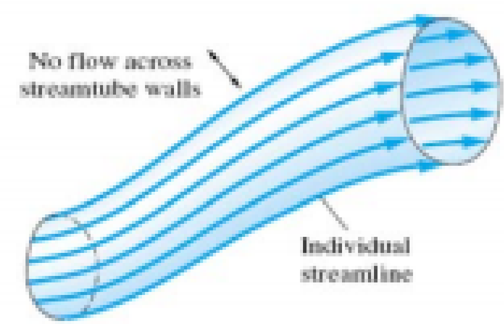






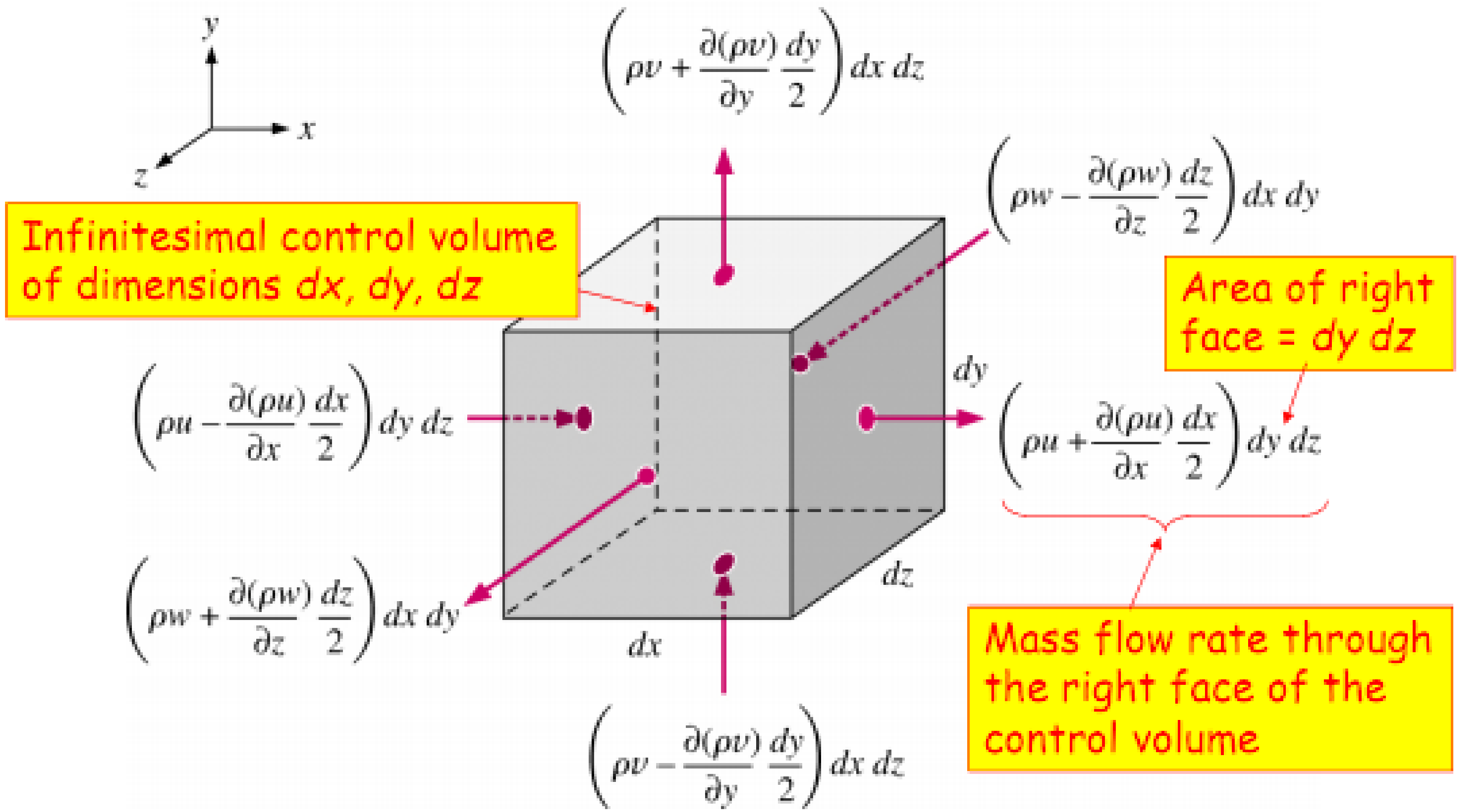


(a)



(b)

Conservation of Mass Differential CV



Conservation of Mass Differential CV

- Now, sum up the mass flow rates into and out of the 6 faces of the CV

Net mass flow rate into CV:

$$\sum_{in} \dot{m} \approx \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\,dz + \left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\,dz + \left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\,dy$$

Net mass flow rate out of CV:

Plug into integral conservation of mass equation

$$\sum_{out} \dot{m} \approx \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\,dz + \left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\,dz + \left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\,dy$$

Rate of change of the control volume mass:

$$\frac{\partial \rho}{\partial t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{CV} \frac{\partial \rho}{\partial t}\,\mathrm{d}\mathcal{V} = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$

Conservation of Mass Differential CV

- After substitution,

$$\frac{\partial \rho}{\partial t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = -\frac{\partial(\rho u)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \frac{\partial(\rho v)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \frac{\partial(\rho w)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

- Dividing through by volume *dxdydz*

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

Or, if we apply the definition of the divergence of a vector

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0}$$

where

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Conservation of Mass: *Alternative forms*

- Use product rule on divergence term

$$\frac{\partial \rho}{\partial t} + \boxed{\vec{\nabla} \cdot \left(\rho \vec{V} \right)} = \underbrace{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho}_{} + \rho \nabla \cdot \vec{V} = 0$$

$$\boxed{\frac{D\rho}{Dt}} + \rho \nabla \cdot \vec{V} = 0$$

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{V} = 0}$$

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Conservation of Mass: *Cylindrical coordinates*

- There are many problems which are simpler to solve if the equations are written in cylindrical-polar coordinates
- Easiest way to convert from Cartesian is to use vector form and definition of divergence operator in cylindrical coordinates

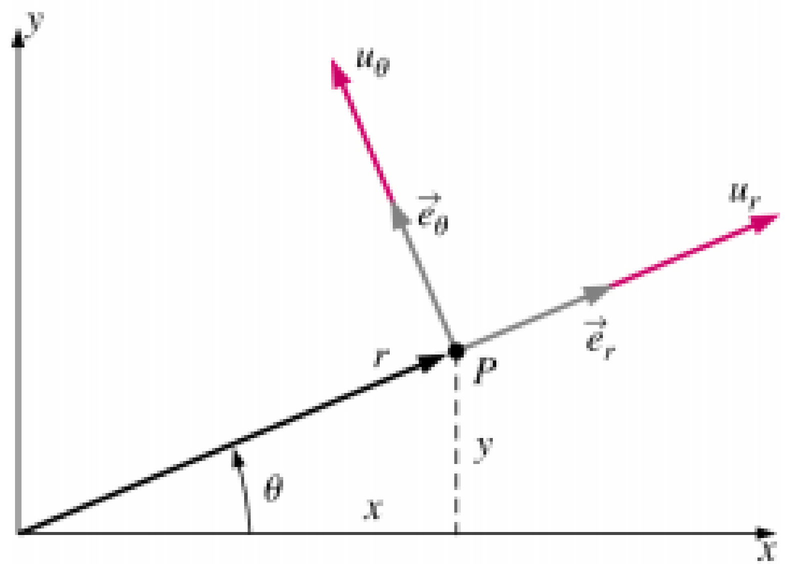
The Divergence Operation

Cartesian coordinates:

$$\vec{\nabla}\cdot(\rho\vec{V}) = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)$$

Cylindrical coordinates:

$$\vec{\nabla}\cdot(\rho\vec{V}) = \frac{1}{r}\frac{\partial(r\rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial \theta} + \frac{\partial(\rho u_z)}{\partial z}$$



Conservation of Mass: *Cylindrical coordinates*

$$\vec{\nabla} = \frac{1}{r}\frac{\partial(r)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$$

$$\vec{V} = U_r\hat{e}_r + U_\theta\hat{e}_\theta + U_z\hat{e}_z$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho U_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho U_\theta)}{\partial \theta} + \frac{\partial(\rho U_z)}{\partial z} = 0}$$

Conservation of Mass: *Special Cases*

- Steady compressible flow

$$\cancel{\frac{\partial \rho}{\partial t}} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0$$

$$\boxed{\vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0}$$

Cartesian $$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Cylindrical $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho U_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho U_\theta)}{\partial \theta} + \frac{\partial(\rho U_z)}{\partial z} = 0$$

Conservation of Mass: *Special Cases*

- **Incompressible flow**

ρ = constant, and hence $$\frac{\partial \rho}{\partial t} = 0$$

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\boxed{\vec{\nabla} \cdot \vec{V} = 0}$$

Cartesian $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Cylindrical $$\frac{1}{r}\frac{\partial (rU_r)}{\partial r} + \frac{1}{r}\frac{\partial (U_\theta)}{\partial \theta} + \frac{\partial (U_z)}{\partial z} = 0$$

Conservation of Mass

- In general, continuity equation cannot be used by itself to solve for flow field, however it can be used to
  1. Determine if a velocity field represents a flow.
  2. Find missing velocity component

Example

For an incompressible flow

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z$$

$$w = ?$$

Determine: w , required to satisfy the continuity equation.

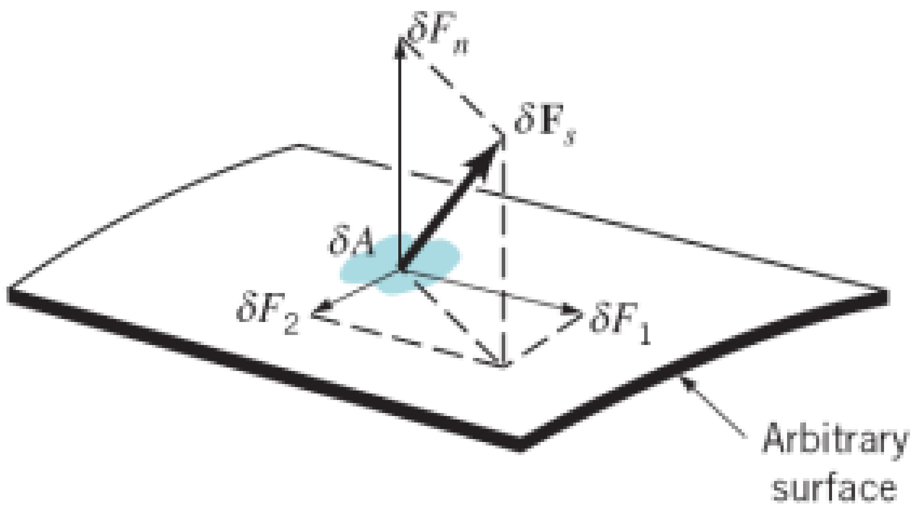
$$\text{Solution}: w = -3xz - \frac{z^2}{2} + c(x, y)$$

Conservation of Momentum

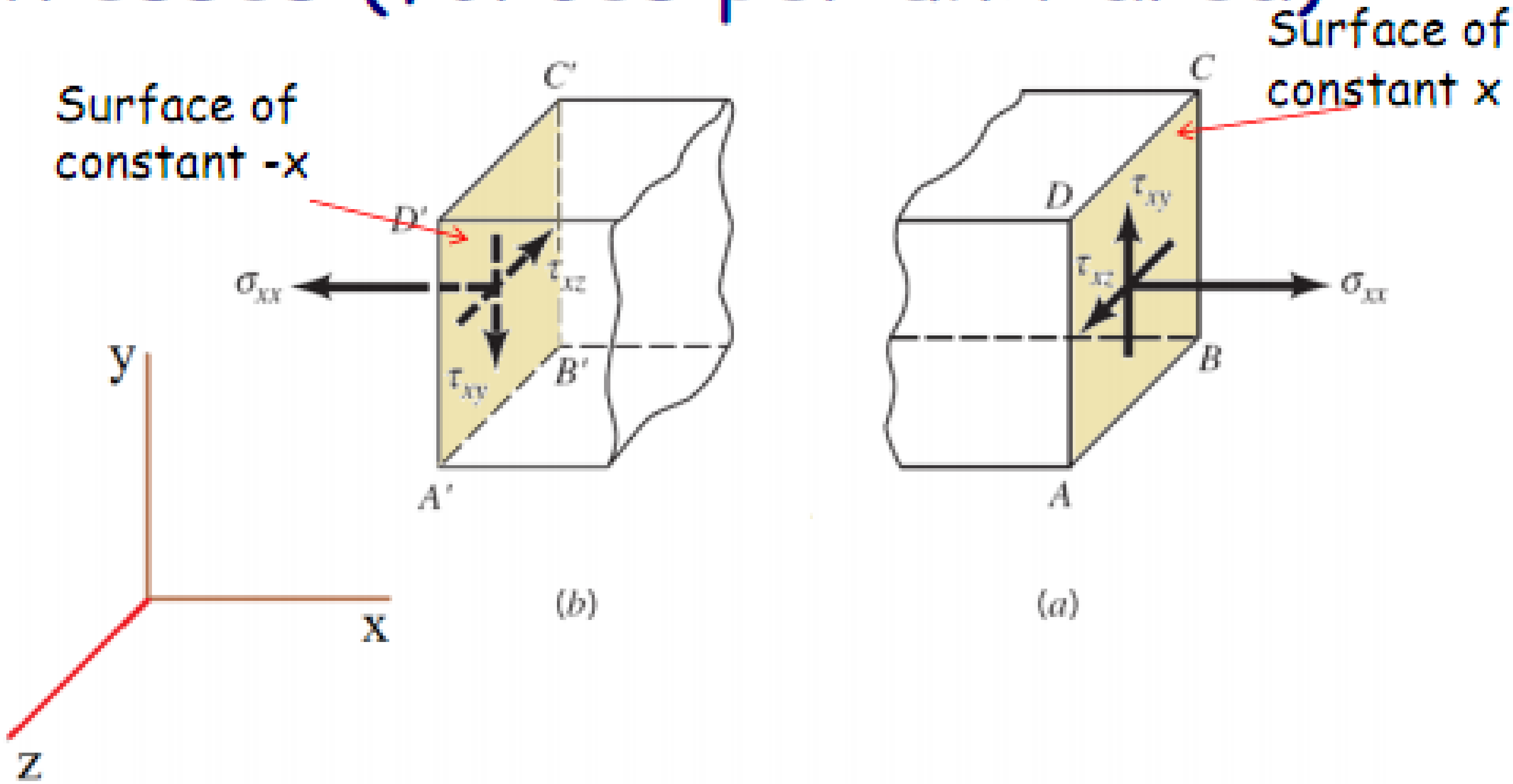
Types of forces:

1. Surface forces: include all forces acting on the boundaries of a medium though direct contact such as pressure, friction,…etc.
2. Body forces are developed without physical contact and distributed over the volume of the fluid such as gravitational and electromagnetic.

- The force δF acting on δA may be resolved into two components, one normal and the other tangential to the area.



Stresses (forces per unit area)

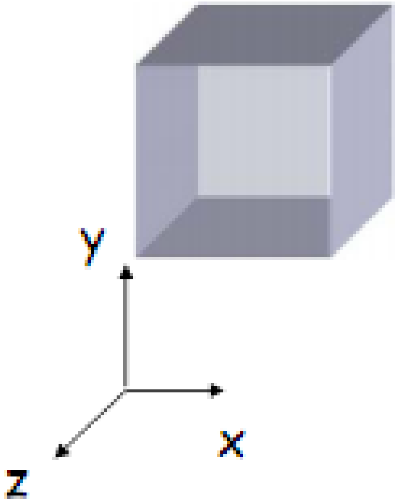


Double subscript notation for stresses.

- First subscript refers to the surface
- Second subscript refers to the direction
- Use σ for normal stresses and τ for tangential stresses

Momentum Conservation

- From Newton's second law: Force = mass * acceleration
- Consider a small element $\delta x\ \delta y\ \delta z$ as shown



The element experiences an acceleration

$$m\frac{D\vec{V}}{Dt}=\rho(\delta x\delta y\delta z)\left(\frac{\partial\vec{V}}{\partial t}+u\frac{\partial\vec{V}}{\partial x}+v\frac{\partial\vec{V}}{\partial y}+w\frac{\partial\vec{V}}{\partial z}\right)$$

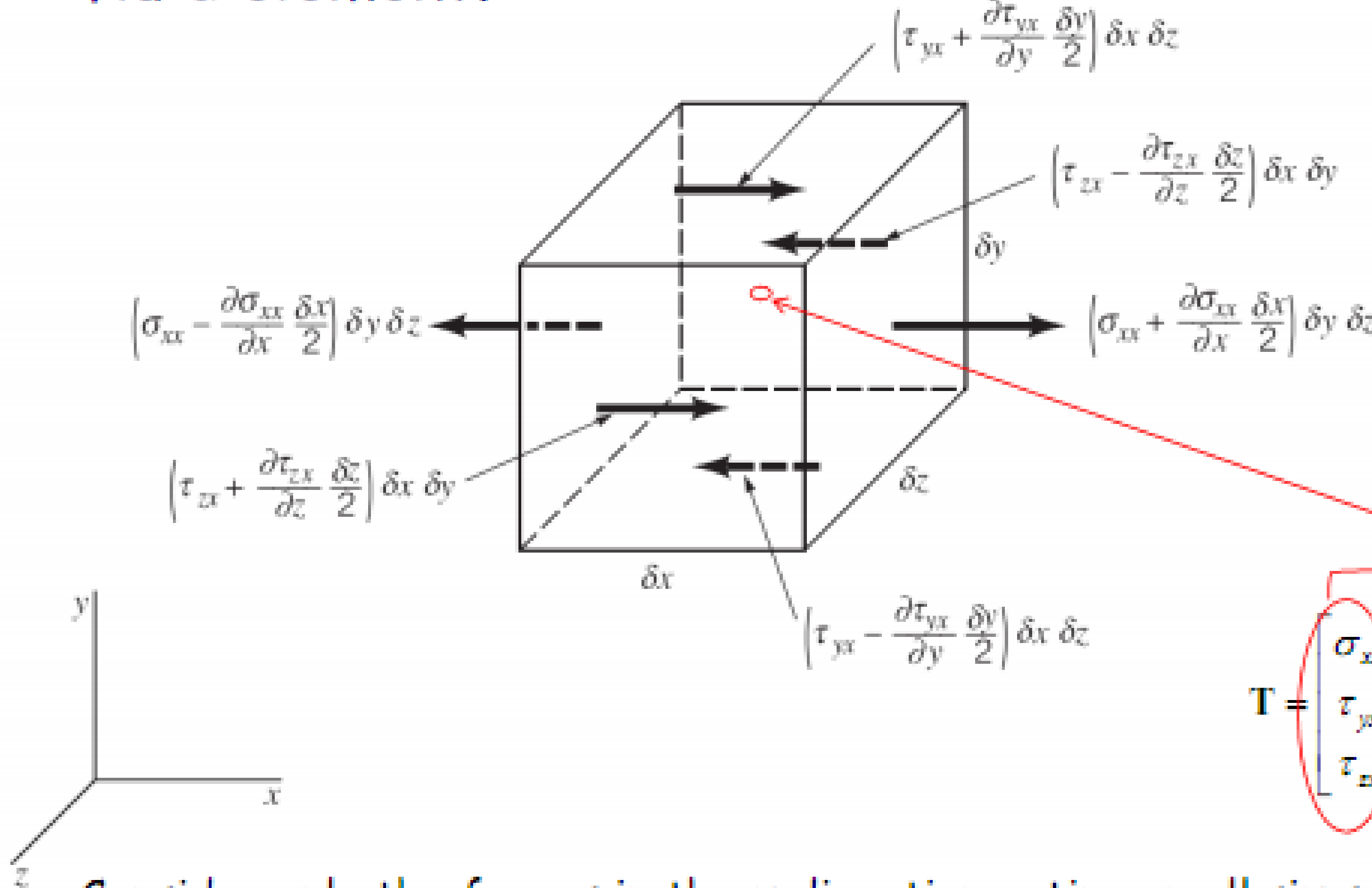
as it is under the action of various forces:

normal stresses, shear stresses, and gravitational force.

- The stresses at the center of the element are presented by the stress tensor:

$$\mathbf{T}=\begin{bmatrix}\sigma_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_{zz}\end{bmatrix}$$

Surface forces in the x direction acting on a fluid element.



$$\mathrm{T} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Consider only the forces in the x direction acting on all six surfaces

Momentum Balance (cont.)

$$\delta F_x = \delta F_{sx} + \delta F_{bx}$$

$$\delta F_x = (\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{\delta x}{2})\delta y\delta z - (\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{\delta x}{2})\delta y\delta z + (\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{\delta y}{2})\delta x\delta z$$

$$-(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{\delta y}{2})\delta x\delta z + (\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2})\delta x\delta y - (\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2})\delta x\delta y + \rho g_x \delta x\delta y\delta z$$

Net force acting along the x-direction:

$$\underbrace{\boxed{\frac{\partial \sigma_{xx}}{\partial x}\delta x\delta y\delta z}}_{\text{Normal stress}} + \underbrace{\boxed{\frac{\partial \tau_{yx}}{\partial x}\delta x\delta y\delta z + \frac{\partial \tau_{zx}}{\partial x}\delta x\delta y\delta z}}_{\text{Shear stresses}} + \underbrace{\boxed{\rho g_x \delta x\delta y\delta z}}_{\text{Body force}}$$

$$\delta F_x = \delta m a_x \Rightarrow \delta F_{bx} + \delta F_{sx} = \delta m a_x$$

The differential equation of motion in the x-direction is:

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - - - - - - -(a)$$

Similar equations can be obtained for the other two directions

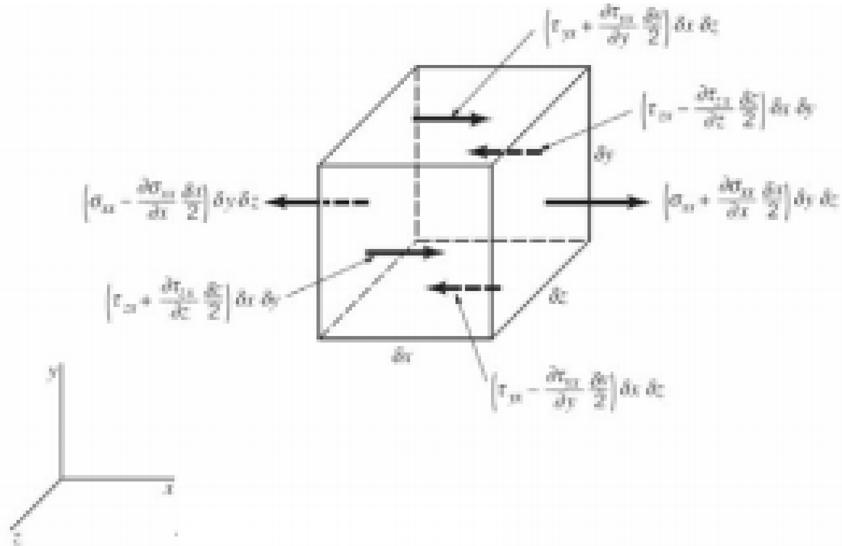
Forces acting on the element

Surface forces

$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \delta x \delta y \delta z$$

$$\delta F_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) \delta x \delta y \delta z$$

$$\delta F_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \delta x \delta y \delta z$$



Body forces

$$\delta F_{bx} = \rho g_x \delta x \delta y \delta z$$

$$\delta F_{by} = \rho g_y \delta x \delta y \delta z$$

$$\delta F_{bz} = \rho g_z \delta x \delta y \delta z$$

Equation of Motion (momentum eq.s)

$$
\begin{aligned}
\delta F_x &= \delta m a_x \qquad & & \delta F_{bx} + \delta F_{sx} = \delta m a_x \\
\delta F_x &= \delta m a_x \qquad & \Rightarrow \qquad & \delta F_{by} + \delta F_{sy} = \delta m a_y \\
\delta F_z &= \delta m a_z \qquad & & \delta F_{bz} + \delta F_{sz} = \delta m a_z
\end{aligned}
$$

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - - - - - - - (a)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) - - - - - - - (b)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) - - - - - - (c)$$

General differential equation of motion for a fluid.

Unknowns ----- stresses Velocities

Conservation of Linear Momentum

- Unfortunately, this equation is not very useful
  - 10 unknowns
    - Stress tensor, $\mathbf{T}$: 6 independent components

$$\mathrm{T} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

    - Note:

$$\tau_{xy} = \tau_{yx} \,, \quad \tau_{xz} = \tau_{zx} \quad and \quad \tau_{yz} = \tau_{zy}$$

    - Density ρ *:1*
    - Velocity, $\vec{V}$: 3 independent components
  - 4 equations (continuity + 3 momentum)
  - 6 more equations required to close problem!

Navier-Stokes Equations

- For Newtonian fluids the stress tensor components have some definitions.
- Substituting the stress tensor components into the equation of motion with produce the well-known Navier-Stokes equations that are suitable only for Newtonian Fluids:
- Stress tensor components:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

&

$$\sigma_{xx} = -\left(p + \frac{2}{3}\mu\nabla\cdot\vec{V}\right) + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -\left(p + \frac{2}{3}\mu\nabla\cdot\vec{V}\right) + 2\mu\frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -\left(p + \frac{2}{3}\mu\nabla\cdot\vec{V}\right) + 2\mu\frac{\partial w}{\partial z}$$

Navier-Stokes Equations

$$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}[\mu(2\frac{\partial u}{\partial x}-\frac{2}{3}\vec{\nabla}.\vec{V})]+\frac{\partial}{\partial y}[\mu(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})]$$
$$+\frac{\partial}{\partial z}[\mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})] \qquad (a)$$

x-momentum

$$\rho\frac{Dv}{Dt}=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}[\mu(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})]+\frac{\partial}{\partial y}[\mu(2\frac{\partial v}{\partial y}-\frac{2}{3}\vec{\nabla}.\vec{V})]$$
$$+\frac{\partial}{\partial z}[\mu(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})] \qquad (b)$$

y-momentum

$$\rho\frac{Dw}{Dt}=\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}[\mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]+\frac{\partial}{\partial y}[\mu(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})]$$
$$+\frac{\partial}{\partial z}[\mu(2\frac{\partial w}{\partial z}-\frac{2}{3}\vec{\nabla}.\vec{V})] \qquad (c)$$

z-momentum

Navier-Stokes Equations

- For Newtonian fluids the stress tensor components have some definitions.
- Substituting the stress tensor components into the equation of motion with produce the well-known Navier-Stokes equations that are suitable only for Newtonian Fluids:
- Stress tensor components:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad \sigma_{xx} = -\left(p + \frac{2}{3}\mu \nabla \cdot \vec{V}\right) + 2\mu \frac{\partial u}{\partial x}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad \& \quad \sigma_{yy} = -\left(p + \frac{2}{3}\mu \nabla \cdot \vec{V}\right) + 2\mu \frac{\partial v}{\partial y}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad \sigma_{zz} = -\left(p + \frac{2}{3}\mu \nabla \cdot \vec{V}\right) + 2\mu \frac{\partial w}{\partial z}$$

Navier-Stokes Equations

- For incompressible flow with constant dynamic viscosity:
- x- momentum $$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad (a)$$
- Y- momentum $$\rho\frac{Dv}{Dt}=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad (b)$$
- z-momentum $$\rho\frac{Dw}{Dt}=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \qquad (c)$$
- In vector form, the three equations are given by:

$$\boxed{\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}-\nabla p+\mu\nabla^2\vec{V}}$$

Incompressible NSE written in vector form

Navier-Stokes Equations

- For incompressible fluids, constant μ:
- Continuity equation: $\nabla . V = 0$

$$\frac{\partial}{\partial x}[\mu(2\frac{\partial u}{\partial x}-\cancel{\frac{2}{3}\vec{\nabla}.\vec{V}})]+\frac{\partial}{\partial y}[\mu(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})]+\frac{\partial}{\partial z}[\mu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]=$$

$$\mu\{\frac{\partial}{\partial x}[(2\frac{\partial u}{\partial x})]+\frac{\partial}{\partial y}[(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})]+\frac{\partial}{\partial z}[(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})]\}=$$

$$\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z})=$$

$$\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+\mu\frac{\partial}{\partial x}\cancel{(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z})}=$$

$$\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})=\mu\nabla^2 u$$

Navier-Stokes Equations

- For incompressible flow with constant dynamic viscosity:
- x- momentum $$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}) \qquad (a)$$
- Y- momentum $$\rho\frac{Dv}{Dt}=\rho g_y-\frac{\partial p}{\partial y}+\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}) \qquad (b)$$
- z-momentum $$\rho\frac{Dw}{Dt}=\rho g_z-\frac{\partial p}{\partial z}+\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}) \qquad (c)$$
- In vector form, the three equations are given by:

$$\boxed{\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}-\nabla p+\mu\nabla^2\vec{V}}$$

Incompressible NSE written in vector form

Navier-Stokes Equation

- The Navier-Stokes equations for incompressible flow in vector form:

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

$$\nabla \cdot \vec{V} = 0$$

Incompressible NSE written in vector form

- This results in a *closed* system of equations!
  - 4 equations (continuity and 3 momentum equations)
  - 4 unknowns (U, V, W, p)
- In addition to vector form, incompressible N-S equation can be written in several other forms including:
  - Cartesian coordinates
  - Cylindrical coordinates
  - Tensor notation

Euler Equations

- For inviscid flow (μ = 0) the momentum equations are given by:
- x- momentum $$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})=\rho g_x-\frac{\partial p}{\partial x} \qquad (a)$$
- Y- momentum $$\rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z})=\rho g_y-\frac{\partial p}{\partial y} \qquad (b)$$
- z-momentum $$\rho(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z})=\rho g_z-\frac{\partial p}{\partial z} \qquad (c)$$
- In vector form, the three equations are given by:

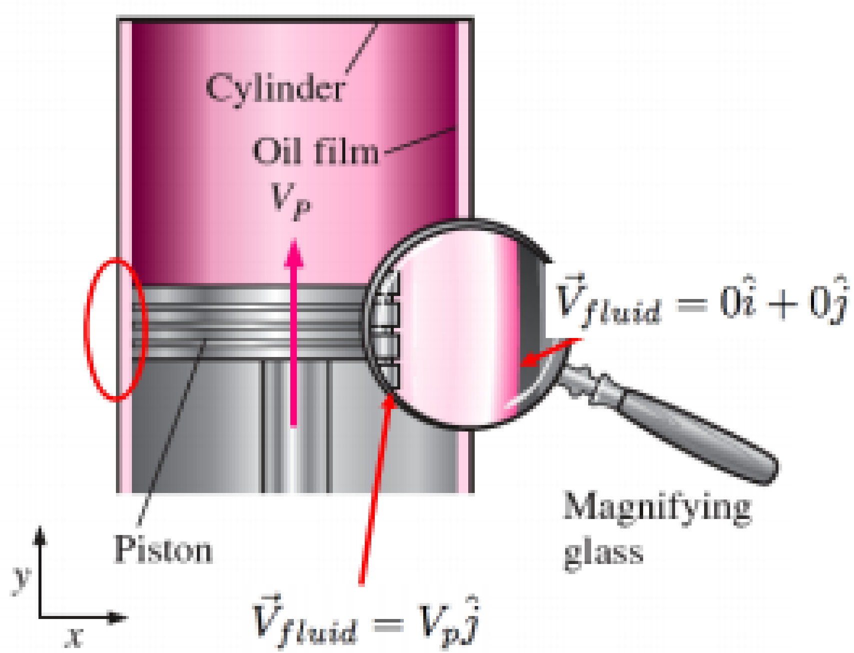
$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}-\nabla p$$

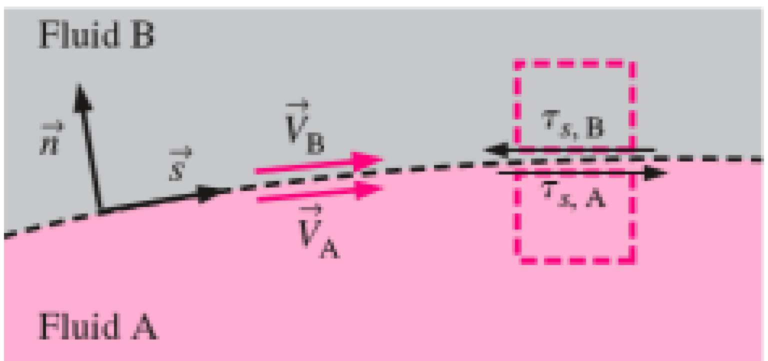
Euler equations written in vector form

Differential Analysis of Fluid Flow Problems

- Now that we have a set of governing partial differential equations, there are 2 problems we can solve
  1. Calculate pressure (*P*) for a known velocity field
  2. Calculate velocity (*U*, *V*, *W*) and pressure (*P*) for known geometry, boundary conditions (BC), and initial conditions (IC)
- There are about 80 known exact solutions to the NSE
- Solutions can be classified by type or geometry, for example:
  1. Couette shear flows
  2. Steady duct/pipe flows (Poisseulle flow)

boundary conditions





No-slip boundary condition

- For a fluid in contact with a solid wall, the velocity of the fluid must equal that of the wall

$$\vec{V}_{fluid} = \vec{V}_{wall}$$

Interface boundary condition

- When two fluids meet at an interface, the velocity and shear stress must be the same on both sides

$$\vec{V}_A = \vec{V}_B \qquad \tau_{s,A} = \tau_{s,B}$$

- If surface tension effects are negligible and the surface is nearly flat

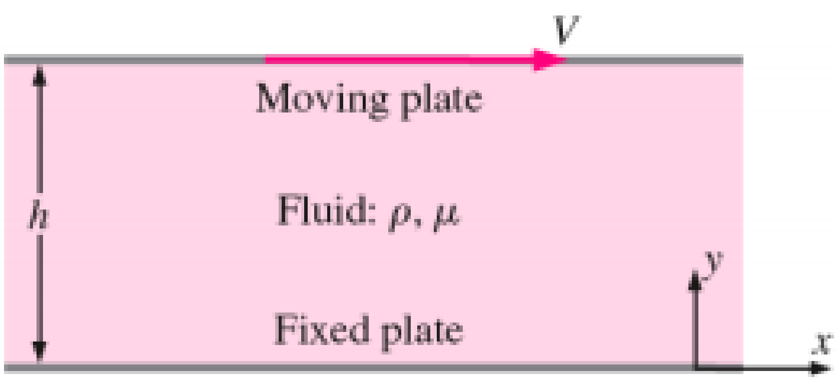
$$P_A = P_B$$

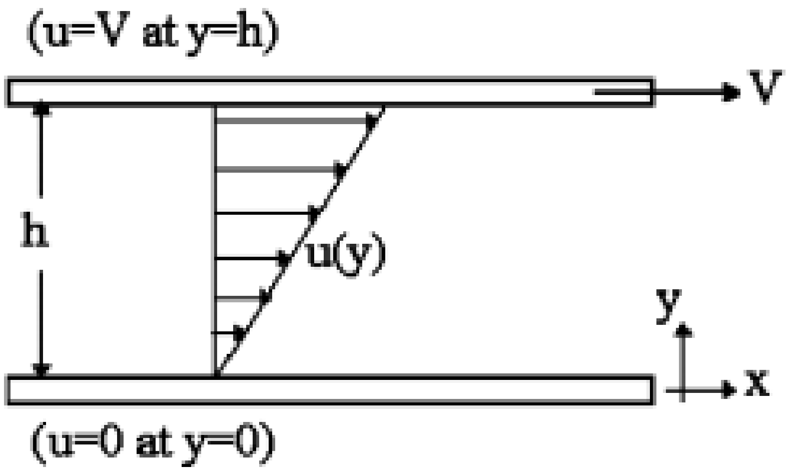
Example exact solution

Fully Developed Couette Flow

- For the given geometry and BC's, calculate the velocity and pressure fields, and estimate the shear force per unit area acting on the bottom plate

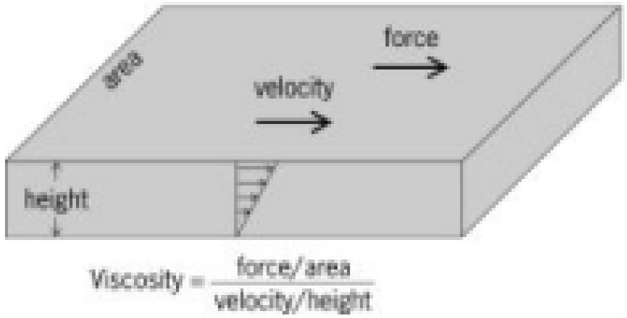
- Step 1: Geometry, dimensions, and properties





Fully Developed Couette Flow

- Step 2: Assumptions and BC's
  - Assumptions
    1. Plates are infinite in x and z
    2. Flow is steady, $\partial/\partial t = 0$
    3. Parallel flow, $v = 0$
    4. Incompressible, Newtonian, laminar, constant properties
    5. No pressure gradient
    6. 2D, $w = 0$, $\partial/\partial z = 0$
    7. Gravity acts in the -y direction, $\boxed{\vec{g} = -g\,\hat{j}\,,\ g_y = -g}$



  - Boundary conditions
    1. Bottom plate (y=0) - no slip condition: $u=0$, $v=0$, $w=0$
    2. Top plate (y=h) : no slip condition: $u=V$, $v=0$, $w=0$

Fully Developed Couette Flow

- Step 3: Simplify

Note: these numbers refer to the assumptions on the previous slide

Continuity

$$\frac{\partial U}{\partial x} + \underbrace{\cancel{\frac{\partial V}{\partial y}}}_{3} + \underbrace{\cancel{\frac{\partial W}{\partial z}}}_{6} = 0$$

$$\boxed{\frac{\partial U}{\partial x} = 0}$$

This means the flow is "*fully developed*" or not changing in the direction of flow

X-momentum

$$\rho\left(\underbrace{\cancel{\frac{\partial U}{\partial t}}}_{2} + \underbrace{\cancel{U\frac{\partial U}{\partial x}}}_{\text{Cont.}} + \underbrace{\cancel{V\frac{\partial U}{\partial y}}}_{3} + \underbrace{\cancel{W\frac{\partial U}{\partial z}}}_{6}\right) = \underbrace{\cancel{-\frac{\partial P}{\partial x}}}_{5} + \underbrace{\cancel{\rho g_x}}_{7} + \mu\left(\underbrace{\cancel{\frac{\partial^2 U}{\partial x^2}}}_{\text{Cont.}} + \frac{\partial^2 U}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2 U}{\partial z^2}}}_{6}\right)$$

$$\boxed{\frac{\partial^2 U}{\partial y^2} = 0}$$

Fully Developed Couette Flow

- Step 3: Simplify, cont.

Y-momentum

$$\rho\left(\overset{\boxed{2,3}}{\cancel{\frac{\partial V}{\partial t}}}+U\overset{\boxed{3}}{\cancel{\frac{\partial V}{\partial x}}}+V\overset{\boxed{3}}{\cancel{\frac{\partial V}{\partial y}}}+W\overset{\boxed{3,6}}{\cancel{\frac{\partial V}{\partial z}}}\right)=-\frac{\partial P}{\partial y}+\rho g_y+\mu\left(\overset{\boxed{3}}{\cancel{\frac{\partial^2 V}{\partial x^2}}}+\overset{\boxed{3}}{\cancel{\frac{\partial^2 V}{\partial y^2}}}+\overset{\boxed{3}}{\cancel{\frac{\partial^2 V}{\partial z^2}}}\right)$$

$$\boxed{\frac{\partial p}{\partial y}=-\rho g_y} \Longrightarrow \mathrm{p} = \mathrm{p(y)}$$

Z-momentum

$$\rho\left(\overset{\boxed{2,6}}{\cancel{\frac{\partial W}{\partial t}}}+U\overset{\boxed{6}}{\cancel{\frac{\partial W}{\partial x}}}+V\overset{\boxed{6}}{\cancel{\frac{\partial W}{\partial y}}}+W\overset{\boxed{6}}{\cancel{\frac{\partial W}{\partial z}}}\right)=-\frac{\partial P}{\partial z}+\overset{\boxed{7}}{\cancel{\rho g_z}}+\mu\left(\overset{\boxed{6}}{\cancel{\frac{\partial^2 W}{\partial x^2}}}+\overset{\boxed{6}}{\cancel{\frac{\partial^2 W}{\partial y^2}}}+\overset{\boxed{6}}{\cancel{\frac{\partial^2 W}{\partial z^2}}}\right)$$

$$\boxed{\frac{\partial p}{\partial z}=0} \Longrightarrow \mathrm{p} = \mathrm{p(y)} \Longrightarrow \boxed{\frac{\partial p}{\partial y}=\frac{dp}{dy}-\rho g_y}$$

Fully Developed Couette Flow

- Step 4: Integrate

X-momentum

$$\frac{d^2u}{dy^2} = 0 \xrightarrow{\text{integrate}} \frac{du}{dy} = C_1 \xrightarrow{\text{integrate}} \boxed{u(y) = C_1 y + C_2}$$

y-momentum

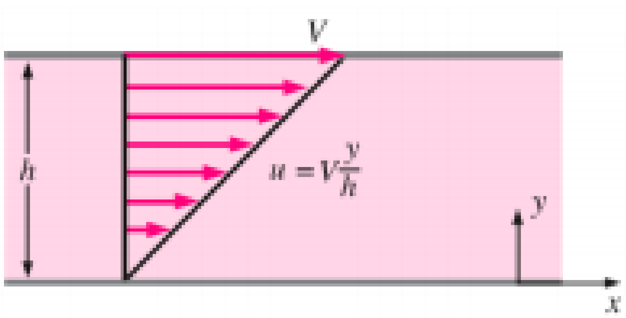
$$\boxed{\frac{dp}{dy} = -\rho g} \xrightarrow{\text{integrate}} \boxed{p(y) = -\rho g y + C_3}$$

Fully Developed Couette Flow

$$\boxed{u(y) = C_1 y + C_2}$$

- Step 5: Apply BC's
  - $y = 0$, $u = 0 = C_1(0) + C_2 \Rightarrow C_2 = 0$
  - $y = h$, $u = V = C_1 h \Rightarrow C_1 = V/h$
  - This gives

$$\boxed{u(y) = V\frac{y}{h}}$$



  - For pressure, no explicit BC, therefore C_3 can remain an arbitrary constant (recall only ∇P appears in NSE).
    - Let $p = p_0$ at $y = 0$ (C_3 renamed p_0)

$$\boxed{p(y) = p_0 - \rho g y}$$

1. Hydrostatic pressure
2. Pressure acts independently of flow

Fully Developed Couette Flow

- Step 6: Verify solution by back-substituting into differential equations
  - Given the solution (u,v,w)=(Vy/h, 0, 0)

$$\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 0, \frac{\partial w}{\partial z} = 0$$

  - Continuity is satisfied

    0 + 0 + 0 = 0

  - X-momentum is satisfied

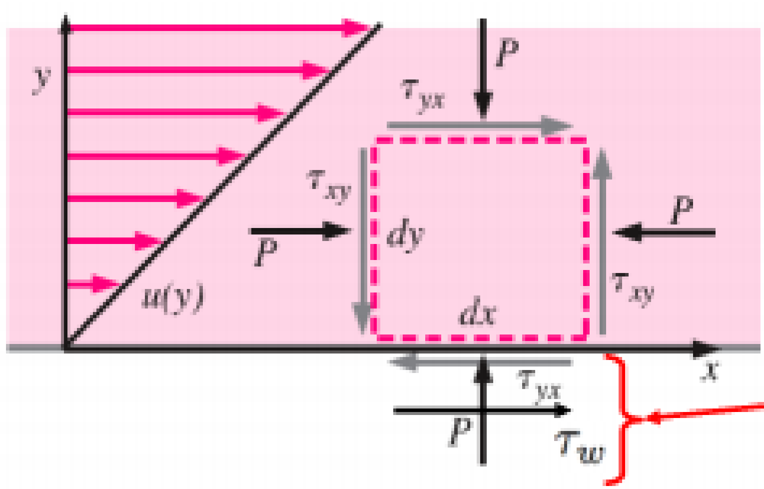
$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$
$$\rho\left(0 + V\frac{y}{h}\cdot 0 + 0\cdot V/h + 0\cdot 0\right) = -0 + \rho\cdot 0 + \mu\left(0+0+0\right)$$
$$0 = 0$$

Fully Developed Couette Flow

- Finally, calculate shear force on bottom plate

$$\boxed{u(y) = V\frac{y}{h}} \Rightarrow \tau_{xy} = \tau_{yx} = \mu\left(\cancel{\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial y}\right) = \mu\frac{V}{h}$$

Shear force per unit area acting on the wall

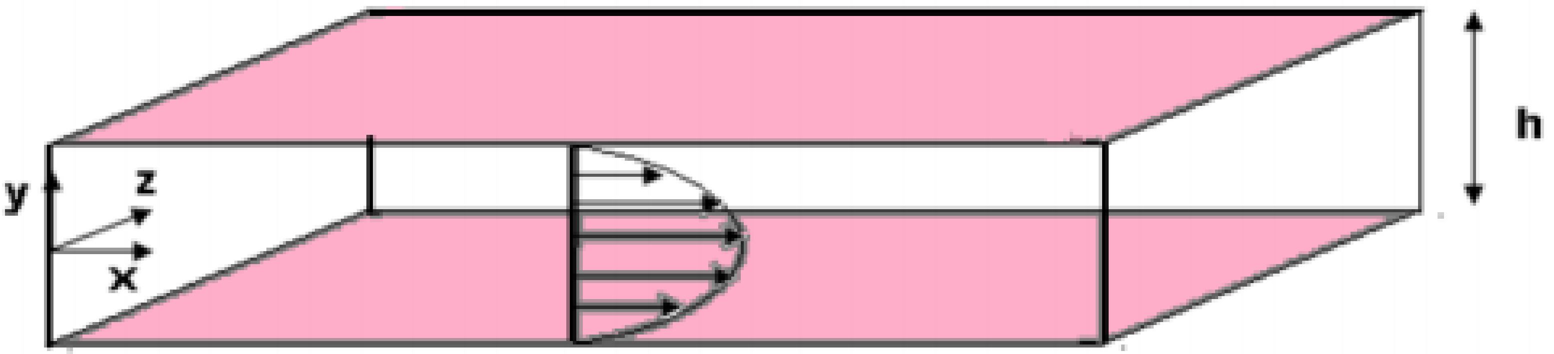


$$\boxed{\frac{\vec{F}}{A} = \tau_w = \mu\frac{V}{h}\hat{i}}$$

Note that τ_w is equal and opposite to the shear stress acting on the fluid τ_{yx} (Newton's third law).

Parallel Plates (Poiseuille Flow)

- Given: A steady, fully developed, laminar flow of a Newtonian fluid in a rectangular channel of two parallel plates where the width of the channel is much larger than the height, h, between the plates.



- Find: The velocity profile and shear stress due to the flow.

Assumptions:

- Entrance Effects Neglected
- No-Slip Condition
- No vorticity/turbulence

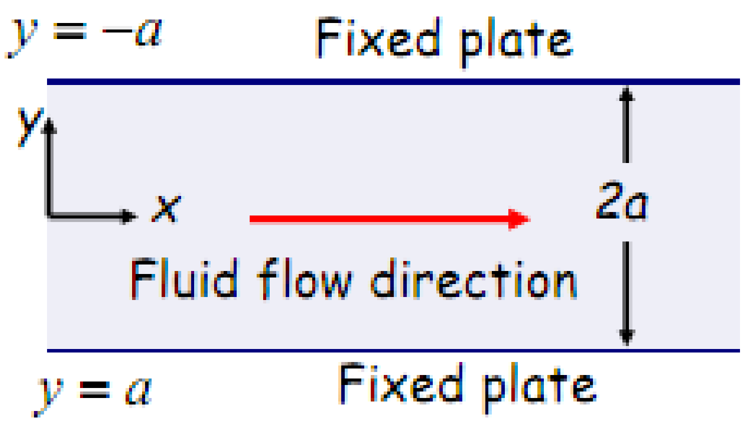
Additional and Highlighted Important Assumptions

- The width is very large compared to the height of the plate.
- No entrance or exit effects.
- Fully developed flow.
- THEREFORE…
  - Velocity can only be dependent on vertical location in the flow (*u*)
  - $v = w = 0$
  - The pressure drop is constant and in the x-direction only.

$$\frac{\partial p}{\partial x} = \text{Constant} = \frac{\Delta p}{L}, \text{ where } L \text{ is a length in } x.$$

Boundary Conditions

- No Slip Condition Applies
  - Therefore, at y = -a and y = +a, **u=v=w** = 0
- The bounding walls in the **z** direction are often ignored. If we don't ignore them we also need:
  - **z** = -W/2 and **z** = +W/2, **u=v=w** = 0, where W is the width of the channel.



Incompressible Newtonian Stress Tensor

$$\tau = \begin{pmatrix} \cancel{2\mu \frac{\partial u}{\partial x}} & \mu\left(\cancel{\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial y}\right) & \mu\left(\cancel{\frac{\partial w}{\partial x}} + \cancel{\frac{\partial u}{\partial z}}\right) \\ \mu\left(\cancel{\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial y}\right) & \cancel{2\mu \frac{\partial v}{\partial y}} & \mu\left(\cancel{\frac{\partial w}{\partial y}} + \cancel{\frac{\partial v}{\partial z}}\right) \\ \mu\left(\cancel{\frac{\partial w}{\partial x}} + \frac{\partial u}{\partial z}\right) & \mu\left(\cancel{\frac{\partial w}{\partial y}} + \cancel{\frac{\partial v}{\partial z}}\right) & \cancel{2\mu \frac{\partial w}{\partial z}} \end{pmatrix}$$

Now, we cancel terms out based on our assumptions.

This results in our new tensor:

$$T = \begin{pmatrix} 0 & \mu\left(\frac{\partial u}{\partial y}\right) & 0 \\ \mu\left(\frac{\partial u}{\partial y}\right) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Navier-Stokes Equations

In Vector Form for incompressible flow:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}\right) + \rho\mathbf{v} \bullet \nabla\mathbf{v} = -\nabla \mathrm{p} + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

Which we expand to component form:

x - component :

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial \mathrm{p}}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + \rho g_x$$

y - component :

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial \mathrm{p}}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] + \rho g_y$$

z - component :

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial \mathrm{p}}{\partial z} + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] + \rho g_z$$

Reducing Navier-Stokes

x - component :

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]+\rho g_x$$

y - component :

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]+\rho g_y$$

z - component :

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right]+\rho g_z$$

N-S equation therefore reduces to

x - component: $$-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+\rho g_x=0$$

y - component: $$-\frac{\partial p}{\partial y}+\rho g_y=0$$

z - component: $$-\frac{\partial p}{\partial z}+\rho g_z=0$$

Ignoring gravitational effects, we get

x - component: $$-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}=0$$

y - component: $-\frac{\partial p}{\partial y}=0$ ⇒ *p* is not a function of *y*

z - component: $-\frac{\partial p}{\partial z}=0$ ⇒ *p* is not a function of *z*

⇒ *p* is a function of *x* only

Rewriting (4), we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \qquad \text{——(5)}$$

p is a function of x only and μ is a constant and therefore RHS is a function of x only

u is a function of y only and therefore LHS is a function of y only

Therefore (5) gives, function of (y) = function of (x) = constant

It means $\frac{\Delta p}{L} = -\frac{\partial p}{\partial x}$ = constant

That is, pressure gradient in the x-direction is a constant.

LHS = left hand side of the equation
RHS = right hand side of the equation

Rewriting (5), we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{-\Delta p}{\mu L} \quad \text{——(6)}$$

where $\frac{\Delta p}{L} = -\frac{\partial p}{\partial x}$ is the constant pressure gradient in the x-direction

Since u is only a function of y, the partial derivative becomes an ordinary derivative.

Therefore, (5) becomes

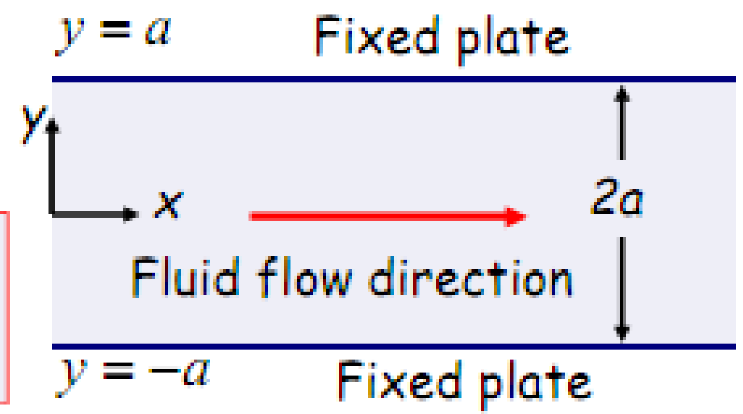
$$\frac{d^2 u}{dy^2} = -\frac{\Delta p}{\mu L} \quad \text{——(7)}$$

Integrating (6), we get

$$u=-\frac{\Delta p}{\mu L}\frac{y^2}{2}+C_1 y+C_2 \quad \text{——} \quad (8)$$

where C_1 and C_2 are constants to be determined using the boundary conditions given below:

$$\left.\begin{array}{lll} u=0 & \text{at} & y=a \\ u=0 & \text{at} & y=-a \end{array}\right\} \boxed{\text{no-slip boundary condition}}$$

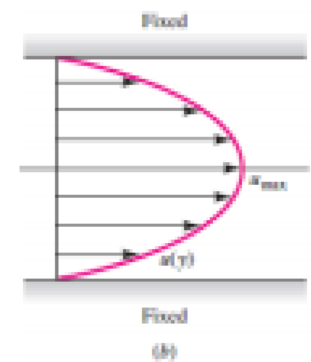


Substituting the boundary conditions in (8), we get

$$C_1=0 \quad \text{and} \quad C_2=\frac{\Delta p}{\mu L}\frac{a^2}{2}$$

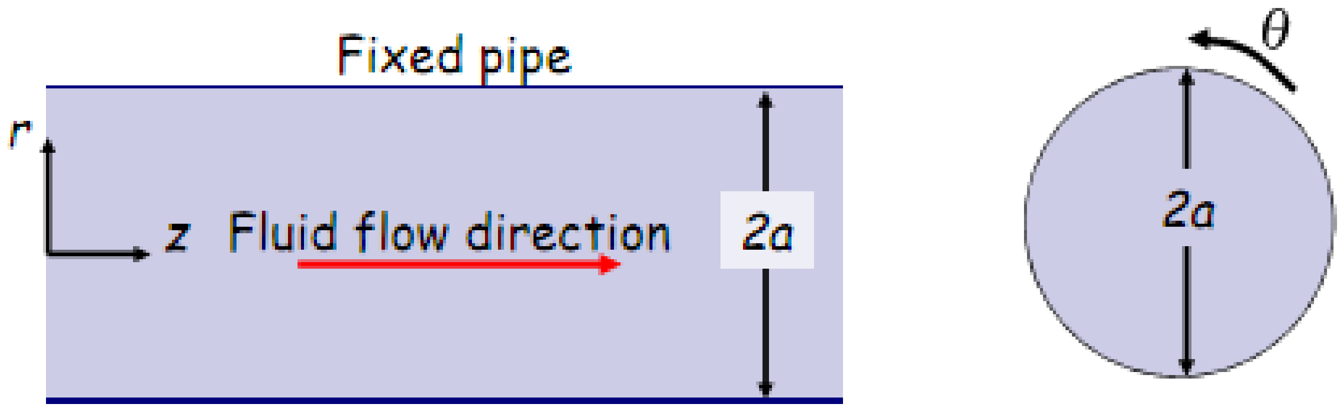
Therefore, (8) reduces to

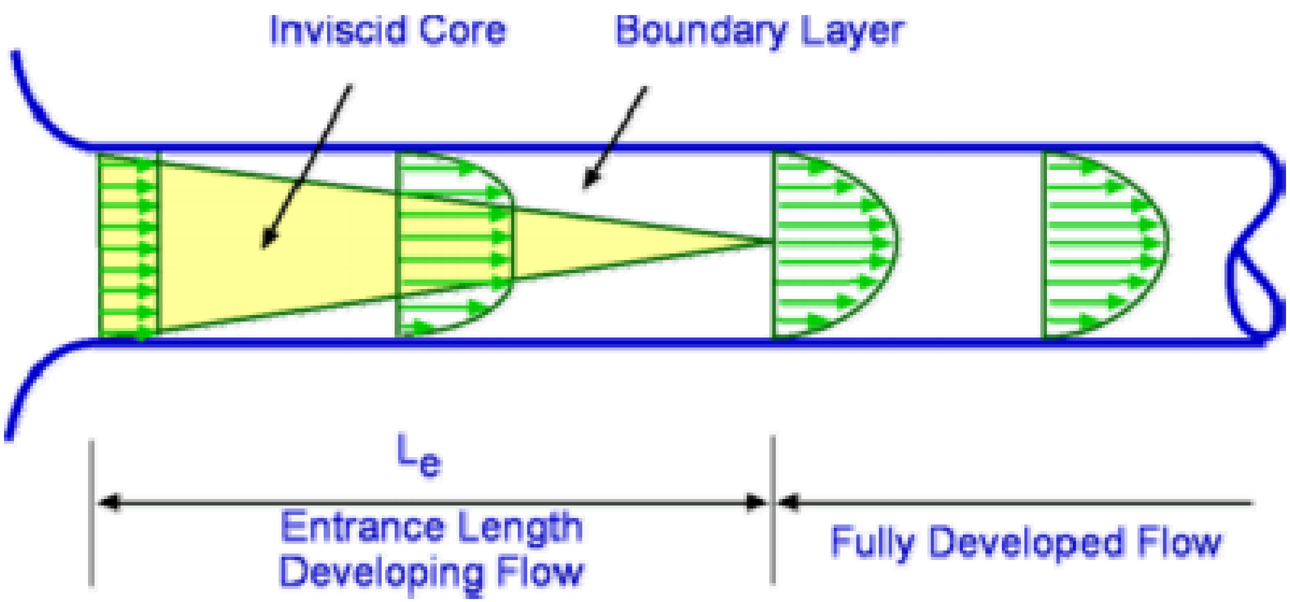
$$\boxed{u=\frac{\Delta p}{2\mu L}\left(a^2-y^2\right)}$$



Parabolic velocity profile

Steady, incompressible flow of Newtonian fluid in a pipe
- fully developed pipe Poisuille flow





Laminar Flow in pipes

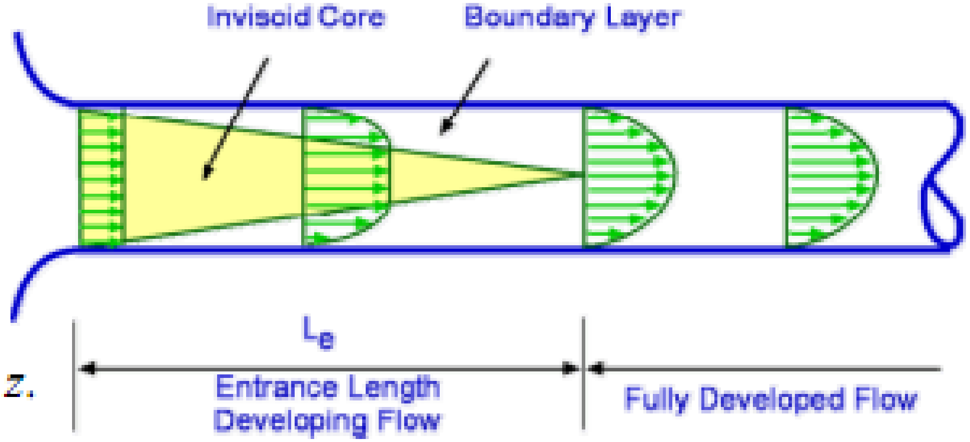
Steady Laminar pipe flow.

Assumptions:

$v_r = v_\theta = 0$

$v_z = v_z(r)$

$\frac{\partial p}{\partial z} = \frac{dp}{dz}$ *i.e.* pressure is a linear of z.



The above "assumptions" can be obtained from the single assumption of "fully developed" flow.

In fully developed pipe flow, all velocity components are assumed to be unchanging along the axial direction, and axially symmetric i.e.:

Now look at the z-momentum equation.

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r\frac{\partial v_z}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}} + \cancel{v_z\frac{\partial v_z}{\partial z}}\right)$$

$$= -\frac{dp}{dz} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{d z^2}}\right)$$

Results from Mass/Momentum

The removal of the indicated terms yields:

$$-\frac{1}{\mu}\frac{dp}{dz}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=0$$

Or rearranging, and substituting for the pressure gradient:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=\frac{1}{\mu}\frac{dp}{dz}$$

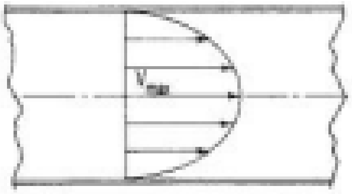
In a typical situation, we would have control over *dp/dz*. That is, we can induce a pressure gradient by altering the pressure at one end of the pipe. We will therefore take it as the input to the system (similar to what an electrical engineer might do in testing a linear system).

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=\frac{r}{\mu}\frac{dp}{dz}$$

Laminar flow in pipes

Integrating once gives:

$$\left(r\frac{dv_z}{dr}\right)=\frac{r^2}{2\mu}\frac{dp}{dz}+C_1$$



But at r = 0 velocity = max or $dv_z/dr = 0$ $\longrightarrow$ $C_1 = 0$

$$\left(r\frac{dv_z}{dr}\right)=\frac{r^2}{2\mu}\frac{dp}{dz}\quad\xrightarrow{\text{Divide by r}}\quad\left(\frac{dv_z}{dr}\right)=\frac{r}{2\mu}\frac{dp}{dz}$$

Integrating one more time gives:

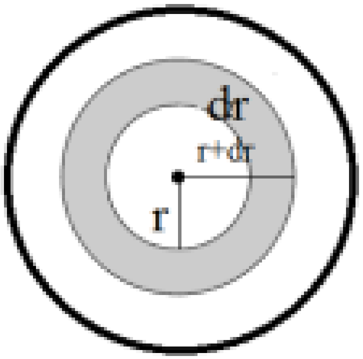
$$v_z=\frac{r^2}{4\mu}\frac{dp}{dz}+C_2$$

at pipe wall r = a velocity = 0 $\longrightarrow 0=\dfrac{a^2}{4\mu}\dfrac{dp}{dz}+C_2 \longrightarrow C_2=-\dfrac{a^2}{4\mu}\dfrac{dp}{dz}$

Laminar flow in pipes

Substituting in the velocity equation gives:

$$v_z = \frac{r^2}{4\mu}\frac{dp}{dz} - \frac{a^2}{4\mu}\frac{dp}{dz} = -\frac{dp}{dz}\left(\frac{a^2 - r^2}{4\mu}\right)$$



Volume flow rate is obtained from:

$$Q = \int_{cross-section} v_z dA \longrightarrow Q = \int_{r=0}^{r=a} 2\pi r v_z dr$$

$$Q = \frac{\pi\left(-\frac{dp}{dz}\right)}{2\mu}\int_{r=0}^{r=a} r\left(a^2 - r^2\right)dr \longrightarrow Q = \frac{\pi a^4}{8\mu}\left\{-\frac{dp}{dz}\right\}$$

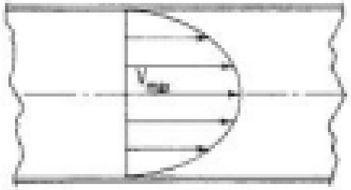
$$\longrightarrow Q = \frac{\pi D^4}{128\mu}\left\{-\frac{dp}{dz}\right\}$$

Laminar flow in pipes

Average velocity, v_{av}= Q/area

$$v_{z,ave} = \frac{\pi D^4}{128\mu}\left\{-\frac{dp}{dz}\right\} / \frac{\pi D^2}{4} = \frac{D^2}{32\mu}\left\{-\frac{dp}{dz}\right\}$$

Maximum velocity at r = 0:



$$v_{z,\max} = -\frac{dp}{dz}\left(\frac{a^2}{4\mu}\right) = -\frac{dp}{dz}\left(\frac{D^2}{16\mu}\right)$$

$V_{avergae}$ = ½ V_{max}

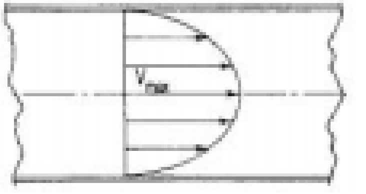
Shear stress:

$$\tau = \mu \frac{dv_z}{dr}\Big|_{r=a} = \mu\left(-\frac{dp}{dz}\right)\left(\frac{-2a}{4\mu}\right) = \mu\frac{dp}{dz}\left(\frac{D}{4\mu}\right) = 8\mu\left(\frac{v_{z,ave}}{D}\right)$$

Laminar flow in pipes

Drag Coefficient:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho v^2{}_{ave}} = \frac{16\mu v_{ave}}{\rho v_{ave}^2 D} = \frac{16}{R_e}$$



Coefficient of friction: $f = 4\ C_f$

$$f = \frac{64}{R_e}$$