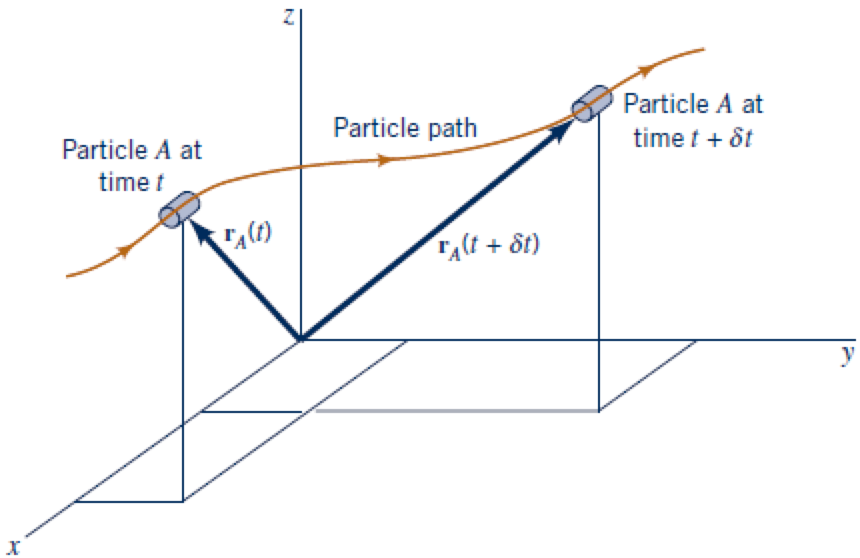
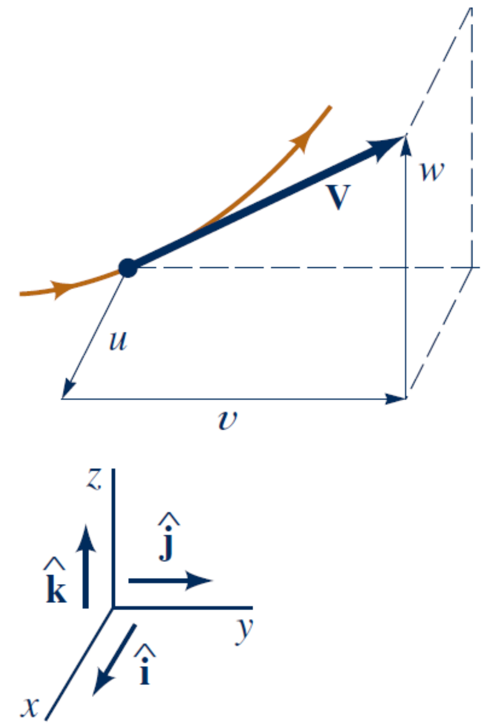
SPC 307
Introduction to Aerodynamics

Lecture 6
Potential Flow

April 2, 2017



■ **Figure 4.1** Particle location in terms of its position vector.



$$d\mathbf{r}_A/dt = \mathbf{V}_A$$

$$\mathbf{V} = u(x, y, z, t)\hat{\mathbf{i}} + v(x, y, z, t)\hat{\mathbf{j}} + w(x, y, z, t)\hat{\mathbf{k}}$$

GIVEN A velocity field is given by $\mathbf{V} = (V_0/\ell)(-x\hat{\mathbf{i}} + y\hat{\mathbf{j}})$ where V_0 and ℓ are constants.

FIND At what location in the flow field is the speed equal to V_0? Make a sketch of the velocity field for $x \geq 0$ by drawing arrows representing the fluid velocity at representative locations.

SOLUTION

The x, y, and z components of the velocity are given by $u = -V_0 x/\ell$, $v = V_0 y/\ell$, and $w = 0$ so that the fluid speed, V, is

$$V = (u^2 + v^2 + w^2)^{1/2} = \frac{V_0}{\ell}(x^2 + y^2)^{1/2} \qquad \textbf{(1)}$$

The speed is $V = V_0$ at any location on the circle of radius ℓ centered at the origin $[(x^2 + y^2)^{1/2} = \ell]$ as shown in Fig. E4.1*a*. **(Ans)**

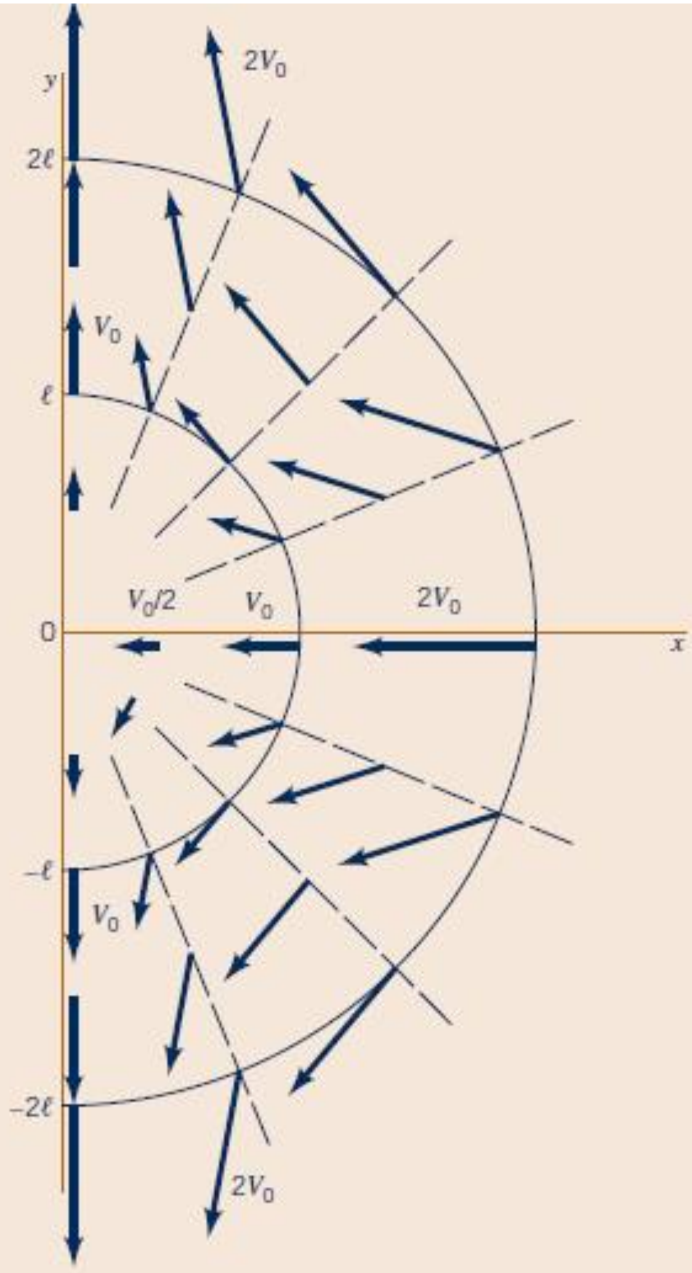
The direction of the fluid velocity relative to the x axis is given in terms of $\theta = \arctan(v/u)$ as shown in Fig. E4.1*b*. For this flow

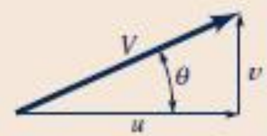
$$\tan\theta = \frac{v}{u} = \frac{V_0 y/\ell}{-V_0 x/\ell} = \frac{y}{-x}$$

Thus, along the x axis $(y = 0)$ we see that $\tan\theta = 0$, so that $\theta = 0°$ or $\theta = 180°$. Similarly, along the y axis $(x = 0)$ we obtain $\tan\theta = \pm\infty$ so that $\theta = 90°$ or $\theta = 270°$. Also, for $y = 0$ we find $\mathbf{V} = (-V_0 x/\ell)\hat{\mathbf{i}}$, while for $x = 0$ we have $\mathbf{V} = (V_0 y/\ell)\hat{\mathbf{j}}$, indicating (if $V_0 > 0$) that the flow is directed away from the origin along the y axis and toward the origin along the x axis as shown in Fig. E4.1*a*.

By determining $\mathbf{V}$ and θ for other locations in the x–y plane, the velocity field can be sketched as shown in the figure. For example, on the line $y = x$ the velocity is at a 45° angle relative to the x axis ($\tan \theta = v/u = -y/x = -1$). At the origin $x = y = 0$ so that $\mathbf{V} = 0$. This point is a stagnation point. The farther from the origin the fluid is, the faster it is flowing (as seen from Eq. 1). By careful consideration of the velocity field it is possible to determine considerable information about the flow.

COMMENT The velocity field given in this example approximates the flow in the vicinity of the center of the sign shown in Fig. E4.1*c*. When wind blows against the sign, some air flows over the sign, some under it, producing a stagnation point as indicated.


y
$2V_0$
2ℓ
V_0
ℓ
$V_0/2$
V_0
$2V_0$
0
x
$-\ell$
V_0
-2ℓ
$2V_0$

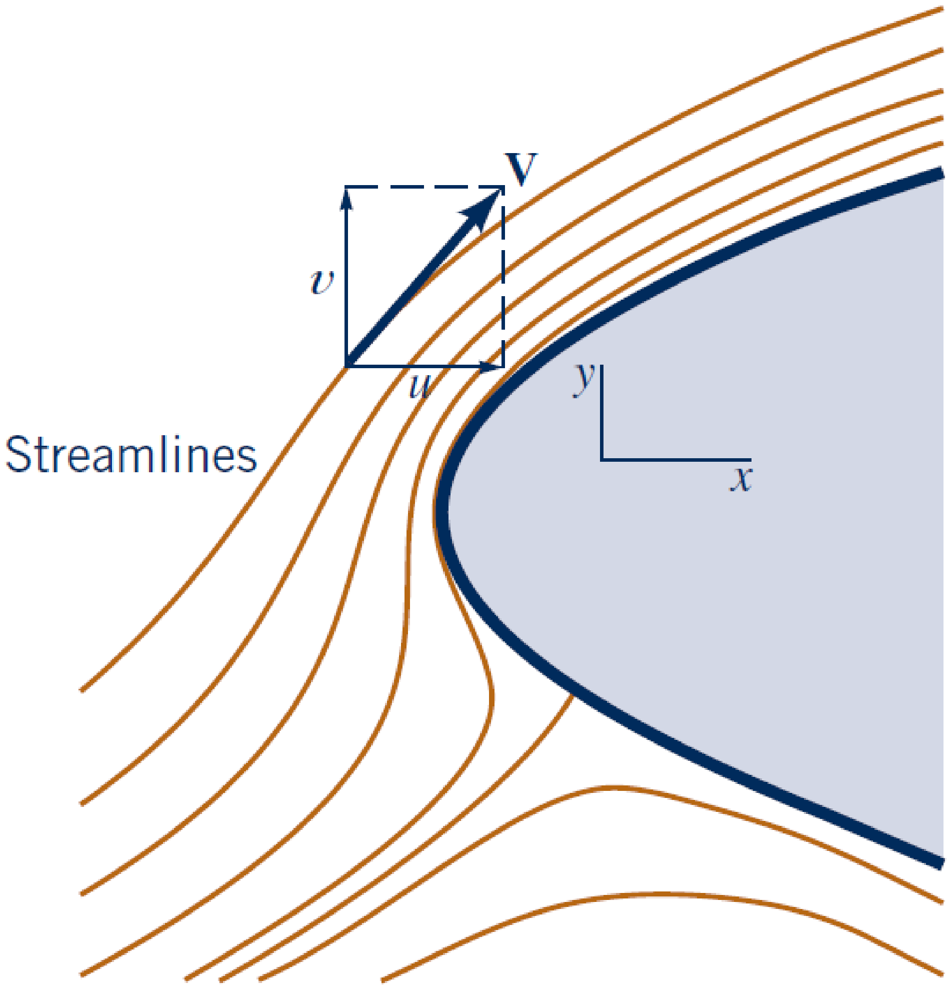

V
θ
v
u

(b)

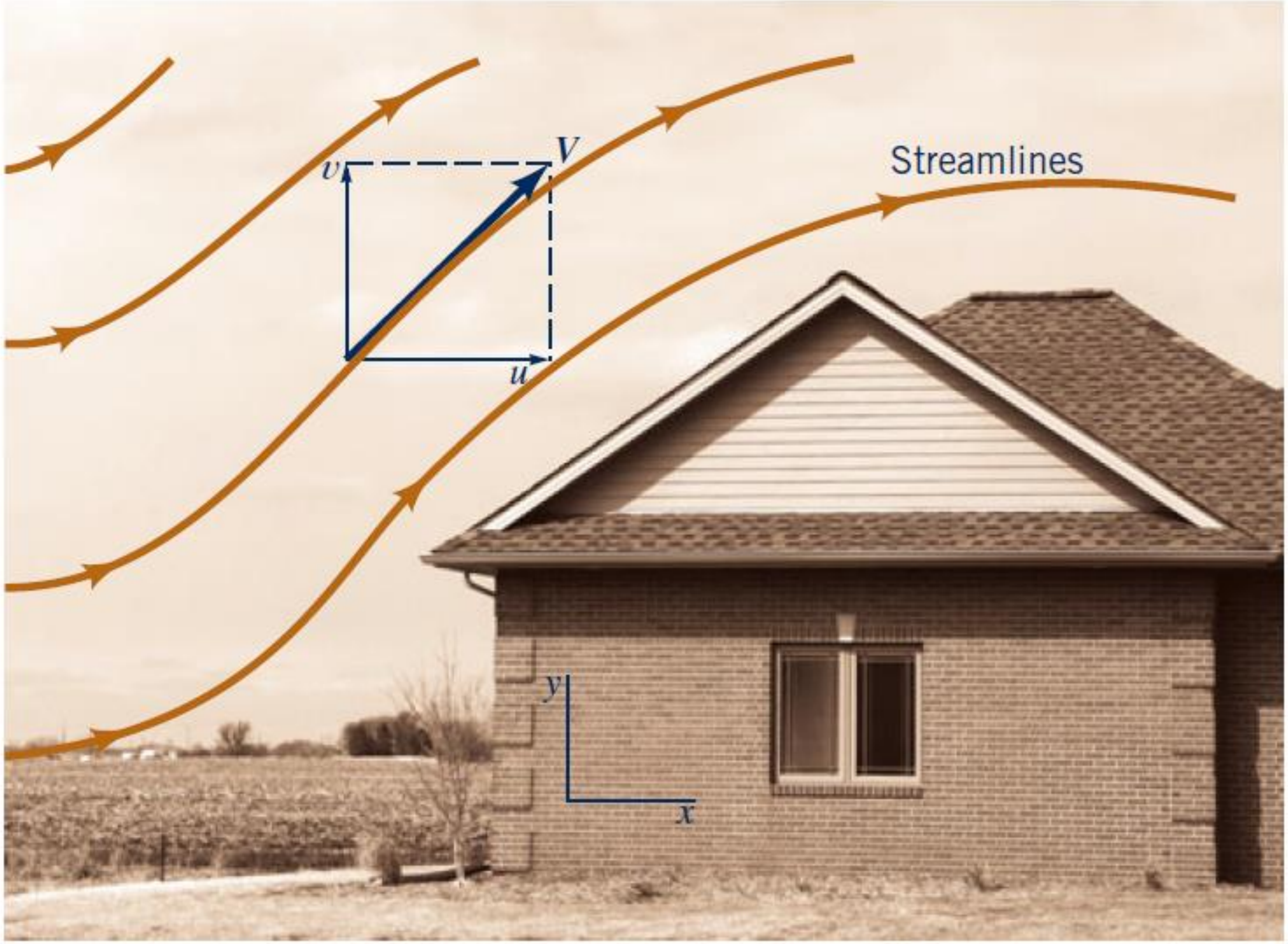

y
x
V = 0
STONE
RD

(c)

Stream Lines



Stream Lines



The Stream Function, ψ(x,y)

- Consider the continuity equation for an incompressible 2D flow

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

- Substituting the clever transformation, ψ(x,y)
- Defined as:

$$u=\frac{\partial \psi}{\partial y} \qquad v=-\frac{\partial \psi}{\partial x}$$

- Gives

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial y})+\frac{\partial}{\partial y}(-\frac{\partial \psi}{\partial x})=0$$

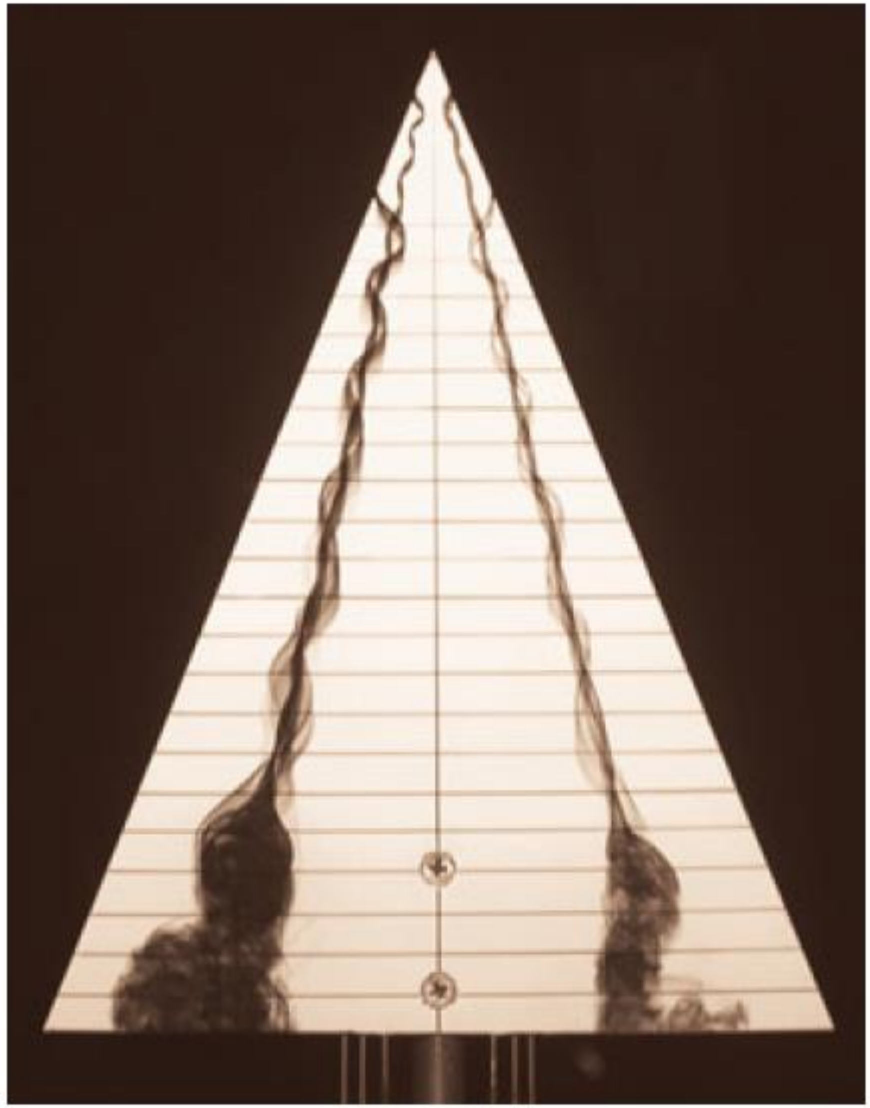
This is true for any smooth function ψ(x,y)

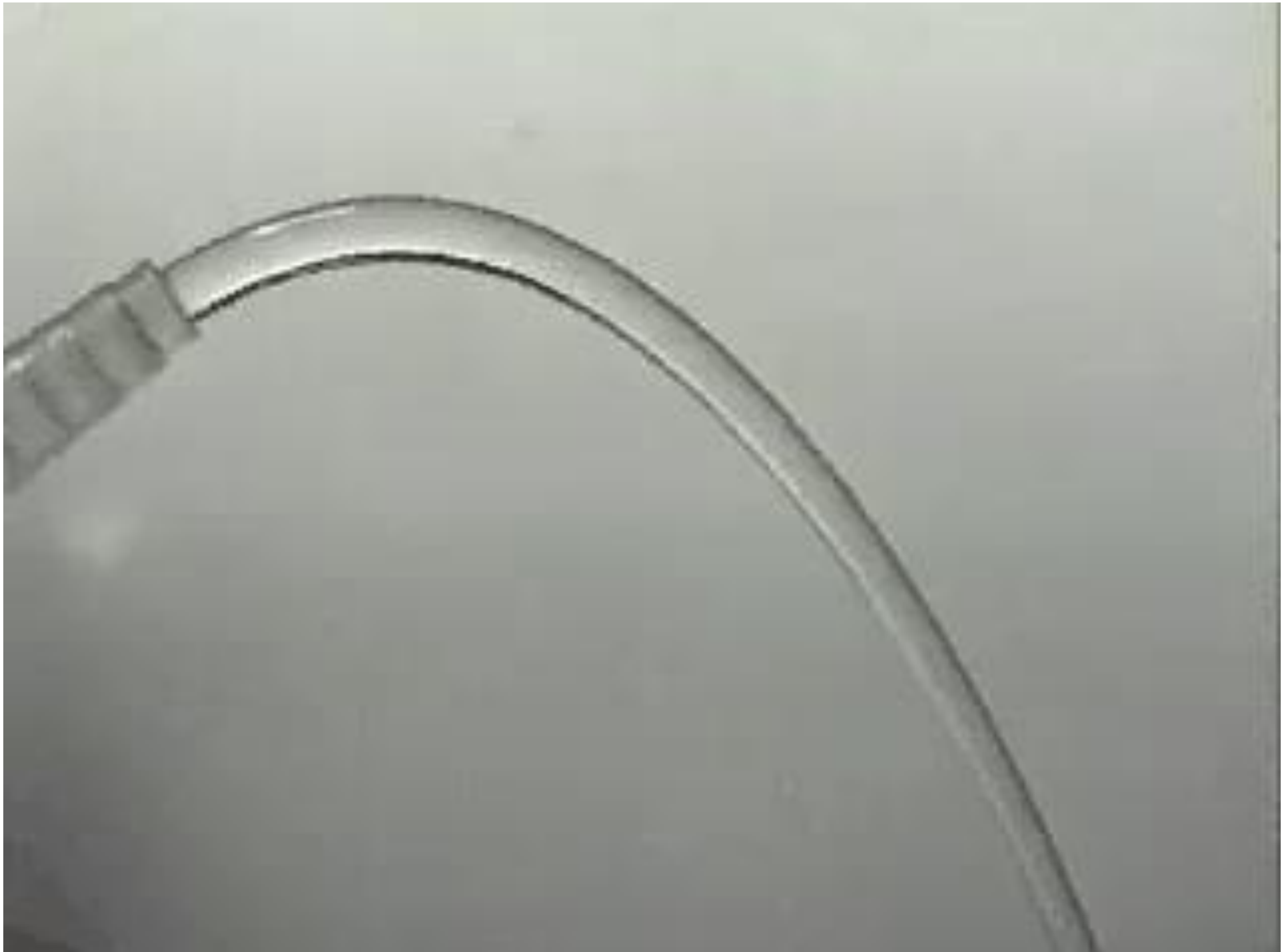
so that it always satisfies the continuity eq.

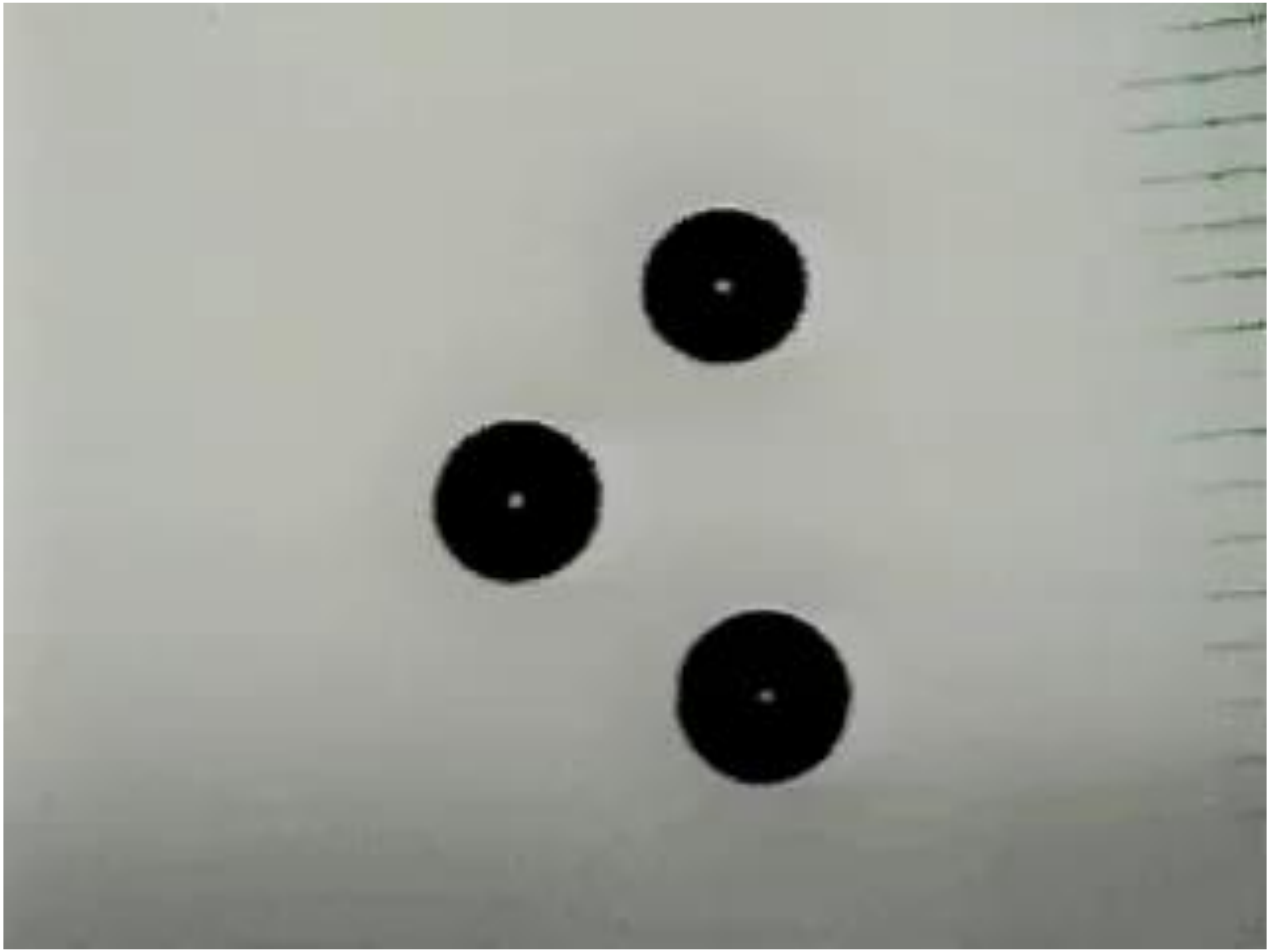
$$\left.\begin{matrix} u \\ v \end{matrix}\right\} \text{two unknows} \xrightarrow{\text{using stream function}} \psi \; \} \text{ one unknow}$$

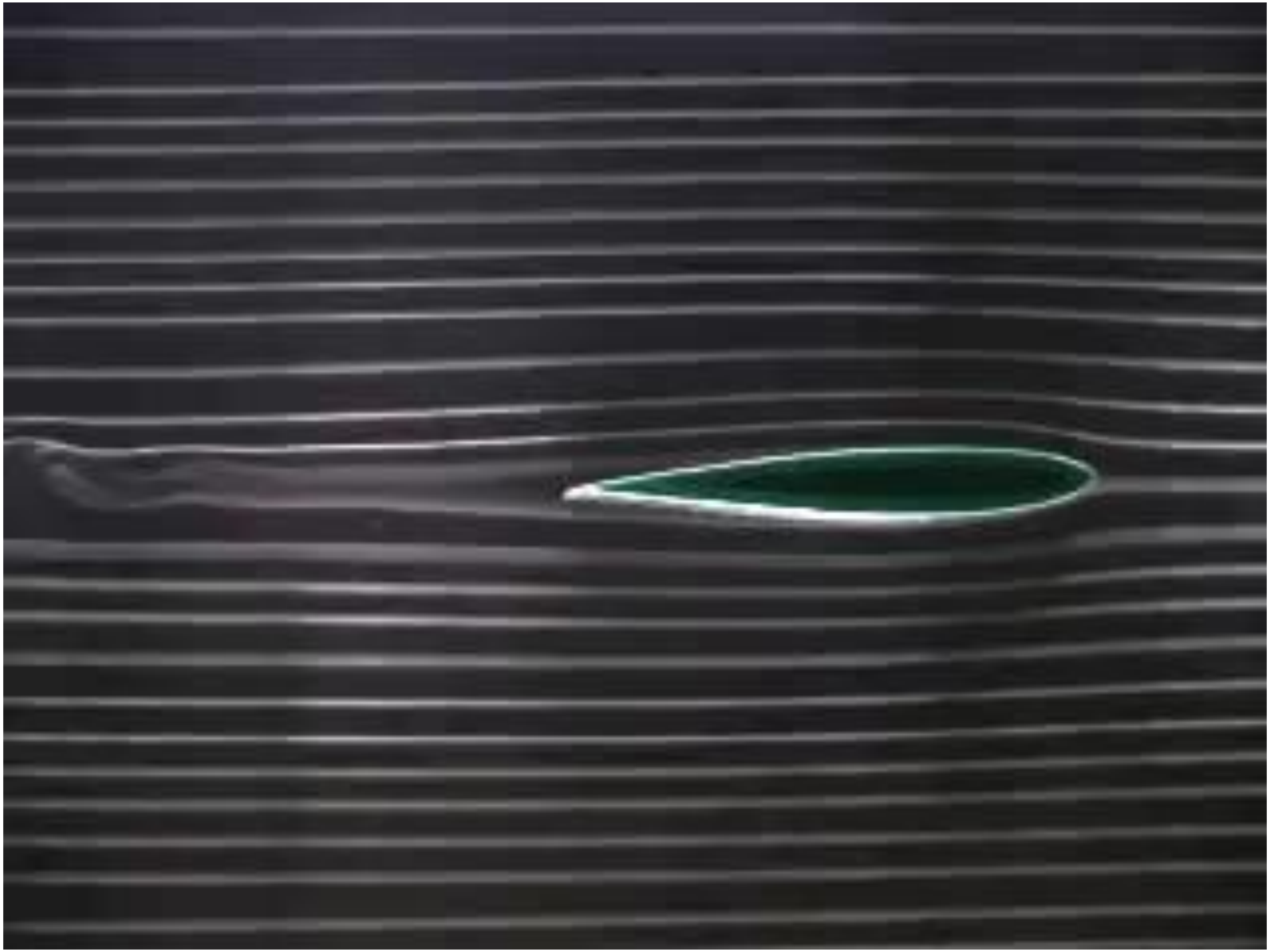
$$\Updownarrow$$

conservation of mass will always be satisfied









GIVEN Consider the two-dimensional steady flow discussed in Example 4.1, $\mathbf{V} = (V_0/\ell)(-x\hat{\mathbf{i}} + y\hat{\mathbf{j}})$.

FIND Determine the streamlines for this flow.

SOLUTION

Since

$$u = (-V_0/\ell)x \text{ and } v = (V_0/\ell)y \qquad \textbf{(1)}$$

it follows that streamlines are given by solution of the equation

$$\frac{dy}{dx} = \frac{v}{u} = \frac{(V_0/\ell)y}{-(V_0/\ell)x} = -\frac{y}{x}$$

in which variables can be separated and the equation integrated to give

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

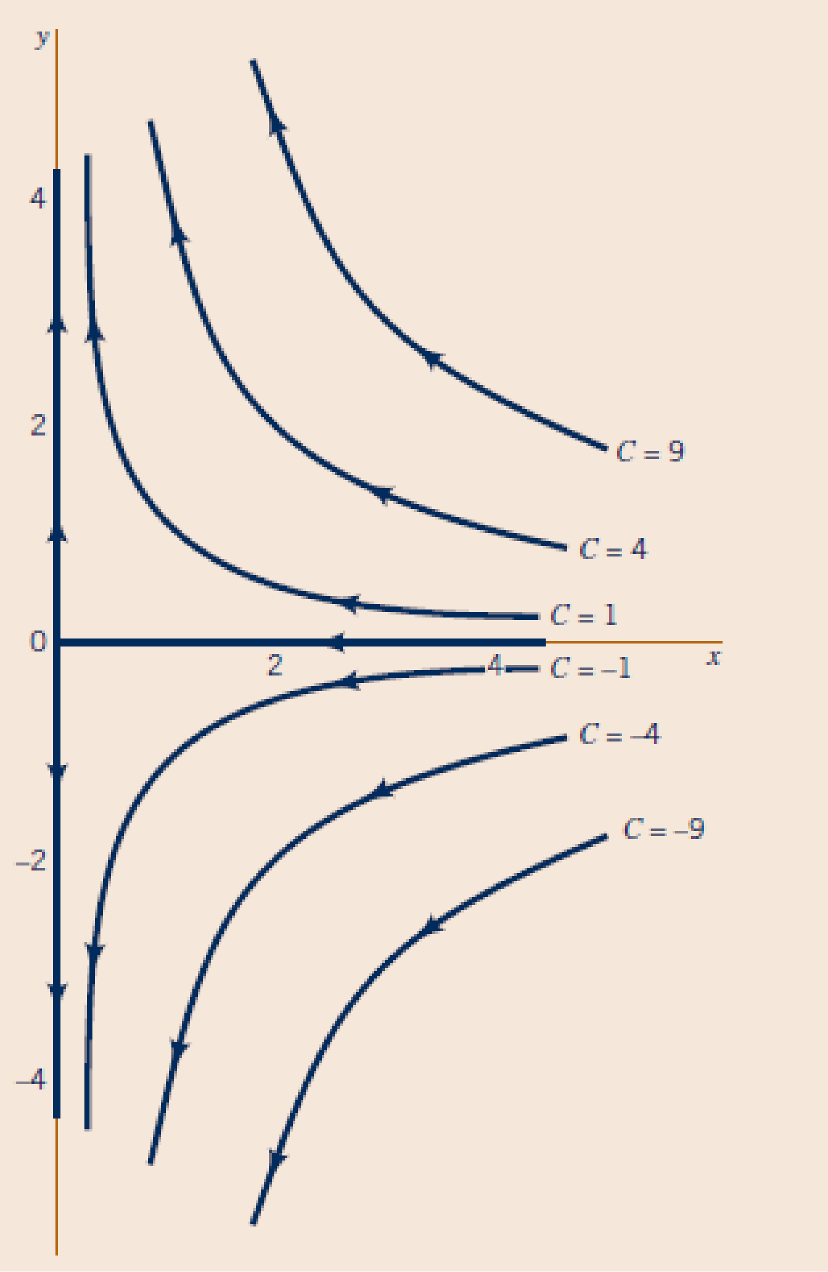
or

$$\ln y = -\ln x + \text{constant}$$

Thus, along the streamline

$$xy = C, \qquad \text{where } C \text{ is a constant} \qquad \textbf{(Ans)}$$

By using different values of the constant C, we can plot various lines in the x–y plane—the streamlines. The streamlines for $x \geq 0$ are plotted in Fig. E4.2. A comparison of this figure with Fig. E4.1*a* illustrates the fact that streamlines are lines tangent to the velocity field.

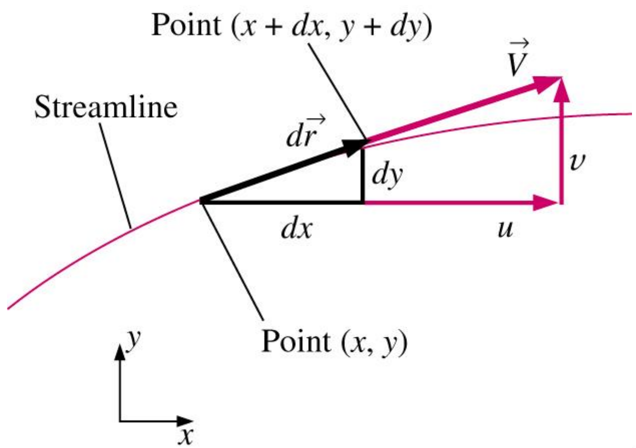

y
4
2
0
–2
–4
2
4
x
C = 9
C = 4
C = 1
C = –1
C = –4
C = –9

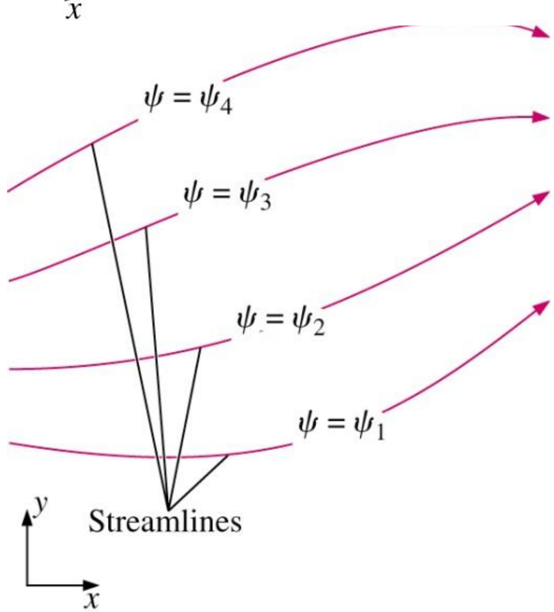
The Stream Function, ψ

- Why do this?
  - Single variable ψ replaces (u,v).
  - Once ψ is known, (u,v) can be determined.
  - Physical significance
    1. Curves of constant ψ are streamlines of the flow
    2. Difference in ψ between streamlines is equal to volume flow rate between streamlines

The Stream Function: *Physical Significance*

1. Curves of constant ψ are streamlines of the flow





Recall that the streamline equation is given by:

$$\frac{dy}{dx} = \frac{v}{u} \Rightarrow -v\,dx + u\,dy = 0$$

$$\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = 0$$

$$d\psi = 0$$

$\therefore$ Change in ψ along streamline is zero

2. Difference in ψ between streamlines is equal to volume flow rate between streamlines

- Let dq represent the volume rate of flow per unit width perpendicular to the x–y plane passing between the two streamlines.
- *From conservation of mass we know* that the inflow, *dq*, crossing the arbitrary surface *AC* must equal the net outflow through surfaces *AB* and *BC*. Thus,

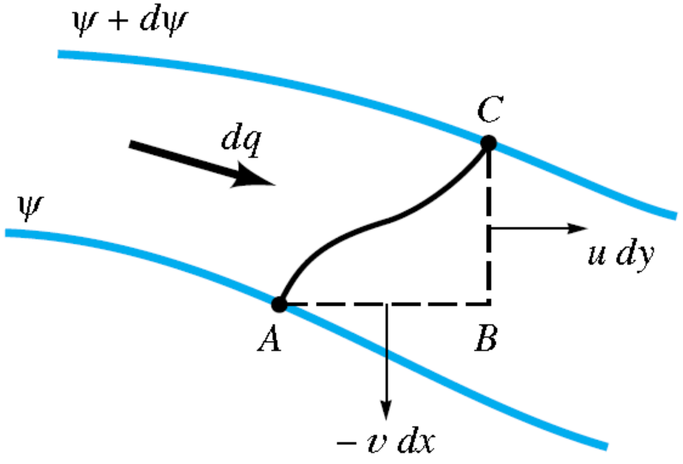
$$dq = u\,dy - v\,dx$$

- or in terms of the stream function

$$dq = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx \quad \Rightarrow \quad dq = d\psi$$

- Thus, the volume rate of flow, *q*, between two streamlines can be determined by integration to yield

$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$



Prove

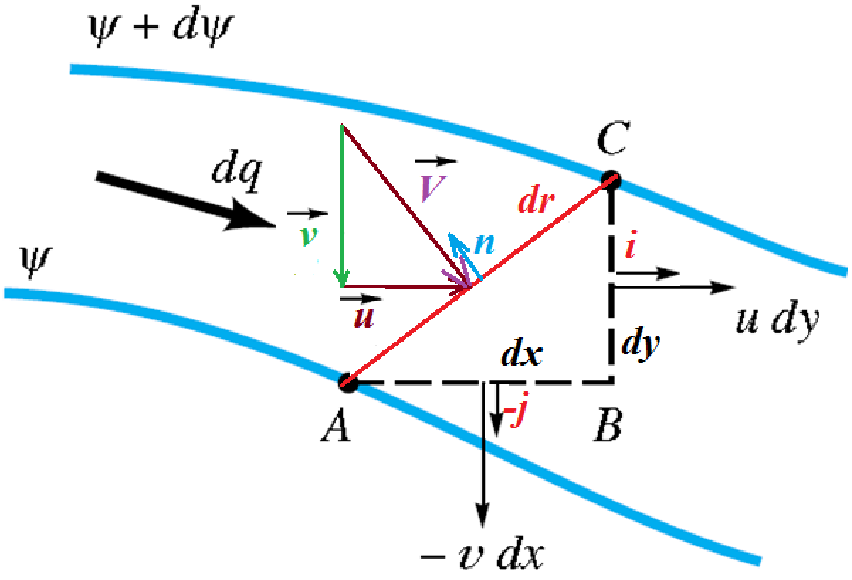
- Volume flow rate per unit depth=

$$dq = (\vec{V} \bullet \hat{n}) * area = (\vec{V} \bullet \hat{n}) * dr$$

- Now, in terms of velocity components

$$\vec{V} = u\hat{i} + v\hat{j}$$

$$dq = (u\hat{i} \bullet \hat{i}) * dy + (v\hat{j} \bullet -\hat{j}) * dx$$
$$= udy - vdx = d\Psi$$



Example Stream Function

- The velocity components in a steady, incompressible, two-dimensional flow field are

$$u = 2y \qquad v = 4x$$

Determine the corresponding stream function and show on a sketch several streamlines.

Indicate the direction of flow along the streamlines.

From the definition of the stream function

$$u = \frac{\partial \psi}{\partial y} = 2y \qquad v = -\frac{\partial \psi}{\partial x} = 4x$$

$$\psi = y^2 + f_1(x) \qquad \psi = -2x^2 + f_2(y)$$

$$\psi = -2x^2 + y^2 + C$$

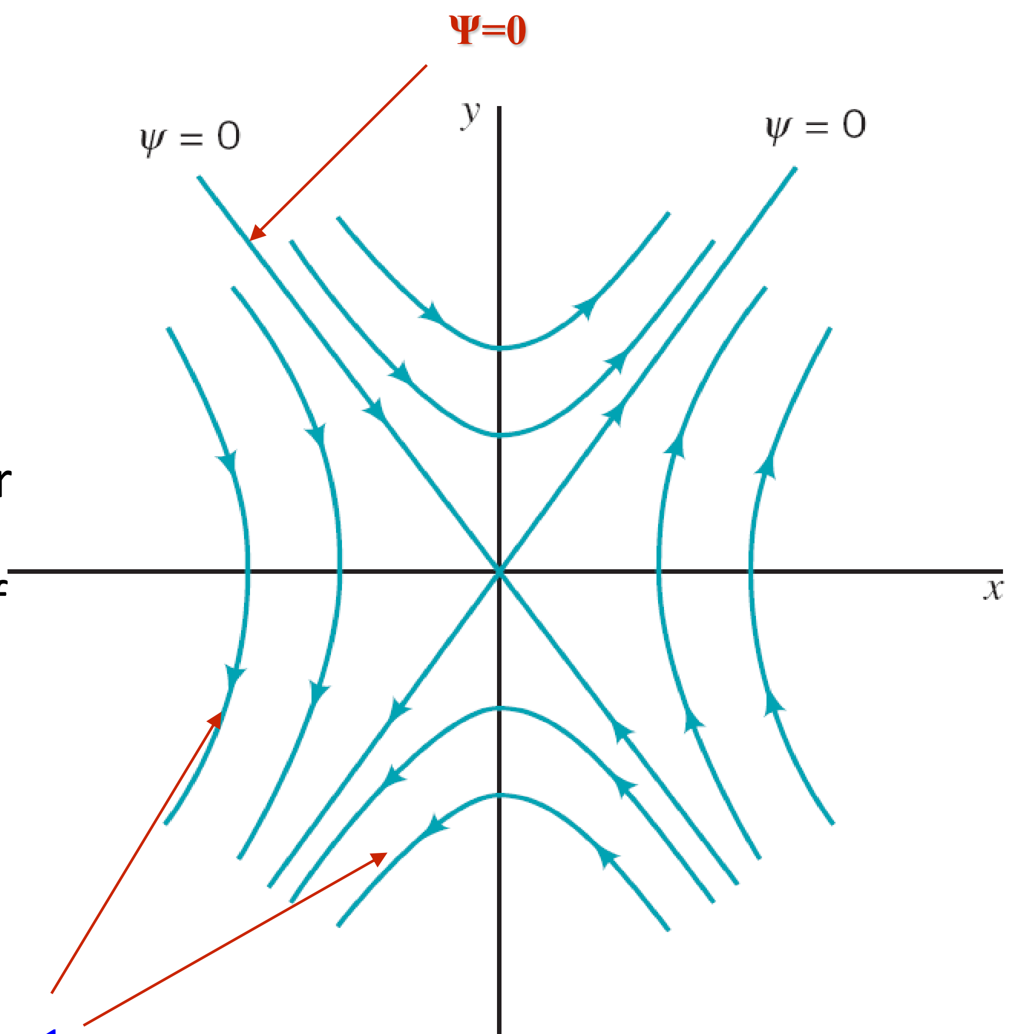
For simplicity, we set C = 0

Example Solution

$$\psi = -2x^2 + y^2$$

- Streamlines can now be determined by setting ψ=const. and plotting the resulting curve.
- With the above expression for ψ, the value of ψ at the origin is zero so that the equation of the streamline passing through the origin is

  $0 = -2x^2 + y^2$



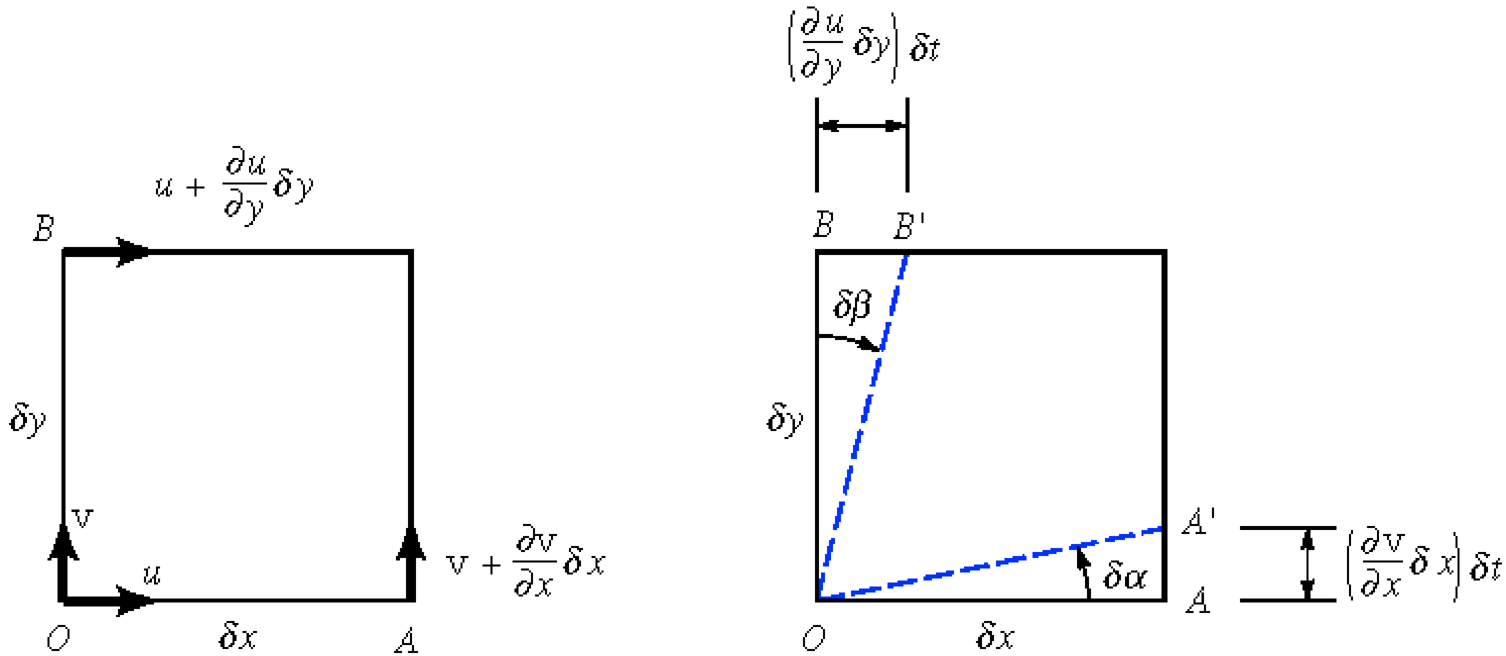
- For

$\Psi \neq 0$

$$\frac{y^2}{\psi} - \frac{x^2}{\psi/2} = 1$$

which we recognize as the equation of a hyperbola

Rate of rotation (angular velocity)



Consider the **rotation** about **z-axis** of the rectangular element **δx-δy**

The **rotation** of the side **δx**

$$\tan(\delta\alpha) \approx \delta\alpha = \frac{\left(\frac{\partial v}{\partial x}\delta x\right)\delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t$$

Rate of rotation (angular velocity)

Angular Velocity of OA

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta \alpha}{\delta t} = lim_{\delta t \to 0} \frac{\frac{\partial v}{\partial x} \delta t}{\delta t} = \frac{\partial v}{\partial x}$$

The **rotation** of the side **δy**

$$\tan(\delta \beta) \approx \delta \beta = \frac{\left(\frac{\partial u}{\partial y} \delta y\right) \delta t}{\delta y} = \frac{\partial u}{\partial y} \delta t$$

$$\frac{\frac{\partial u}{\partial y} \delta y \Big) \delta t}{\delta y} = \frac{\partial u}{\partial y} \delta t$$

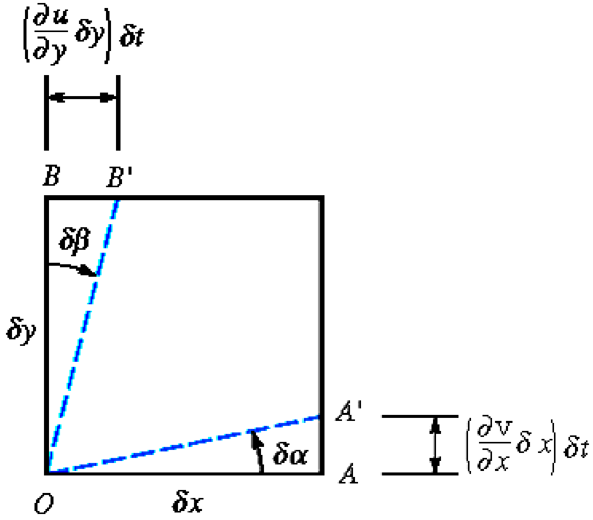
Angular Velocity of OB

$$\omega_{OB} = \lim_{\delta t \to 0} \frac{\delta \beta}{\delta t} = lim_{\delta t \to 0} \frac{\frac{\partial u}{\partial y} \delta t}{\delta t} = \frac{\partial u}{\partial y}$$

Rate of rotation (angular velocity)

- The rotation of the element about the z axis is defined as the average of the angular velocities of the two mutually perpendicular lines OA and OB. If counterclockwise rotation is considered to be positive, then:
- Average **rotation** about **z**-axis
- Average **rotation** about **x**-axis,
- Average **rotation** about **y**-axis,
- Rotation Vector

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

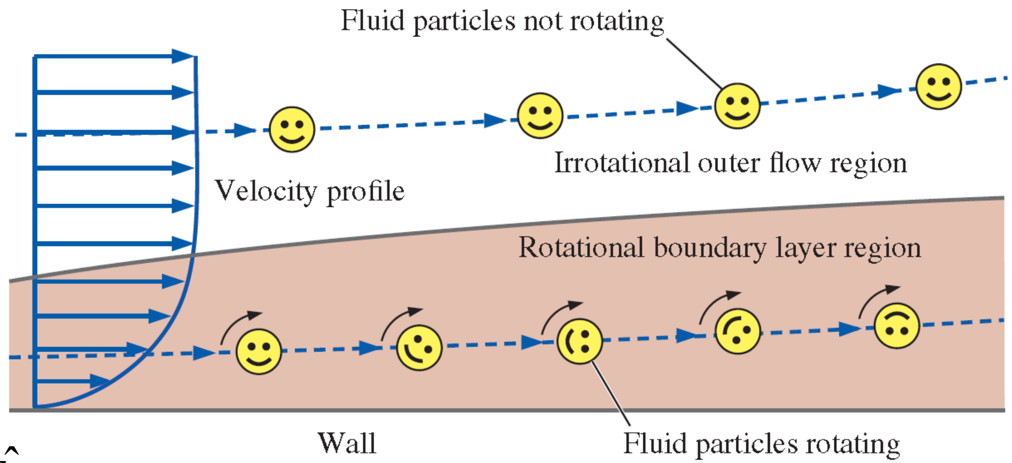


$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\therefore \vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\right]$$

Rotational and Irrotational Flows



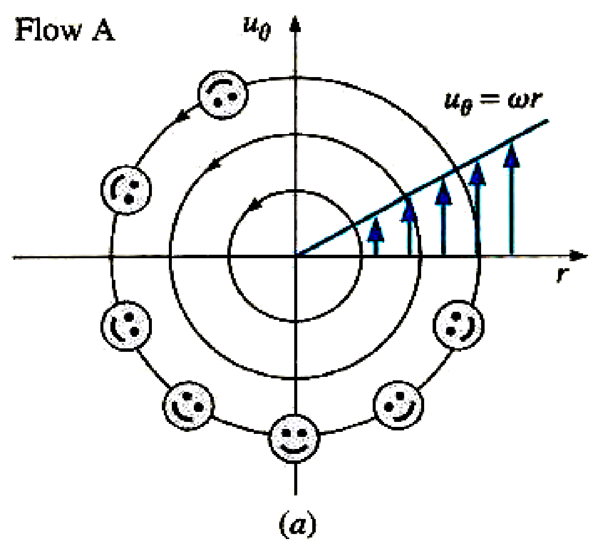
- The vorticity is defined as:

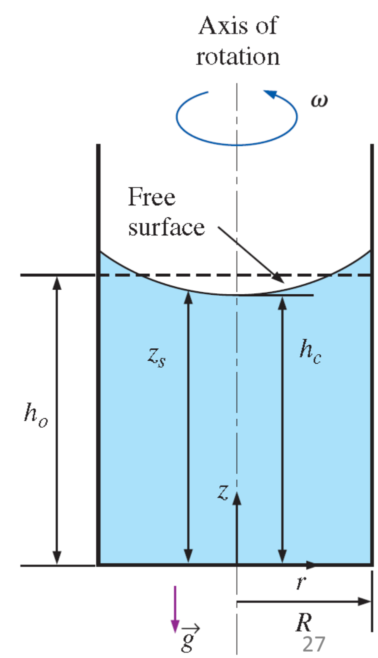
$$\zeta = 2\vec{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

For irrotational flow $\zeta = \varpi = 0$

Examples: Rotational flow:

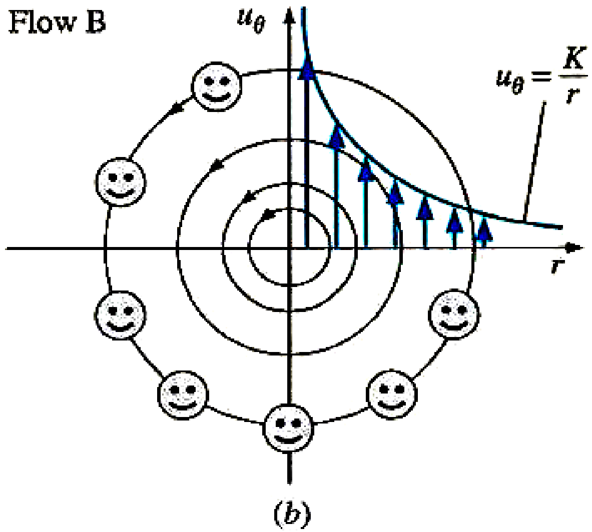
Solid-Body Rotation (Forced Vortex): $u_\theta = \omega r$

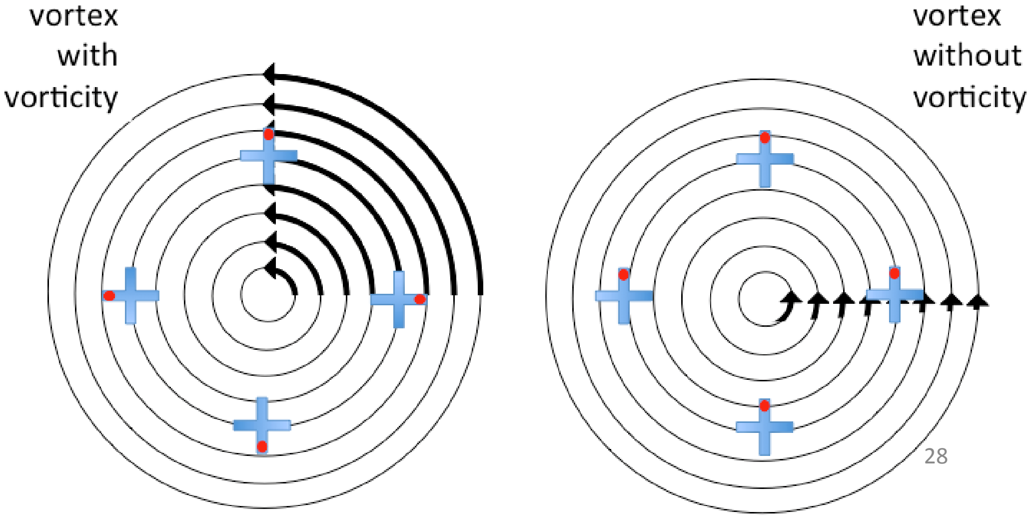




Rotational and Irrotational Flows

- Examples: Irrotational flow:
- Free Vortex: $u_\theta = K/r$






Fluxo
Rotacional
IHR

Mathematical Representation

- Vorticity is the curl of the velocity vector
- For 3-D vorticity in Cartesian coordinates:

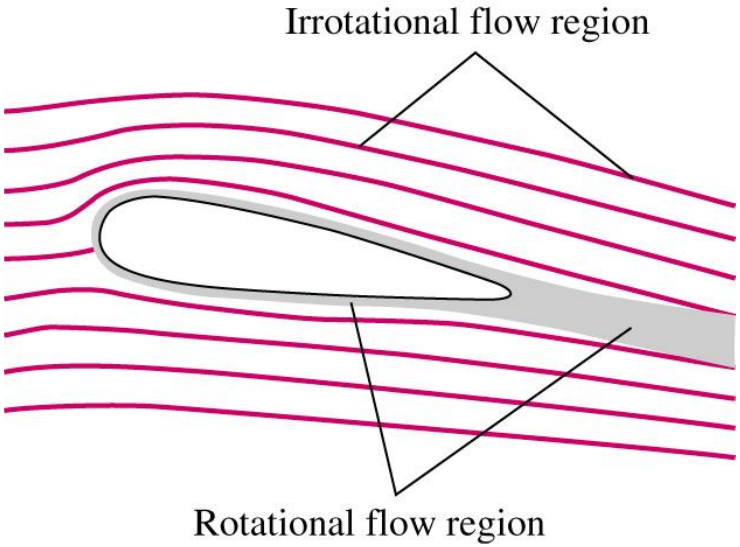
$$\nabla \times \vec{V} \equiv \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \vec{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \vec{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

The horizontal relative vorticity (about z axis) is found by eliminating terms with vertical (ω) components:

$$\vec{k} \bullet \left(\nabla \times \vec{V}\right) \equiv \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Potential Function, ϕ

$$\vec{V} = \nabla \phi$$



- Irrotational approximation: vorticity is negligibly small

$$\vec{\zeta} = \nabla \times \vec{V} \cong 0$$

- In general, inviscid regions are also irrotational, but there are situations where inviscid flow are rotational, e.g., solid body rotation.

What are the implications of irrotational approximation. Look at continuity and momentum equations.

Use the vector identity where ϕ is a scalar function

$$\nabla \times \nabla \phi = 0$$

Since the flow is irrotational where

$$\nabla \times \vec{V} = 0$$

$$\vec{V} = \nabla \phi$$

ϕ is a scalar potential function

Irrotational Flow Approximation

- Therefore, regions of irrotational flow are also called regions of <u>potential flow.</u>
- From the definition of the gradient operator ∇

Cartesian

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}$$

Cylindrical

$$u_r = \frac{\partial \phi}{\partial r}, \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \quad u_z = \frac{\partial \phi}{\partial z}$$

- Substituting into the continuity equation for incompressible flow gives:

$$\nabla \cdot \vec{V} = \nabla \cdot \nabla \phi = \boxed{\nabla^2 \phi = 0}$$

Irrotational Flow Approximation

- This means we only need to solve **1 linear scalar equation** to determine all 3 components of velocity!

$$\nabla^2 \phi = 0$$

Laplace Equation

- Luckily, the Laplace equation appears in numerous fields of science, engineering, and mathematics. This means there are well developed tools for solving this equation.

Momentum equation

If we can compute ϕ from the Laplace equation (which came from continuity) and velocity from the definition , why do we $\vec{V} = \nabla \phi$ e NSE? $\Rightarrow$ the answer: To compute Pressure.

To begin analysis, apply irrotational approximation to viscous term of the incompressible NSE

$$\mu \nabla^2 \vec{V} = \mu \nabla^2 (\nabla \phi) = \mu \nabla \underbrace{(\nabla^2 \phi)}_{= 0} = 0$$

Irrotational Flow Approximation

- Therefore, the NSE reduces to the **Euler equation** for irrotational flow
- Instead of integrating to find P, use vector identity to derive Bernoulli equation

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla \right) \vec{V} \right] = -\nabla P + \rho \vec{g}$$

$$\left(\vec{V} \cdot \nabla \right) \vec{V} = \nabla \left(\frac{V^2}{2} \right) - \vec{V} \times \left(\nabla \times \vec{V} \right) = \nabla \left(\frac{V^2}{2} \right) - \vec{V} \times \vec{\zeta}$$

Irrotational Flow Approximation

- This allows the steady Euler equation to be written as

$$= -g\vec{k} = \nabla\left(gz\right)$$

$$\nabla\left(\frac{V^2}{2}\right) - \vec{V}\times\vec{\zeta} = -\frac{1}{\rho}\nabla P + \vec{g}$$

$$\nabla\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right) = \vec{V}\times\vec{\zeta}$$

- This form of Bernoulli equation is valid for inviscid and irrotational flow since we've shown that NSE reduces to the Euler equation.
- However,

Inviscid

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = C \quad \textbf{along a streamline}$$

Irrotational ($\zeta = \vec{0}$)

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = C \quad \textbf{everywhere}$$

Irrotational Flow Approximation

- Therefore, the process for irrotational flow
  1. Calculate ϕ from Laplace equation (from continuity)
  2. Calculate velocity from definition $\vec{V} = \nabla \phi$
  3. Calculate pressure from Bernoulli equation (der omentum equation)

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \frac{p_\infty}{\rho} + \frac{V_\infty^{\ 2}}{2} + gz_\infty$$

$$P = P_\infty + \rho \left[\frac{V_\infty^2 - V^2}{2} + g\left(z_0 - z\right) \right]$$

Valid for 3D or 2D

Irrotational Flow Approximation

2D Flows

- For 2D flows, we can also use the stream function ψ
- Recall the definition of stream function for planar (x-y) flows

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

- Since vorticity is zero for irrotational flow,

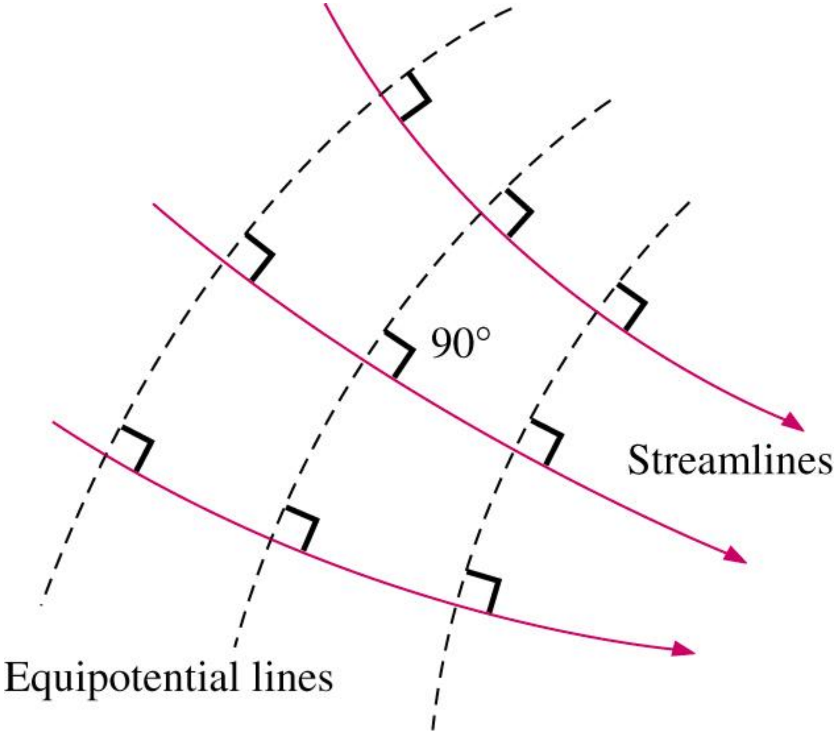
$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0 \quad = \nabla^2 \psi$$

- This proves that the Laplace equation holds for the stream function <u>and</u> the velocity potential

Irrotational Flow Approximation

2D Flows



- Constant values of ψ: streamlines
- Constant values of ϕ: equipotential lines
- ψ and ϕ are mutually orthogonal
- ψ is defined by continuity; $\nabla^2\psi$ results from irrotationality
- ϕ is defined by irrotationality; $\nabla^2\phi$ results from continuity

Flow solution can be achieved by solving either $\nabla^2\phi$ or $\nabla^2\psi$, however, BCs are easier to formulate for ψ.

Relation between ψ and ϕ lines

- If a flow is incompressible, irrotational, and two dimensional, the velocity field may be calculated using either a potential function or a stream function.
- Using the potential function, the velocity components in Cartesian coordinates are

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}$$

- And

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = udx + vdy$$

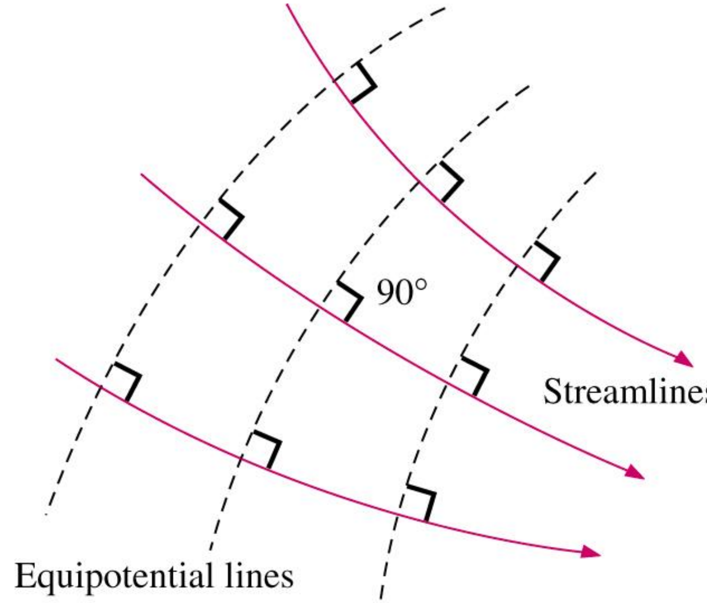
- For lines of constant potential (dϕ = 0), which are called equipotential lines:

$$\left(\frac{dy}{dx}\right)_{\phi=c} = -\frac{u}{v}$$

- Since a streamline is everywhere tangent to the local velocity, the slope of a streamline, which is a line of constant ψ, is

Relation between ψ and ϕ lines

$$\left(\frac{dy}{dx}\right)_{\Psi=c} = \frac{v}{u}$$



- Comparing equations of slopes yields:

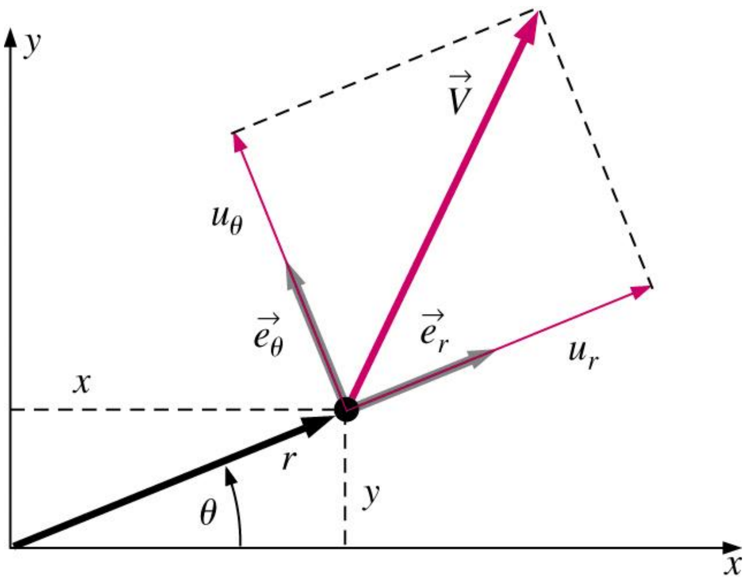
$$\left(\frac{dy}{dx}\right)_{\phi=c} = -\frac{1}{(dy/dx)_{\Psi=c}}$$

- The slope of an equipotential line is the negative reciprocal of the slope of a streamline.
- Therefore, streamlines (ψ = constant) are everywhere orthogonal (perpendicular) to equipotential lines (ϕ = constant).
- This observation is not true, however, at stagnation points, where the components vanish simultaneously.

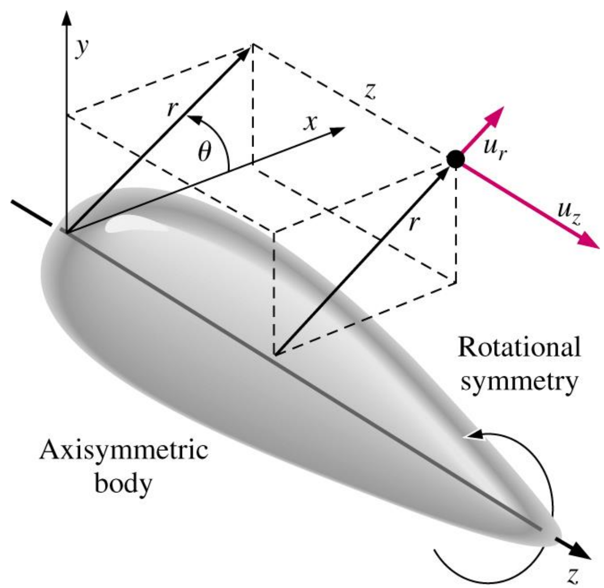
Irrotational Flow Approximation

2D Flows

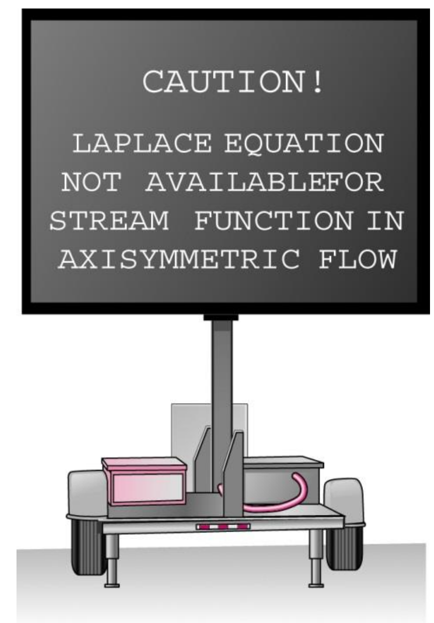
- Similar derivation can be performed for cylindrical coordinates (except for $\nabla^2\psi$ for axisymmetric flow)
  - Planar, cylindrical coordinates: flow is in (r,θ) plane
  - Axisymmetric, cylindrical coordinates : flow is in (r,z) plane



Planar



Axisymmetric



Irrotational Flow Approximation

2D Flows

TABLE 10–2

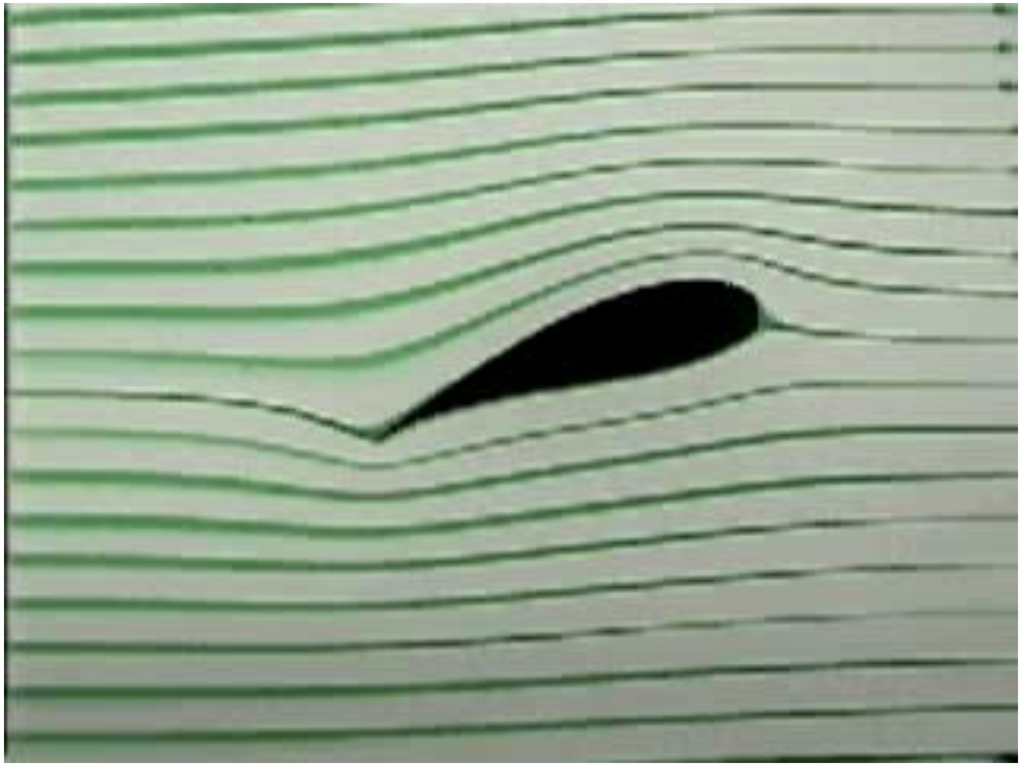
Velocity components for steady, incompressible, irrotational, two-dimensional regions of flow in terms of velocity potential function and stream function in various coordinate systems

Description and Coordinate System	Velocity Component 1	Velocity Component 2
Planar; Cartesian coordinates	$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$	$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$
Planar; cylindrical coordinates	$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$	$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$
Axisymmetric; cylindrical coordinates	$u_r = \frac{\partial \phi}{\partial r} = -\frac{1}{r}\frac{\partial \psi}{\partial z}$	$u_z = \frac{\partial \phi}{\partial z} = \frac{1}{r}\frac{\partial \psi}{\partial r}$

Potential flows Visualization

- Flow fields for which an incompressible fluid is assumed to be frictionless and the motion to be irrotational are commonly referred to as potential flows.
- Paradoxically, potential flows can be simulated by a slowly moving, viscous flow between closely spaced parallel plates.

- For such a system, dye injected upstream reveals an approximate potential flow pattern around a streamlined airfoil shape.
- Similarly, the potential flow pattern around a bluff body is shown. Even at the rear of the bluff body the streamlines closely follow the body shape.
- Generally, however, the flow would separate at the rear of the body, an important phenomenon not accounted for with potential theory.



Irrotational Flow Approximation

2D Flows

- **Method of Superposition**
  1. Since $\nabla^2\phi = 0$ is linear, a linear combination of two or more solutions is also a solution, e.g., if ϕ_1 and ϕ_2 are solutions, then ($A\phi_1$), ($A+\phi_1$), ($\phi_1+\phi_2$), ($A\phi_1+B\phi_2$) are also solutions
  2. Also true for ψ in 2D flows ($\nabla^2\psi = 0$)
  3. Velocity components are also additive

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial\left(\phi_1 + \phi_2\right)}{\partial x} = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial x}$$

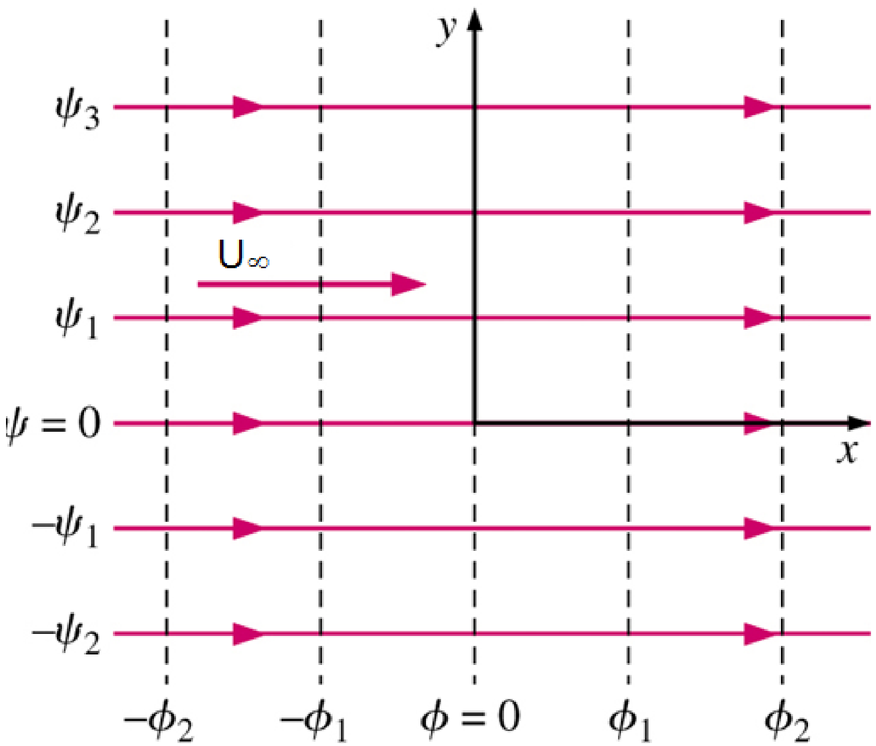
Irrotational Flow Approximation

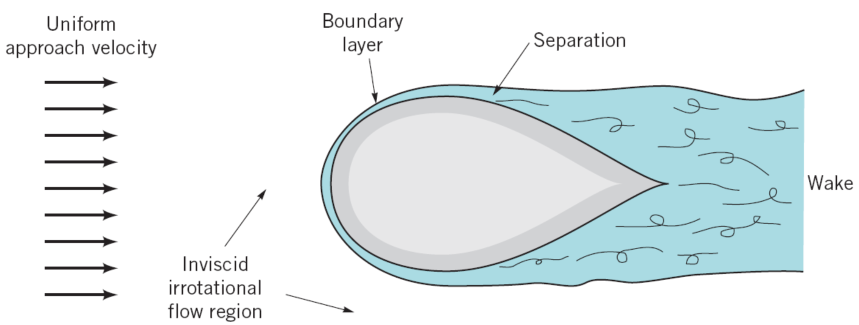
2D Flows

- Given the principal of superposition, there are several elementary planar irrotational flows which can be combined to create more complex flows.
- Elementary Planar Irrotational Flows
  - Uniform stream
  - Line source/sink
  - Line vortex
  - Doublet

Elementary Planar Irrotational Flows

Uniform Stream





- $u = U_\infty$=constant , v = 0, w = 0
- In Cartesian coordinates

$$u = \frac{\partial \psi}{\partial y} = U_\infty \qquad v = -\frac{\partial \psi}{\partial x} = 0$$

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

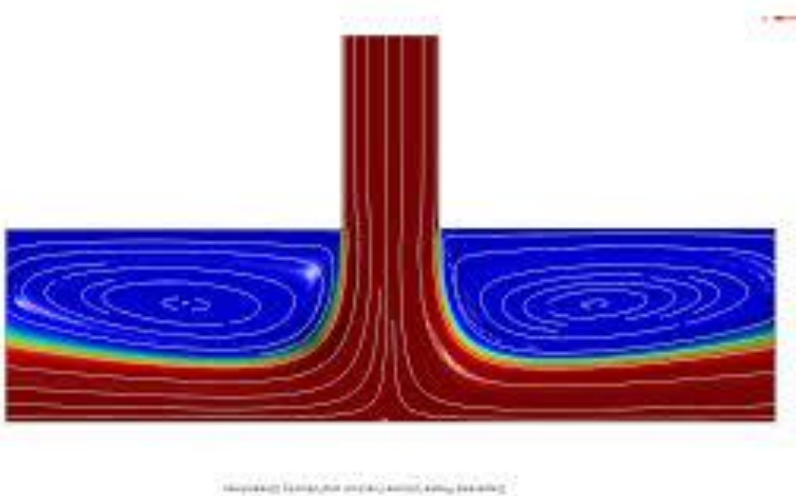
- $\phi = U_\infty x$, $\psi = U_\infty y$
- Conversion to cylindrical coordinates can be achieved using the transformation

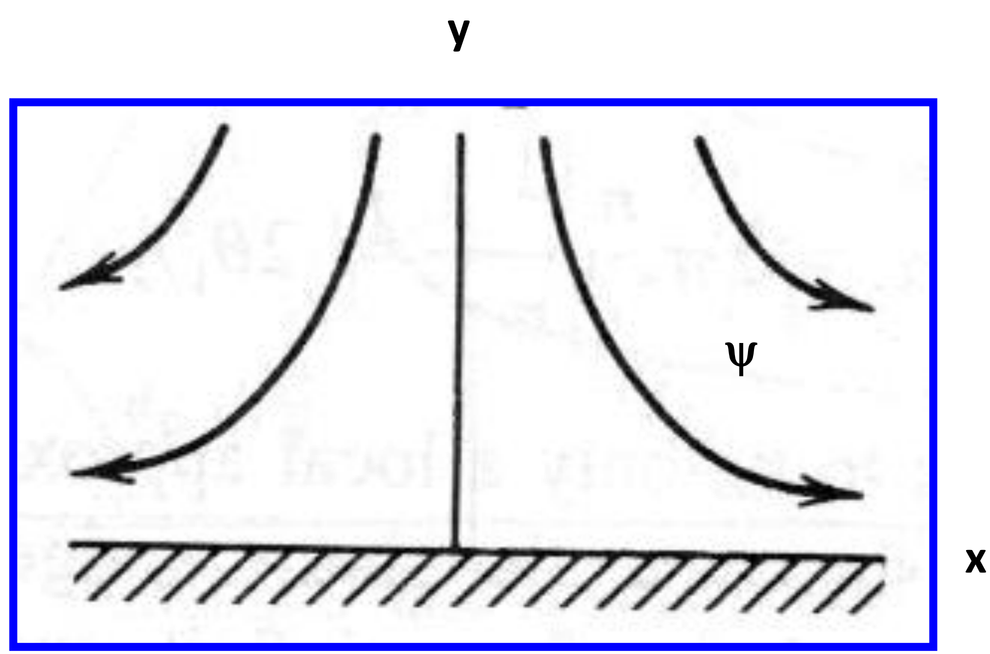
$$x = rcos\theta, \quad y = rsin\theta$$

- $\phi = U_\infty r \cos\theta$, $\psi = U_\infty r \sin\theta$

Stagnation Flow

- The flow is an incoming far field flow which is perpendicular to the wall, and then turns its direction near the wall
- The origin is the stagnation point of the flow. The velocity is zero there.



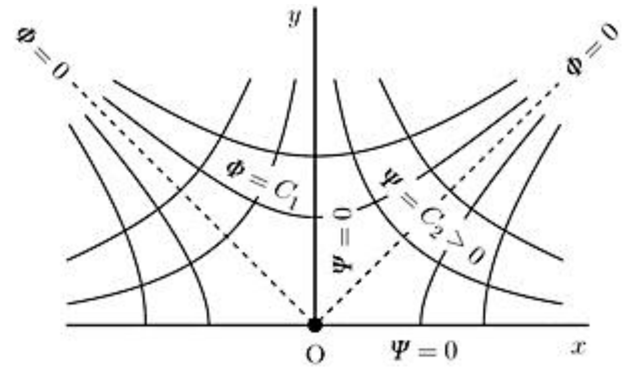


Application: Stagnation Flow

- For a stagnation flow,
- Hence,

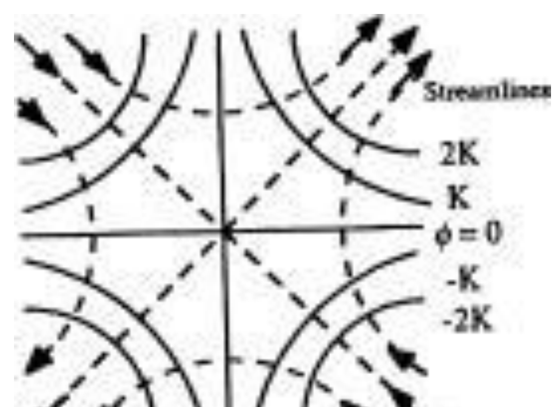
$$\vec{\mathbf{V}} = \left(Bx\,\hat{i} - By\ \hat{j}\right)$$

$$\frac{\partial \phi}{\partial x} = u = Bx, \qquad \frac{\partial \phi}{\partial y} = v = -By$$



- Therefore,
- And $\phi = \frac{B}{2}(x^2 - y^2) = \frac{B}{2}r^2\cos 2\theta$

$$\frac{\partial \psi}{\partial y} = u = Bx, \qquad -\frac{\partial \psi}{\partial x} = v = -By$$

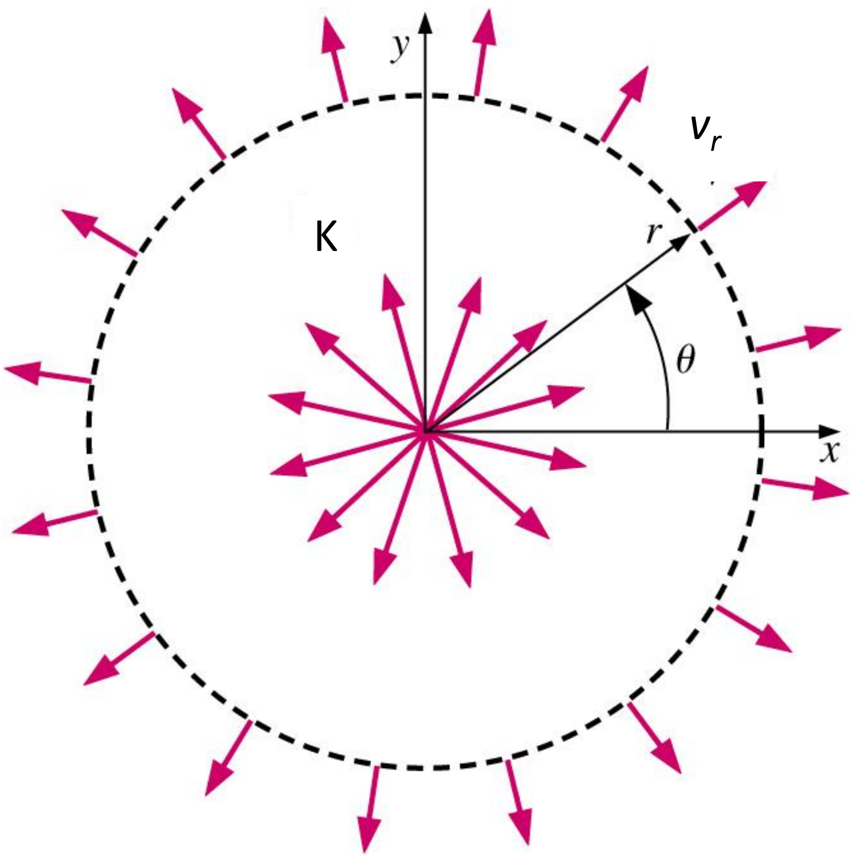


- Therefore

$$\psi = Bxy = \frac{B}{2}r^2\sin 2\theta$$

Elementary Planar Irrotational Flows

Line Source/Sink



- Let's consider fluid flowing radially outward from a line through the origin perpendicular to x-y plane
- from mass conservation:
- The volume flow rate per unit thickness is K
- This gives velocity components

$$v_r = \frac{K}{2\pi r} \quad \text{and} \quad v_\theta = 0$$

$$v_r = \frac{K}{2\pi r} = \frac{\partial \phi}{\partial r} = \frac{\partial \psi}{r \partial \theta}$$

$$\text{and} \quad v_\theta = 0 = \frac{\partial \phi}{r \partial \theta} = -\frac{\partial \psi}{\partial r}$$

Stream function and potential function

$$\frac{\partial \phi}{r\partial \theta} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial r} = \frac{K}{2\pi r}$$

By integration:

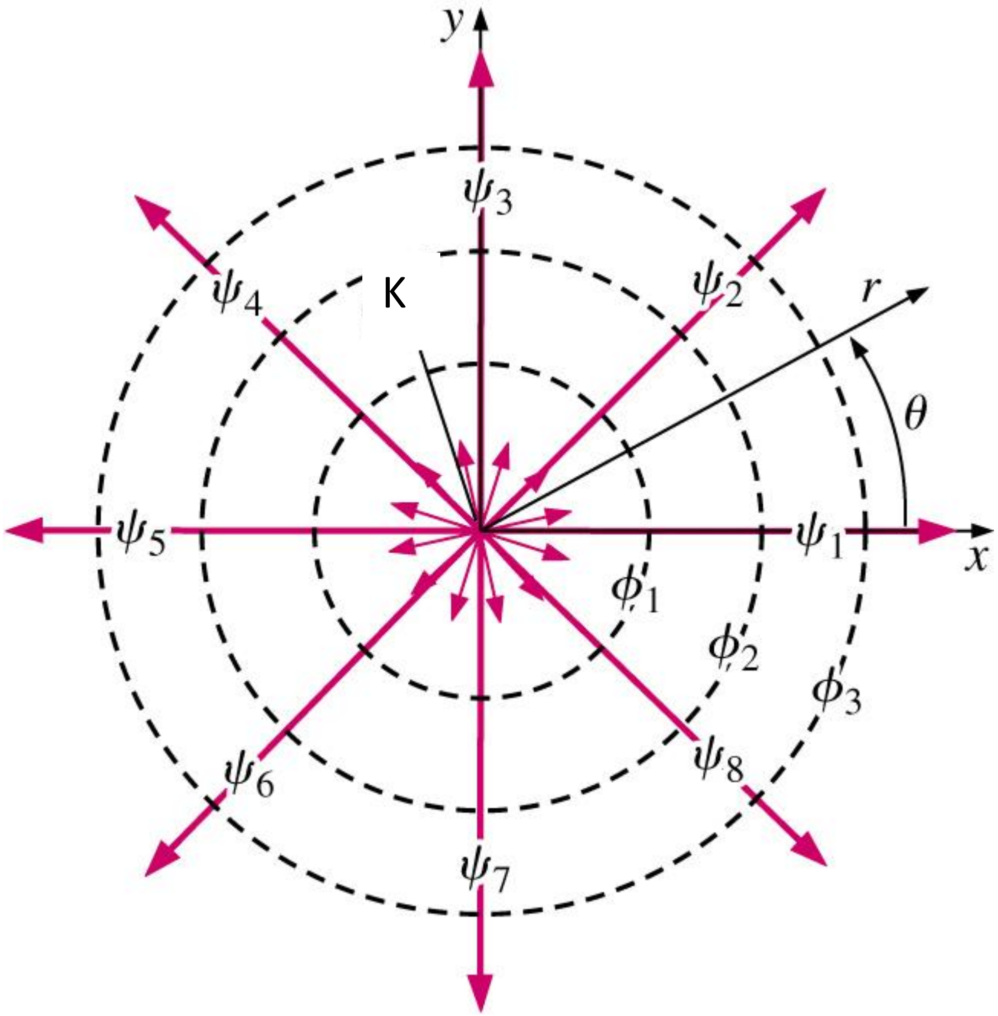
$$\phi = \frac{K}{2\pi} \ln r$$

$$-\frac{\partial \psi}{\partial r} = 0$$

$$\text{and } \frac{\partial \psi}{r\partial \theta} = \frac{K}{2\pi r} \Rightarrow \frac{\partial \psi}{\partial \theta} = \frac{K}{2\pi}$$

By integration:

$$\Psi = \frac{K}{2\pi} \theta$$



Equations are for a source/sink at the origin

Elementary Planar Irrotational Flows

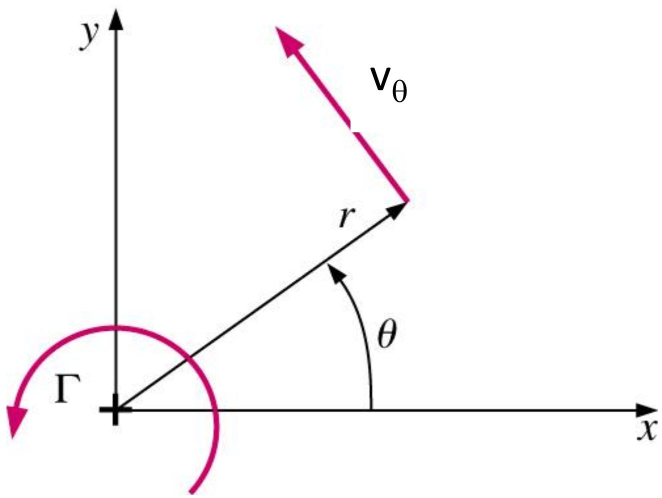
Line (potential) Vortex

- A *potential vortex* is defined as a singularity about which fluid flows with concentric streamlines
- Vortex at the origin. First look at velocity components

$$v_r = \frac{\partial \emptyset}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

$$v_\theta = \frac{1}{r}\frac{\partial \emptyset}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

- These can be integrated to give ϕ and ψ

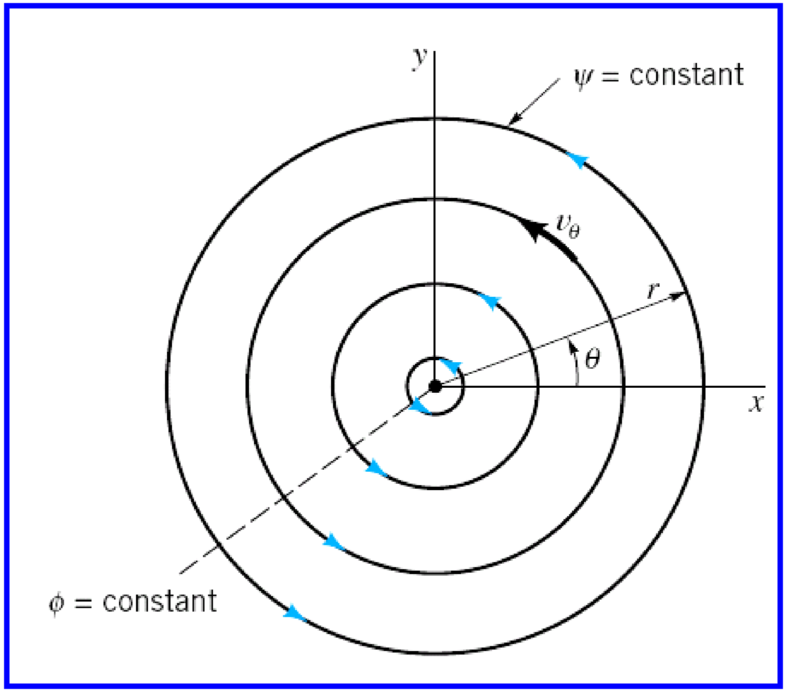


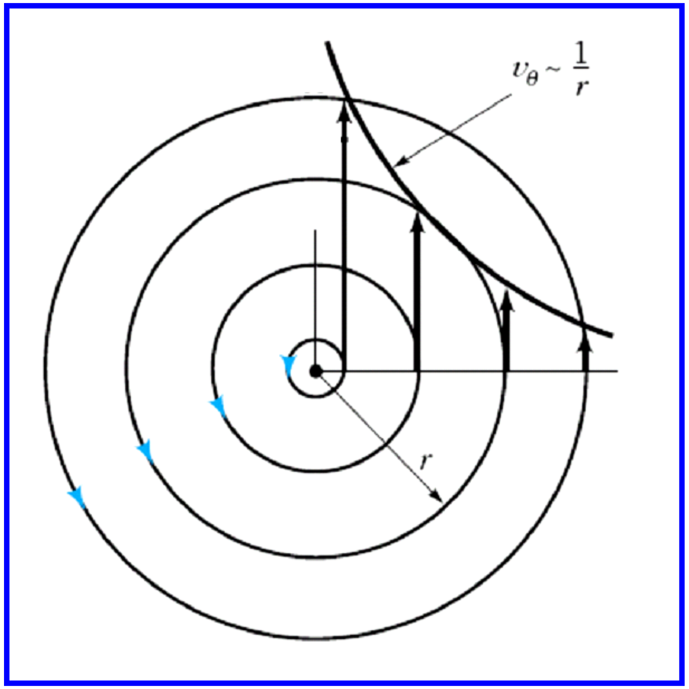
Equations are for a line vortex at the origin where the arbitrary integration constants are taken to be zero at (r,θ)=(1,0)

$$\phi = \frac{\Gamma}{2\pi}\theta \quad \& \quad \psi = -\frac{\Gamma}{2\pi}\ln r$$

Free Vortex

- The potential represents a flow swirling around origin with a constant circulation Γ.
- The magnitude of the flow decreases as 1/r.

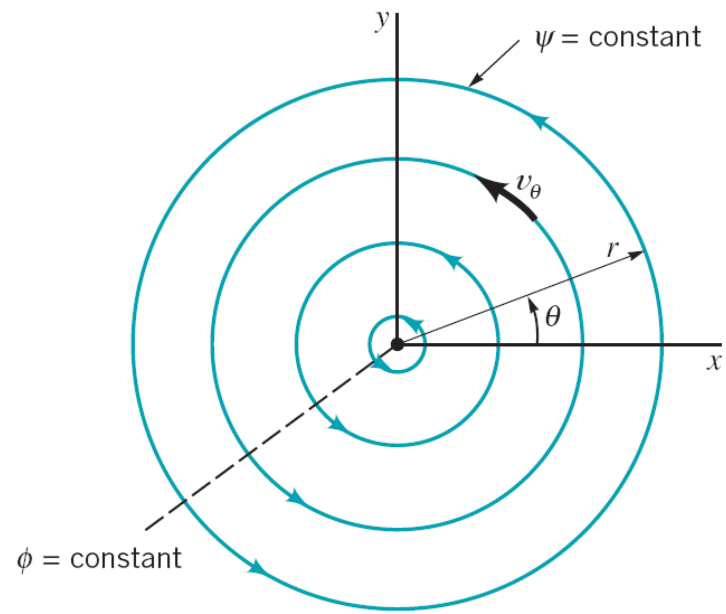




Line Vortex

- now we consider situation when the stream lines are concentric circles i.e. we interchange potential and stream functions:

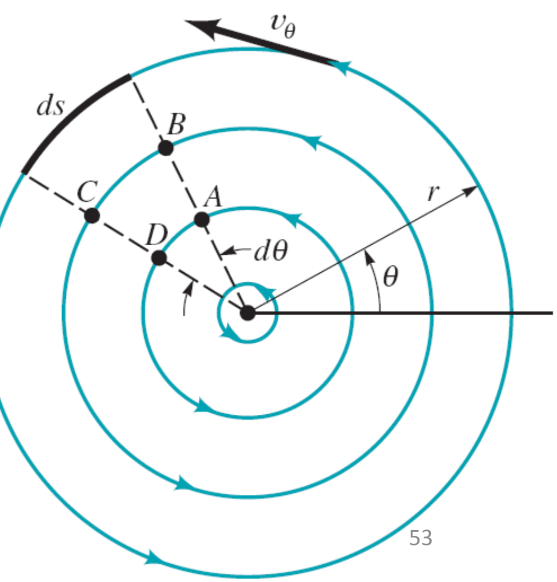
$$\phi = K\theta$$
$$\psi = -K \ln r$$



- circulation

$$\Gamma = \oint_C \mathbf{V} \cdot ds = \oint_C \nabla \phi \cdot ds = \oint_C d\phi = 0$$

- in case of vortex the circulation is zero along any contour except ones enclosing origin

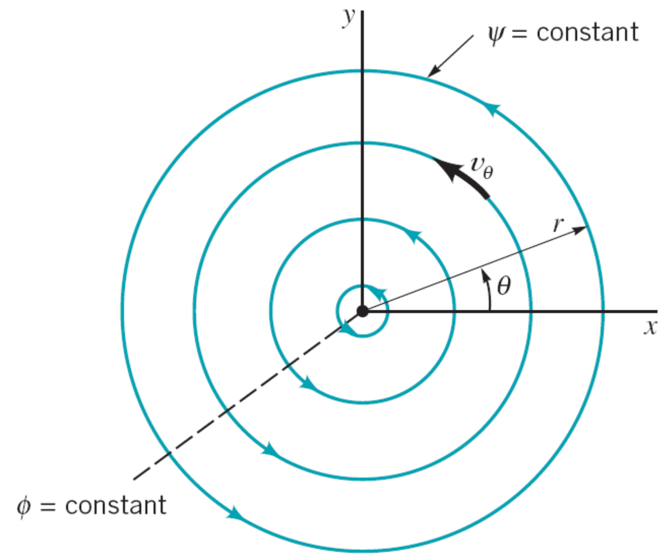


$$\Gamma = \int_0^{2\pi} \frac{K}{r}(r d\theta) = 2\pi K$$

$$\phi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r$$

Shape of the free surface

$$\phi = \frac{\Gamma}{2\pi}\theta \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$



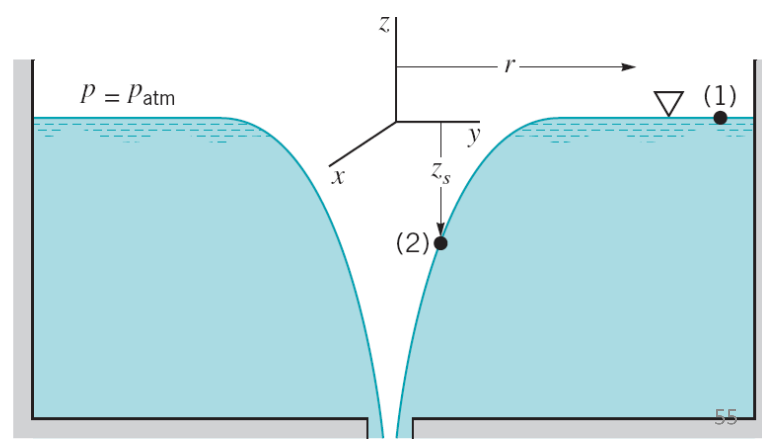
Bernolli's equation

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = const$$

at the free surface p=0:

$$\frac{V_1^2}{2g} = \frac{V_2^2}{2g} + z$$

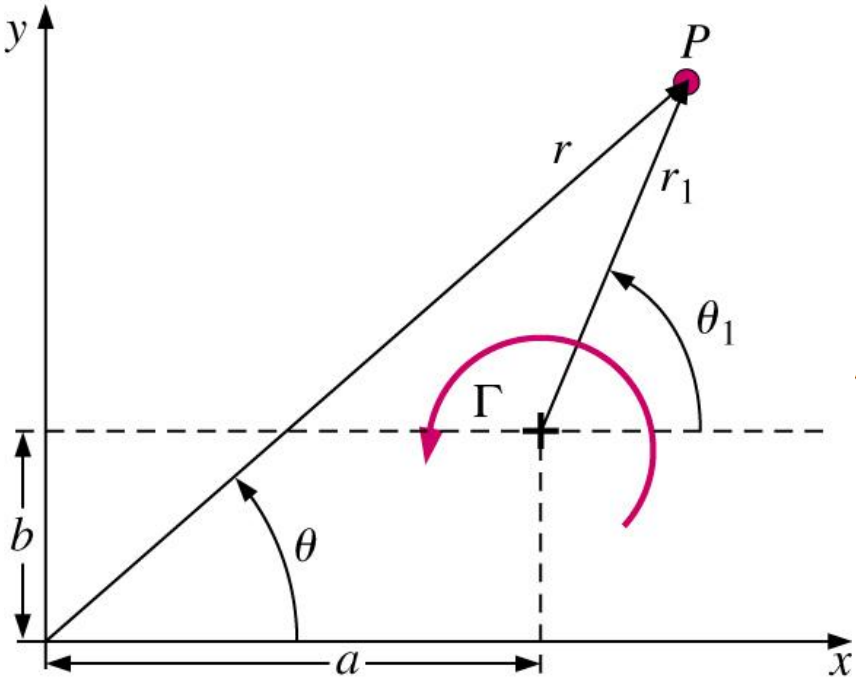
$$z = -\frac{\Gamma^2}{8\pi^2 r^2 g}$$



Elementary Planar Irrotational Flows

Line Vortex

- If vortex is moved to (x,y) = (a,b)



$$\phi = \frac{\Gamma}{2\pi}\theta_1 = \frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y-b}{x-a}\right)$$

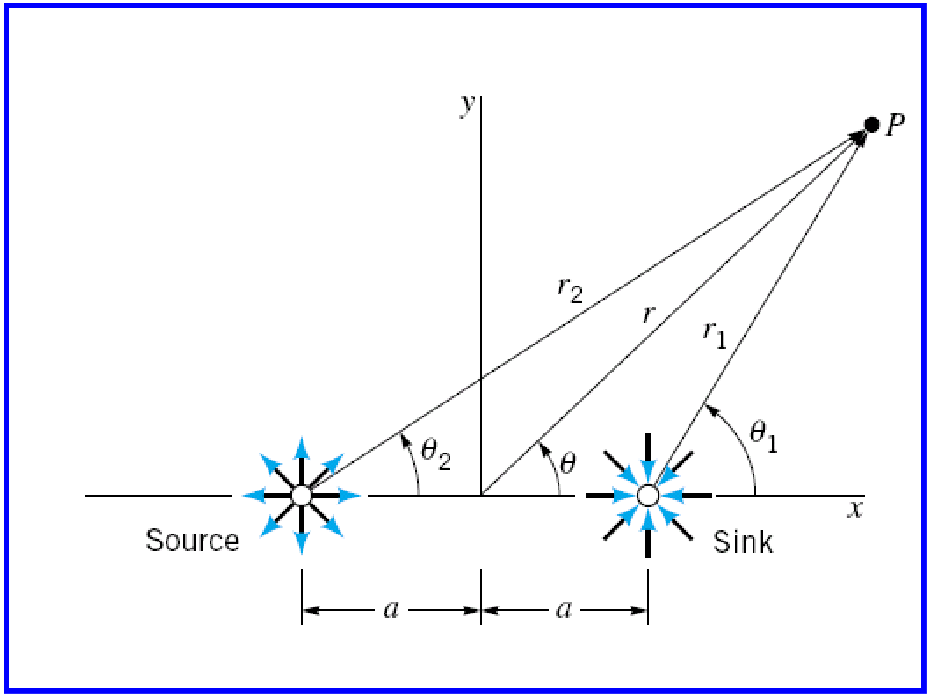
$$\psi = -\frac{\Gamma}{2\pi}\ln r_1 = -\frac{\Gamma}{2\pi}\ln\sqrt{(x-a)^2+(y-b)^2}$$

Source and Sink

- Consider a source of strength K at (-a, 0) and a sink of K at (a, 0)
- For a point P with polar coordinate of (r, θ). If the polar coordinate from (-a,0) to P is (r_2, θ_2) and from (a, 0) to P is (r_1, θ_1),
- Then the stream function and potential function obtained by superposition are given by:

$$\psi = \frac{K}{2\pi}(\theta_2 - \theta_1) \quad ,$$

$$\phi = \frac{K}{2\pi}(\ln r_2 - \ln r_1)$$



Source and Sink

$$\psi = \frac{K}{2\pi}(\theta_1 - \theta_2)$$

- Hence,

$$\tan\left(\frac{2\pi\psi}{K}\right) = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

- Since

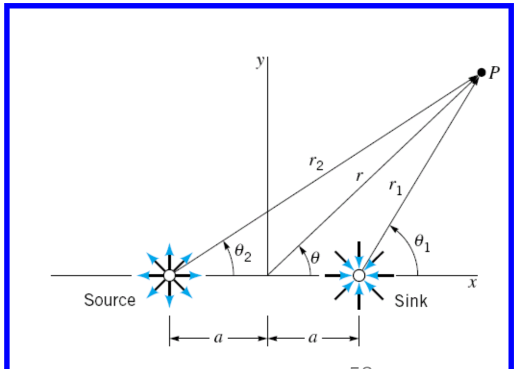
$$\tan\theta_2 = \frac{r\sin\theta}{r\cos\theta + a} \quad \text{and} \quad \tan\theta_1 = \frac{r\sin\theta}{r\cos\theta - a}$$

- We have

$$\tan\left(\frac{2\pi\psi}{K}\right) = \frac{-2ar\sin\theta}{r^2 - a^2}$$

- We have

$$\psi = \frac{K}{2\pi}\tan^{-1}\left(\frac{-2ar\sin\theta}{r^2 - a^2}\right)$$



Source and Sink

- We have

$$\phi = \frac{K}{2\pi}(\ln r_1 - \ln r_2) = \frac{K}{2\pi}(\ln \frac{r_2}{r_1})$$

$$r_2^2 = (r\sin\theta)^2 + (r\cos\theta + a)^2 = r^2 + a^2 + 2ar\cos\theta$$

$$r_1^2 = (r\sin\theta)^2 + (r\cos\theta - a)^2 = r^2 + a^2 - 2ar\cos\theta$$

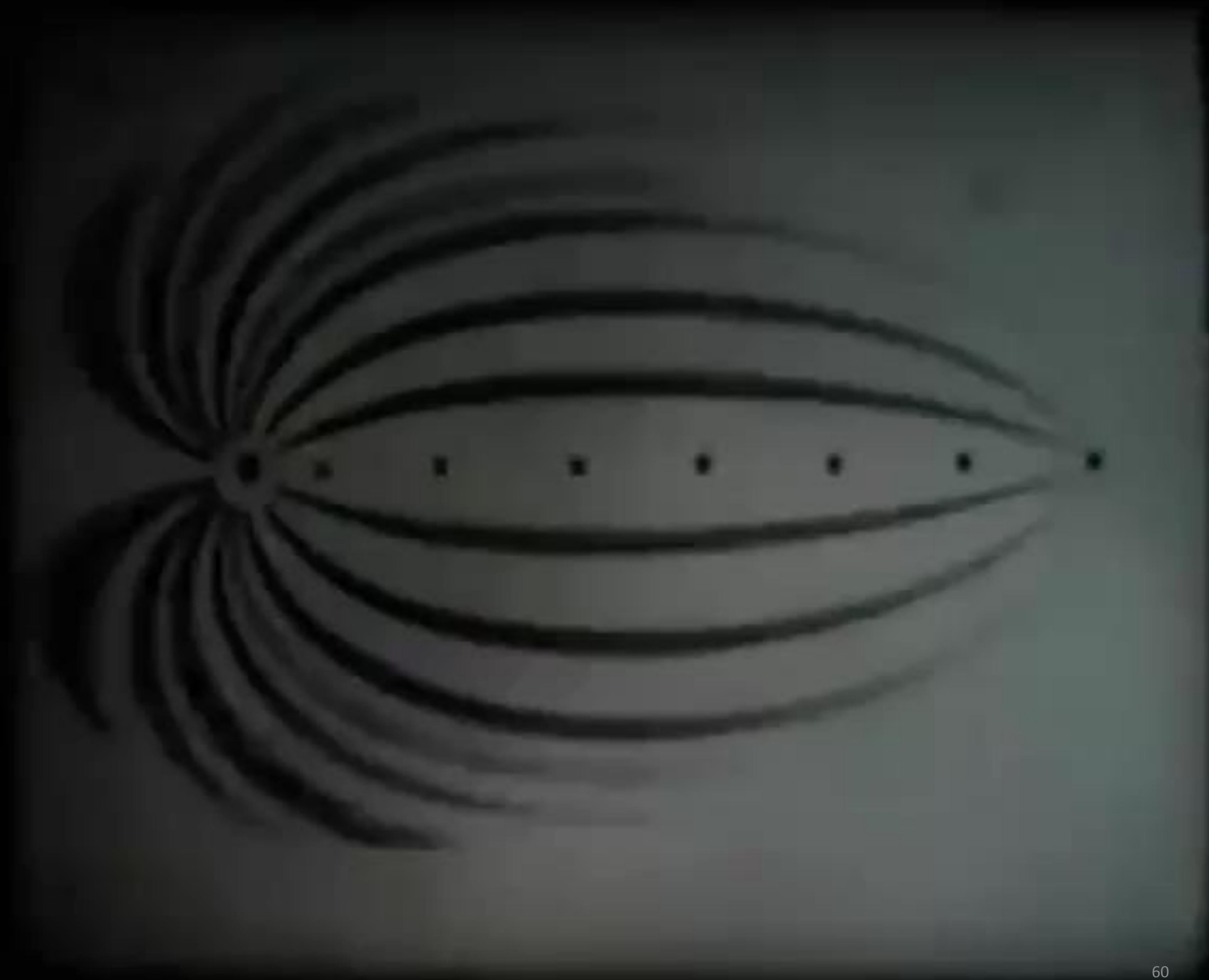
- Therefore,

$$\phi = \frac{K}{2\pi}\ln\left(\frac{\sqrt{r^2 + a^2 + 2ar\cos\theta}}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}\right)$$

- The velocity component are:

$$v_r = \frac{K}{2\pi}\left(\frac{r\cos\theta + a}{r^2 + a^2 + 2ar\cos\theta} - \frac{r\cos\theta - a}{r^2 + a^2 - 2ar\cos\theta}\right)$$

$$v_\theta = \frac{K}{2\pi}\left(\frac{r\sin\theta}{r^2 + a^2 + 2ar\sin\theta} - \frac{r\sin\theta}{r^2 + a^2 - 2ar\sin\theta}\right)$$



Doublet

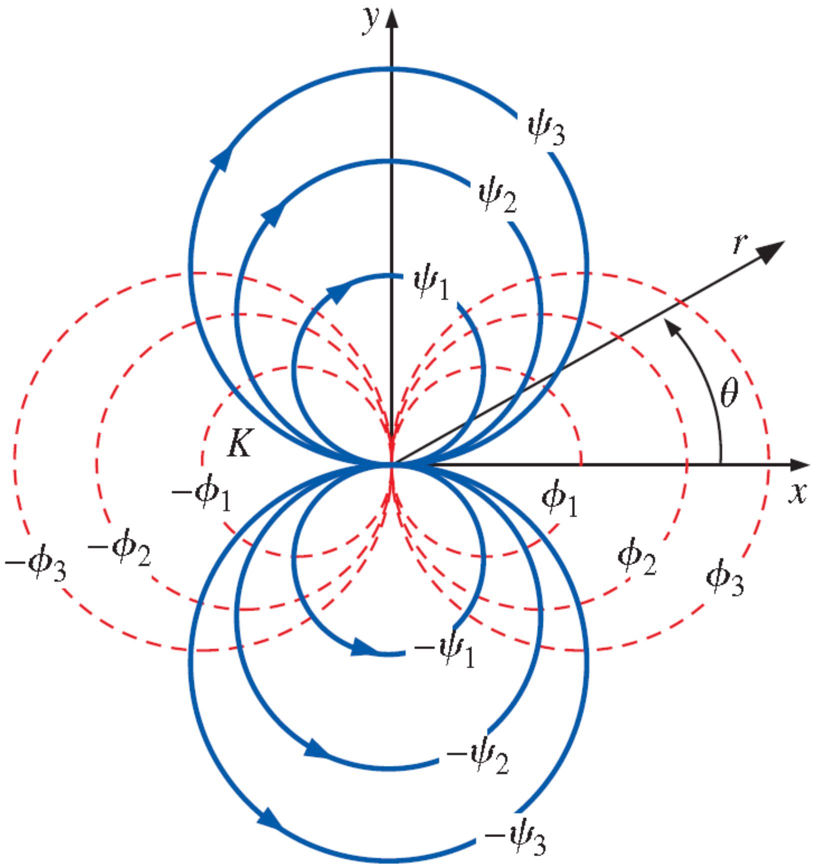
- The doublet occurs when a source and a sink of the same strength are collocated the same location, say at the origin.
- This can be obtained by placing a source at (-a,0) and a sink of equal strength at (a,0) and then letting a → 0, and K→ ∞, with *Ka* kept constant, say *aK/2π=B*
- For source of K at (-a,0) and sink of K at (a,0)

$$\psi = \frac{K}{2\pi}\tan^{-1}\left(\frac{-2ar\sin\theta}{r^2-a^2}\right) \quad \text{and} \quad \phi = \frac{K}{2\pi}\ln\left(\frac{\sqrt{r^2+a^2+2ar\cos\theta}}{\sqrt{r^2+a^2-2ar\cos\theta}}\right)$$

Under these limiting conditions of a → 0, K → ∞ , we have

$$\lim_{a\to 0}\tan^{-1}\left(\frac{-2ar\sin\theta}{r^2-a^2}\right) = \frac{-2a\sin\theta}{r}$$

$$\lim_{a\to 0}\ln\left(\frac{\sqrt{r^2+a^2+2ar\cos\theta}}{\sqrt{r^2+a^2-2ar\cos\theta}}\right) = \frac{2a}{r}\cos\theta$$



Doublet (Summary)

- Adding ψ_1 and ψ_2 together, performing some algebra, Therefore, as a→0 and K→ ∞ with $aK/2\pi=B$ then:

$$\psi = -B\frac{sin\theta}{r}$$

$$\phi = B\frac{cos\theta}{r}$$

B is the doublet strength

The velocity components for a doublet may be found the same way we found them for the source

$$v_r = \frac{\partial \emptyset}{\partial r} = -\frac{Bcos\theta}{r^2}$$

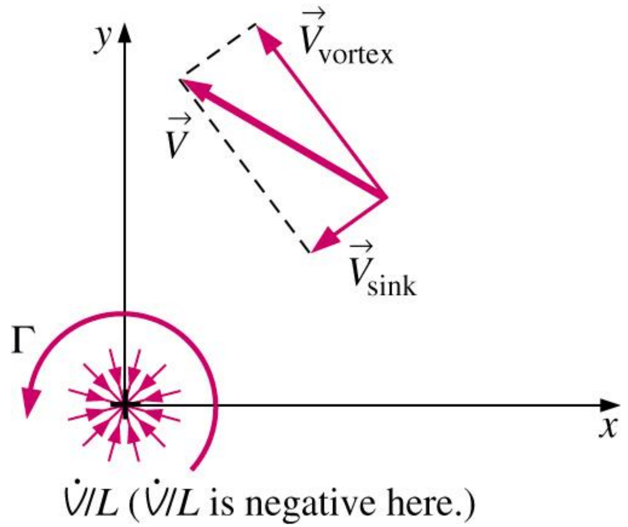
&

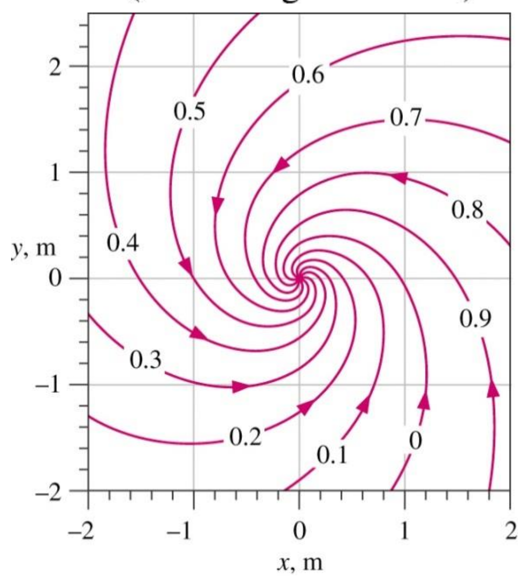
$$v_\theta = \frac{1}{r}\frac{\partial \emptyset}{\partial \theta} = \frac{Bsin\theta}{r^2}$$

Description of Flow Field	Velocity Potential	Stream Function	Velocity Components[a]
Uniform flow at angle α with the x axis (see Fig. 6.16*b*)	$\phi = U(x \cos \alpha + y \sin \alpha)$	$\psi = U(y \cos \alpha - x \sin \alpha)$	$u = U \cos \alpha$ $v = U \sin \alpha$
Source or sink (see Fig. 6.17) $m > 0$ source $m < 0$ sink	$\phi = \frac{m}{2\pi} \ln r$	$\psi = \frac{m}{2\pi} \theta$	$v_r = \frac{m}{2\pi r}$ $v_\theta = 0$
Free vortex (see Fig. 6.18) $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \frac{\Gamma}{2\pi} \theta$	$\psi = -\frac{\Gamma}{2\pi} \ln r$	$v_r = 0$ $v_\theta = \frac{\Gamma}{2\pi r}$
Doublet (see Fig. 6.23)	$\phi = \frac{K \cos \theta}{r}$	$\psi = -\frac{K \sin \theta}{r}$	$v_r = -\frac{K \cos \theta}{r^2}$ $v_\theta = -\frac{K \sin \theta}{r^2}$

Examples of Irrotational Flows Formed by Superposition

Superposition of sink and vortex : bathtub vortex





- Superposition of sink and vortex : bathtub vortex

$$\Psi = \frac{K}{2\pi}\theta - \frac{\Gamma}{2\pi}\ln r$$

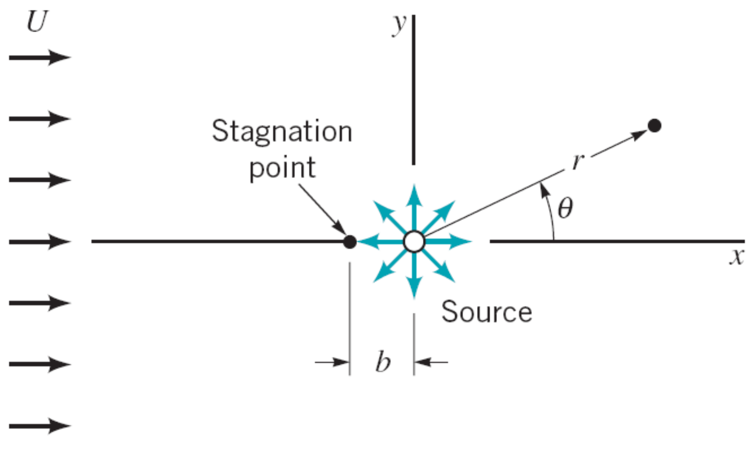
Sink — Vortex

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{K}{2\pi r}$$

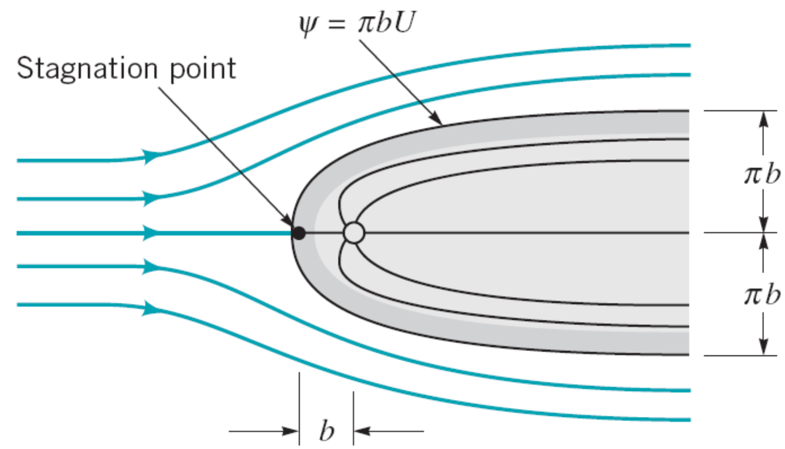
$$v_\theta = -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

Superposition of Source and Uniform Flow

- Assuming the uniform flow U∞ is in x-direction and the source of K stregth at(0,0), the potential and stream functions of the superposed potential flow become:



(a)



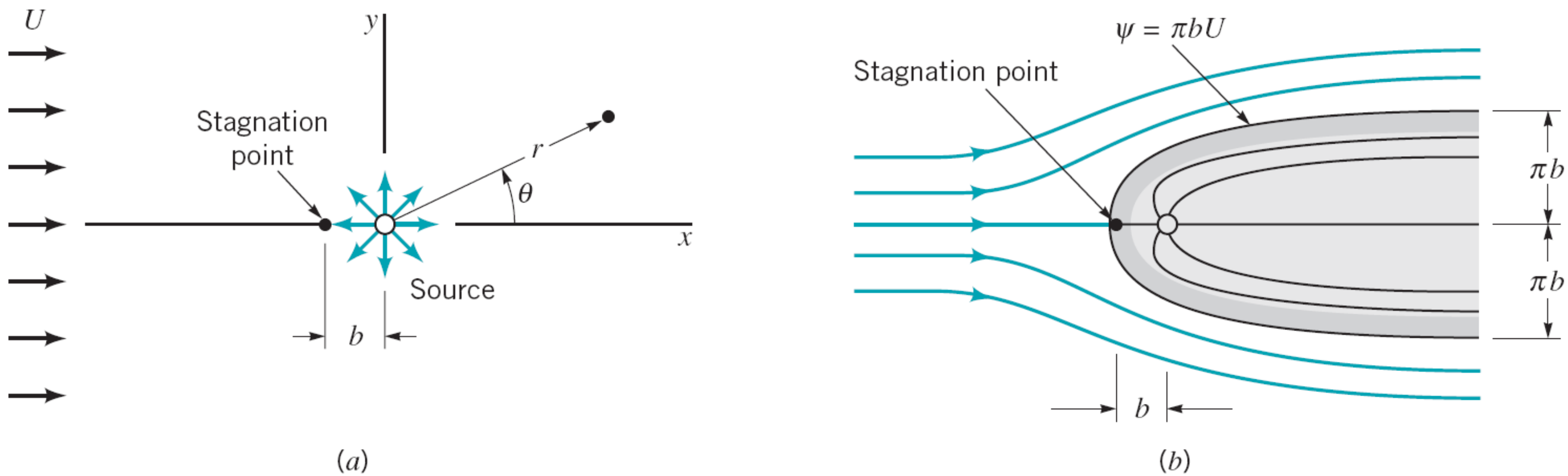
(b)

$$\psi = U_\infty r sin\theta + \frac{K}{2\pi}\theta$$ & $$\emptyset = U_\infty r cos\theta + \frac{K}{2\pi}\ln r$$

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U_\infty cos\theta + \frac{K}{2\pi r}$$ & $$v_\theta = -\frac{\partial\psi}{\partial r} = -U_\infty sin\theta$$

Superposition of Source and Uniform Flow

- Assuming the uniform flow U∞ is in x-direction and the source of K stregth at(0,0), the potential and stream functions of the superposed potential flow become:



$$v_r = \frac{m}{2\pi r}$$

so that the stagnation point will occur at $x = -b$ where

$$U = \frac{m}{2\pi b}$$

or

$$b = \frac{m}{2\pi U}$$

Source in Uniform Stream

- The velocity components are:

$$v_r = \frac{\partial \phi}{\partial r} = U_\infty \cos\theta + \frac{K}{2\pi r} \text{ and } v_\theta = \frac{\partial \phi}{r \partial \theta} = U_\infty \sin\theta$$

- A stagnation point (v_r=v_θ=0) occurs at

$$\theta = \pi \text{ and } r_s = \frac{K}{2\pi U_\infty} \Rightarrow r_s U_\infty = \frac{K}{2\pi}$$

Therefore, the streamline passing through the stagnation point when

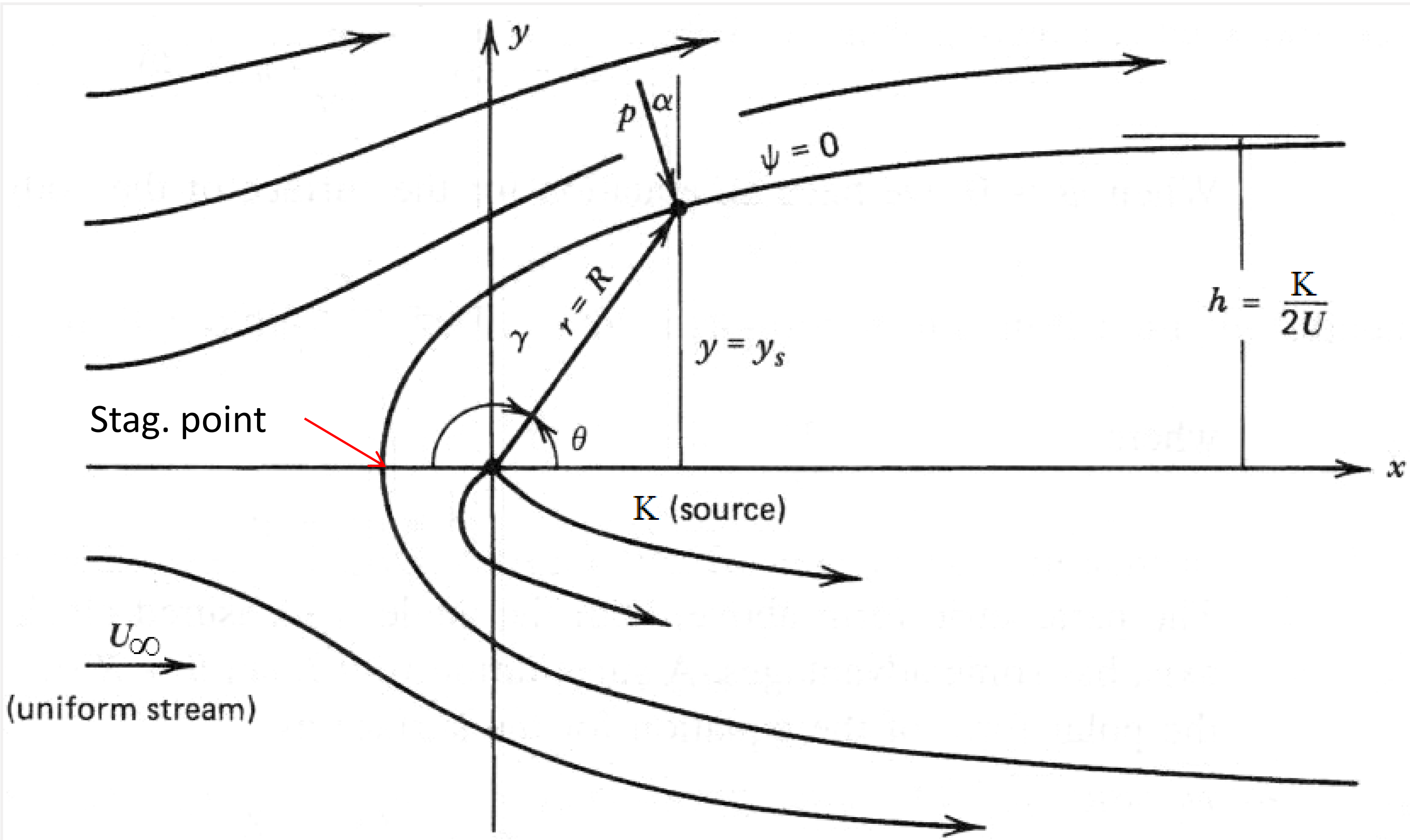
$$\psi = U_\infty r sin\theta + \frac{K}{2\pi}\theta$$

$$\psi_s = \frac{K}{2} = \pi r_s U_\infty$$

- The maximum height of the $\psi = \psi_s = \frac{K}{2}$ curve is

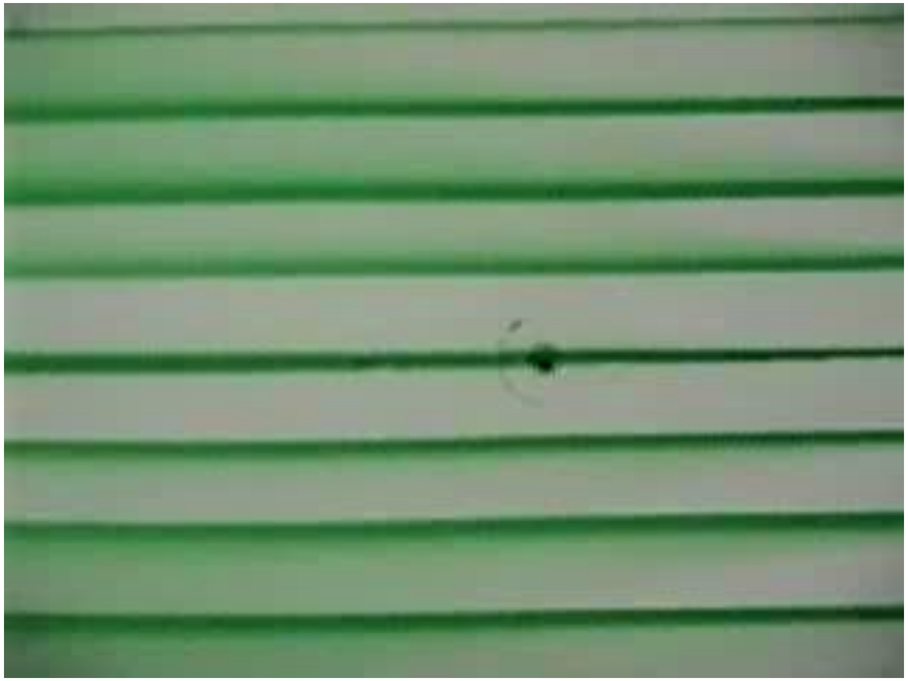
$$h = r\sin\theta = \frac{K}{2U_\infty} \text{ as } \theta \to 0 \text{ and } r \to \infty$$

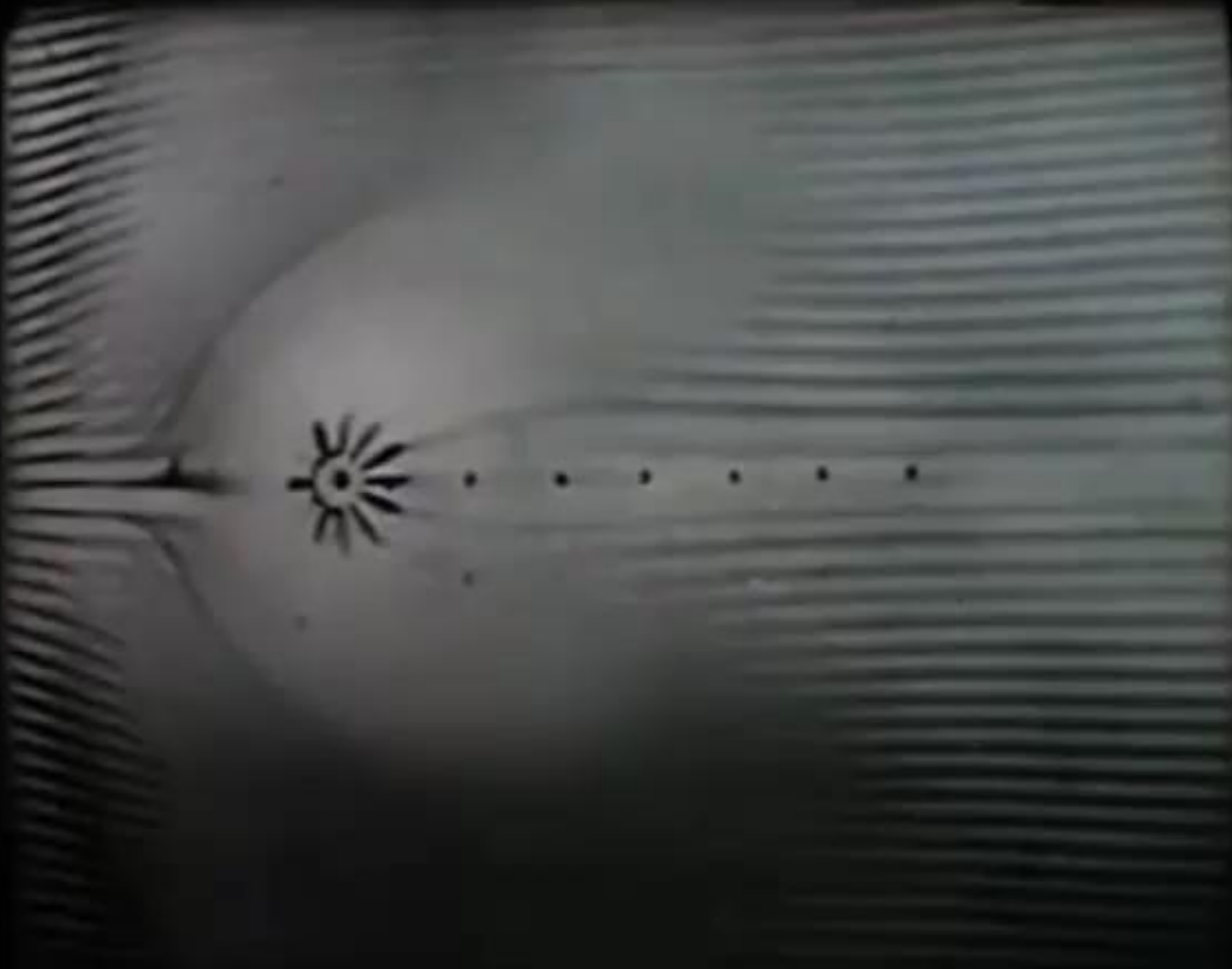
Source in Uniform Stream



Superposition of basic flows

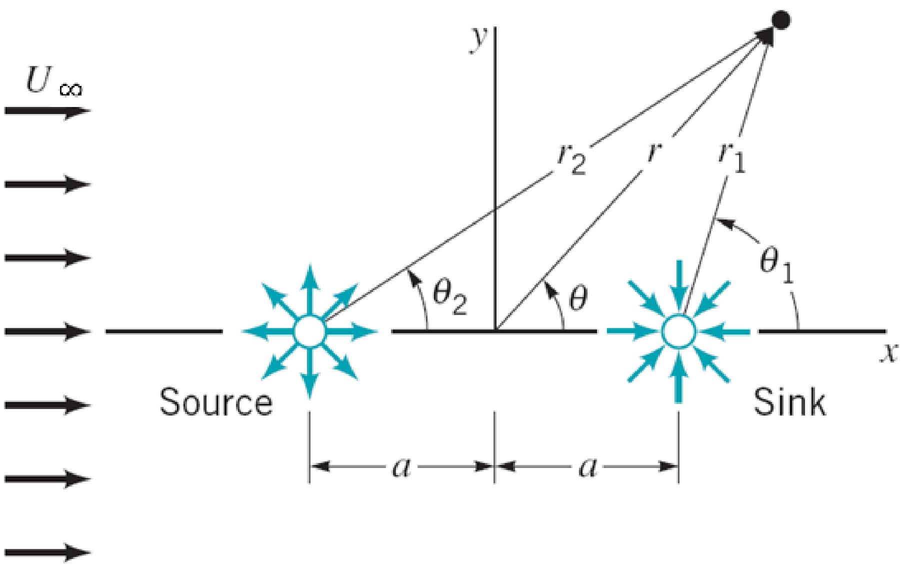
- Streamlines created by injecting dye in steadily flowing water show a uniform flow.
- Source flow is created by injecting water through a small hole.
- It is observed that for this combination the streamline passing through the stagnation point could be replaced by a solid boundary which resembles a streamlined body in a uniform flow.
- The body is open at the downstream end and is thus called a halfbody.

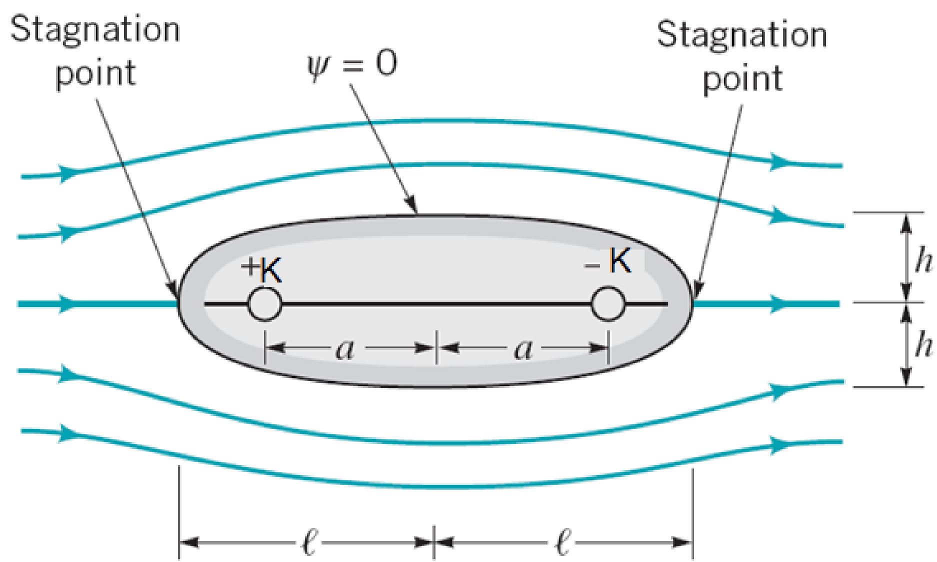




Rankine Ovals

- The 2D Rankine ovals are the results of the superposition of equal strength (K) sink and source at x=a and –a with a uniform flow in x-direction.





$$\psi = U_\infty r sin\theta + \frac{K}{2\pi}(\theta_2 - \theta_1)$$

$$\emptyset = U_\infty r cos\theta + \frac{K}{2\pi}(ln r_2 - \ln r_1)$$

Rankine Ovals

- Equivalently,

$$\phi = U_\infty r\cos\theta + \frac{K}{2\pi}\ln\left(\frac{\sqrt{r^2+a^2+2ar\cos\theta}}{\sqrt{r^2+a^2-2ar\cos\theta}}\right)$$

$$\psi = U_\infty r\sin\theta - \frac{K}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2-a^2}\right)$$

- The velocity components are given by:

$$v_r = \frac{\partial\phi}{\partial r} = \frac{K}{2\pi}\left(\frac{r\cos\theta + a}{r^2+a^2+2ar\cos\theta} - \frac{r\cos\theta - a}{r^2+a^2-2ar\cos\theta}\right)$$

$$v_\theta = \frac{\partial\phi}{r\partial\theta} = \frac{K}{2\pi}\left(\frac{r\sin\theta}{r^2+a^2+2ar\sin\theta} - \frac{r\sin\theta}{r^2+a^2-2ar\sin\theta}\right)$$

- The stagnation points occur at where V = 0 with corresponding $\psi_s = 0$

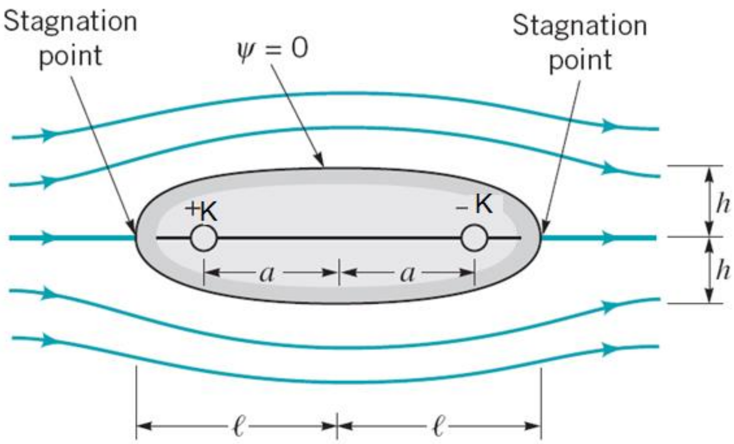
$$x_s = \pm\left(\frac{Ka}{\pi U_\infty} + a^2\right)^{\frac{1}{2}}, \text{ i.e., } \frac{x_s}{a} = \pm\left(\frac{K}{\pi U_\infty a} + 1\right)^{\frac{1}{2}}$$

$$y_s = 0$$

Rankine Ovals

- The maximum height of the Rankine oval is located at $\left(r_0, \frac{\pi}{2}\right)$ when $\psi = \psi_s = 0$,i.e.,

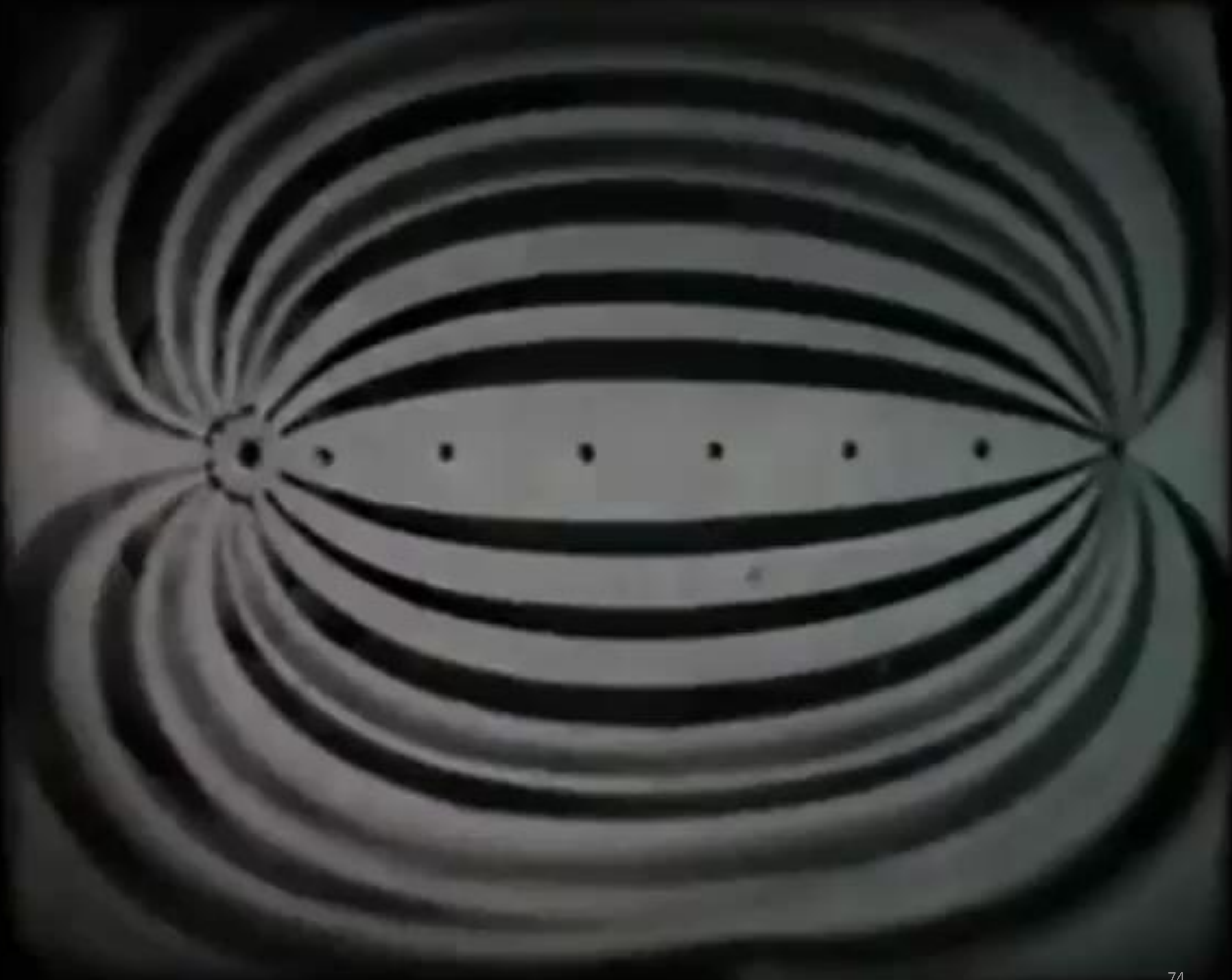
$$\psi = U_\infty r_0 - \frac{K}{2\pi}\tan^{-1}\left(\frac{2ar_0}{r_0^{\,2} - a^2}\right) = 0$$



or

$$\frac{r_0}{a} = \frac{1}{2}\left[\left(\frac{r_0}{a}\right)^2 - 1\right]\tan\left(\frac{2\pi U_\infty a}{K}\frac{r_o}{a}\right)$$

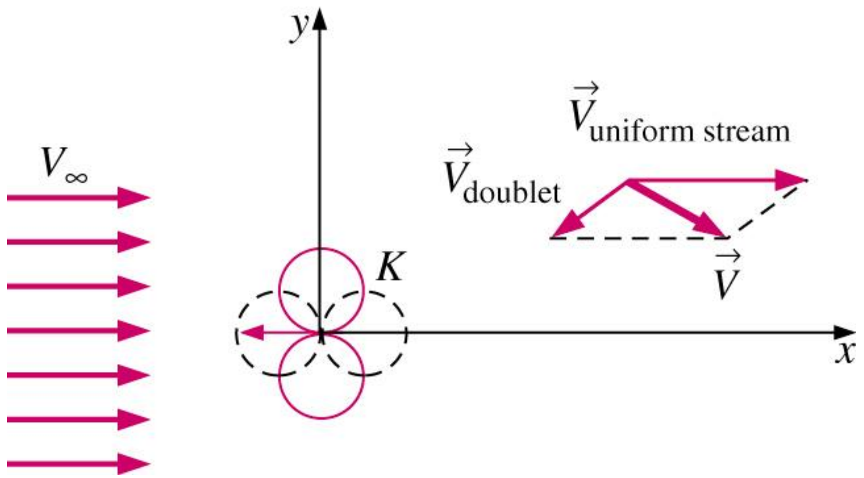
which can only be solved numerically.

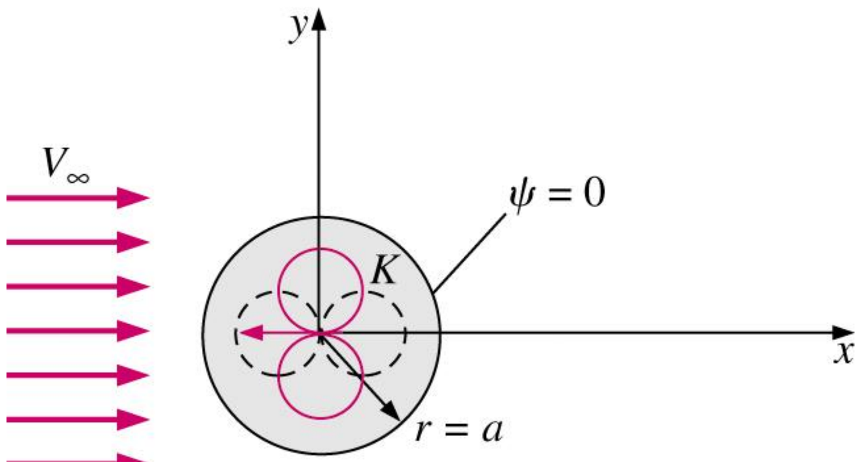


Flow around a Cylinder: Steady Cylinder

- Flow around a steady circular cylinder is the limiting case of a Rankine oval when a→0.
- This becomes the superposition of a uniform parallel flow with a doublet in x-direction.
- Under this limit and with $B = a.K/2\pi$ =constant, the radius of the cylinder is:.

$$R = r_s = \left(\frac{B}{U_\infty}\right)^{\frac{1}{2}}$$





Flow around a Cylinder: Steady Cylinder

- The stream function and velocity potential become:

$$\phi = U_\infty r\cos\theta + \frac{B\cos\theta}{r} = U_\infty r\left(1+\frac{R^2}{r^2}\right)\cos\theta$$

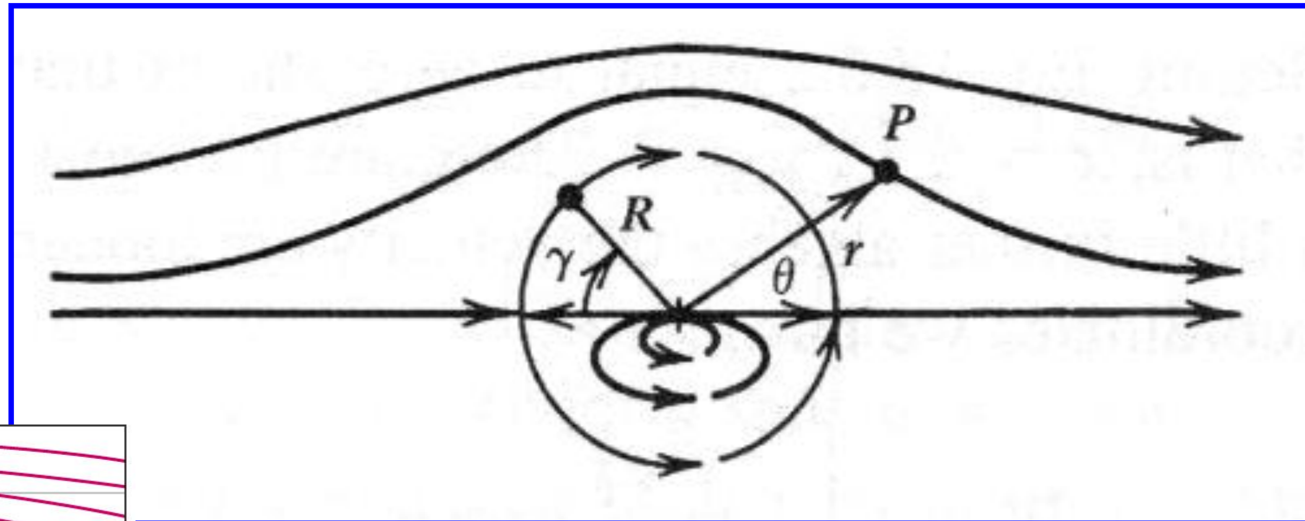
and

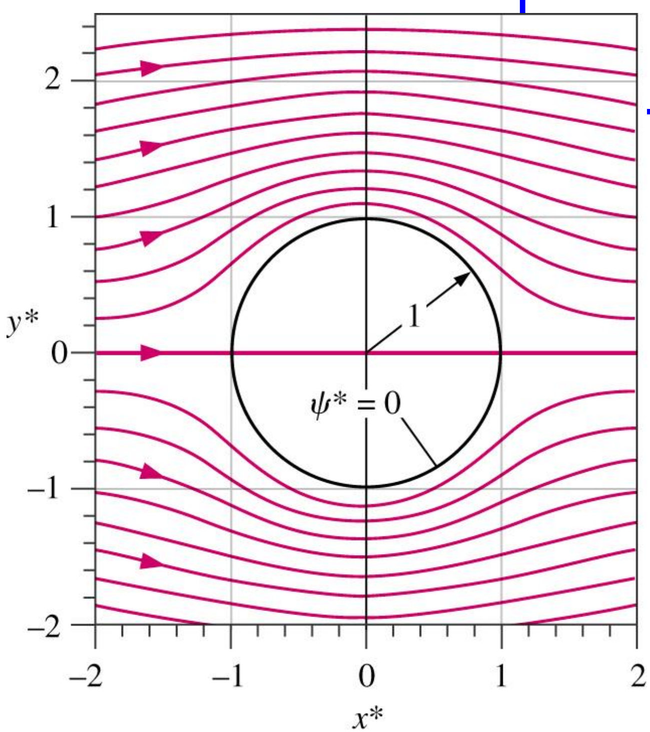
$$\psi = U_\infty r\sin\theta - \frac{B\sin\theta}{r} = U_\infty r\left(1-\frac{R^2}{r^2}\right)\sin\theta$$

- The radial and circumferential velocities are:

$$v_r = \frac{\partial\phi}{\partial r} = \frac{\partial\Psi}{r\partial\theta} = U_\infty\left(1-\frac{R^2}{r^2}\right)\cos\theta \quad \text{and} \quad v_\theta = \frac{\partial\phi}{r\partial\theta} = -\frac{\partial\Psi}{\partial r} = -U_\infty\left(1+\frac{R^2}{r^2}\right)\sin\theta$$

Steady Cylinder

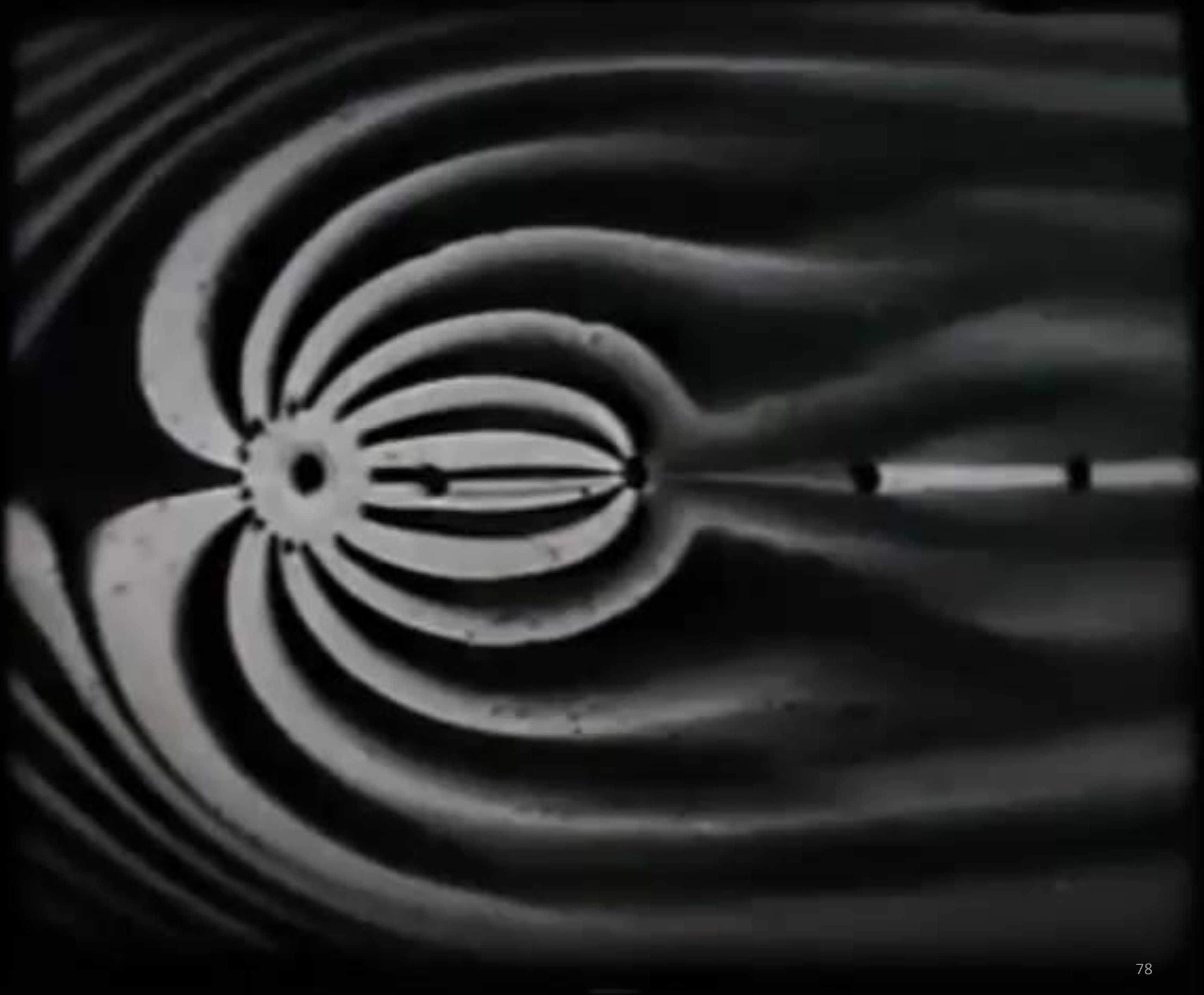




On the cylinder surface (r = R)

$$v_r = 0 \quad \text{and} \quad v_\theta = -2U_\infty \sin\theta$$

Normal velocity (v_r) is zero, Tangential velocity (v_θ) is non-zero ⇒slip condition.



Pressure Distribution on a Circular Cylinder

- Using the irrotational flow approximation, we can calculate and plot the non-dimensional static pressure distribution on the surface of a circular cylinder of radius *R* in a uniform stream of speed U_∞ .
- The pressure far away from the cylinder is p_∞
- Pressure coefficient:

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2}$$

- Since the flow in the region of interest is irrotational, we use the Bernoulli equation to calculate the pressure anywhere in the flow field. Ignoring the effects of gravity
- Bernoulli's equation:

$$\frac{p}{\rho} + \frac{V^2}{2} = cons\tan t = \frac{p_\infty}{\rho} + \frac{U_\infty^2}{2}$$

- Rearranging Cp Eq. , we get

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - \frac{V^2}{U_\infty^2}$$

Pressure Distribution on a Circular Cylinder

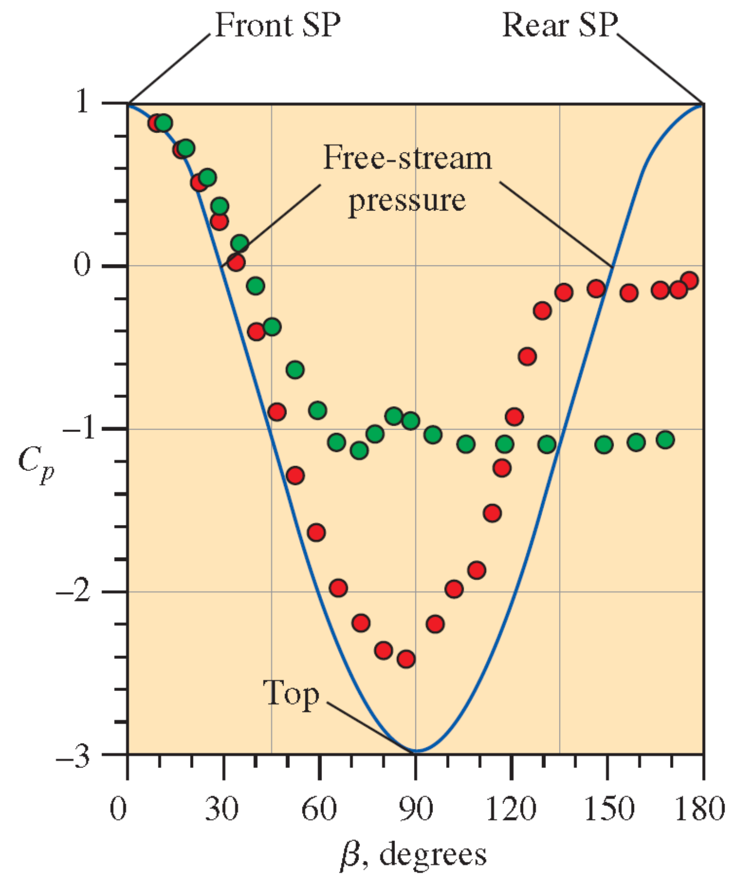
- We substitute our expression for tangential velocity on the cylinder surface, since along the surface $V^2 = v^2_{\theta}$; the Eq. becomes

$$C_p = 1 - \frac{(-2U_\infty^2 \sin\theta)^2}{U_\infty^2} = 1 - 4\sin^2\theta$$

- In terms of angle β, defined from the front of the body, we use the transformation $\beta = \pi - \theta$ to obtain *Cp in terms of angle* β :

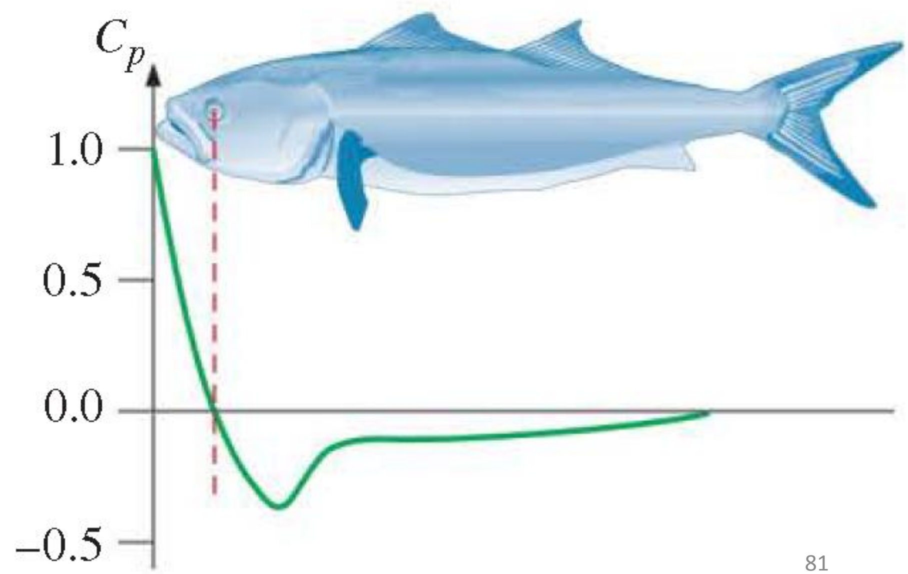
$$C_p = 1 - 4\sin^2\beta$$

- We plot the pressure coefficient on the top half of the cylinder as a function of angle β , solid blue curve.



Pressure distribution on a fish

- Somewhere between the front stagnation point and the aerodynamic shoulder is a point on the body surface where the speed just above the body is equal to *V*, the pressure *P* is equal to P_∞, and *Cp* = 0. This point is called the **zero pressure point**
- At this point, the pressure acting normal to the body surface is the *same* ($P = P_\infty$), regardless of how fast the body moves
- through the fluid.
- This fact is a factor in the location of fish eyes .



Pressure distribution on a fish

- If a fish's eye were located closer to its nose, the eye would experience an increase in water pressure as the fish swims—the faster it would swim, the higher the water pressure on its eye would be. This would cause the soft eyeball to distort, affecting the fish's vision. Likewise, if the eye were located farther back, near the aerodynamic shoulder, the eye would experience a relative *suction* pressure when the fish would swim, again distorting its eyeball and blurring its vision.
- Experiments have revealed that the fish's eye is instead located very close to the zero-pressure point where $P = P_\infty$, and the fish can swim at any speed without distorting its vision.
- Incidentally, the back of the gills is located near the aerodynamic shoulder so that the suction pressure there helps the fish to "exhale."
- The heart is also located near this lowest pressure point to increase the heart's stroke volume during rapid swimming.