SPC 307 - Aerodynamics
Sheet 4 - Solution
Dynamics of an incompressible, inviscid flow field

1. The velocity in a certain flow field is given by the equation

$$\vec{V} = x\,\hat{\imath} + x^2\,\hat{\jmath} + yz\,\hat{k}$$

Determine the expressions for the three rectangular components of acceleration.

From expression for velocity, $u = x$, $v = x^2 z$, $w = yz$

Since

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

then

$$a_x = 0 + (x)(1) + (x^2 z)(0) + (yz)(0)$$

$$= \underline{\underline{x}}$$

Similarly,

$$a_y = \frac{\partial v}{\partial t} + u\cdot\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

and

$$a_y = 0 + (x)(2xz) + (x^2 z)(0) + (yz)(x^2)$$

$$= \underline{\underline{2x^2 z + x^2 yz}}$$

Also,

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

so that

$$a_z = 0 + (x)(0) + (x^2 z)(z) + (yz)(y)$$

$$= x^2 z^2 + y^2 z$$

2. The flow in the plane two-dimensional channel shown in Fif. 1. Gas x- and y-components of velocity given by

$$u = u_0\left(1+\frac{x}{l}\right)\left[1-\left(\frac{y}{Y}\right)^2\right]$$

$$v = u_0\left[\frac{y^3}{lY_0^2}\left(1+\frac{x}{l}\right)^2-\frac{y}{l}\right]$$

Calculate the linear acceleration, rotation, vorticity, rate of volumetric strain, and rate of shear deformation of the flow.

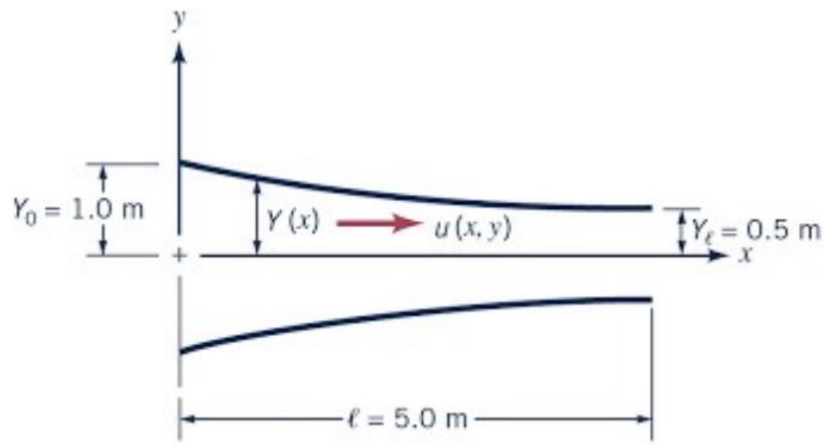

Fig. 1.

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3. If Determine an expression for the vorticity of the flow field described by

$$\vec{V} = -xy^3\,\hat{\imath} + y^4\,\hat{\jmath}$$

Is the flow irrotational?

$$\vec{\zeta} = 2\vec{\omega} \qquad \text{(Eq. 6.17)}$$

From expression for velocity, $u = -xy^3$, $v = y^4$, and $w = 0$, and with

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \text{(Eq. 6.13)}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \text{(Eq. 6.14)}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \text{(Eq. 6.12)}$$

it follows that

$$\omega_x = 0, \quad \omega_y = 0, \quad \text{and} \quad \omega_z = \frac{1}{2}\left[0 - (-3xy^2)\right] = \frac{3}{2}xy^2$$

Thus,

$$\vec{\zeta} = 2\left(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}\right)$$

$$= 2\left[(0)\hat{i} + (0)\hat{j} + \left(\tfrac{3}{2}xy^2\right)\hat{k}\right]$$

$$= \underline{\underline{3xy^2\,\hat{k}}}$$

Since $\vec{\zeta}$ is not zero everywhere the flow is not irrotational. <u>No.</u>

4. The Equation of the x-velocity in fully developed laminar flow between parallel plates is given by

$$u = \frac{1}{2\mu}\left(\frac{\delta p}{\delta x}\right)(y^2 - h^2)$$

The y-velocity is v=0. Determine the volumetric strain rate, the vorticity and the rate of angular deformation. What is the shear stress at the plate surface?

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5. A The stream function for a given two-dimensional flow field is

$$\Psi = 5x^2y - \left(\frac{5}{3}y^3\right)$$

Determine the corresponding velocity potential.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 3x^2 - 3y^2$$

Integrate with respect to y to obtain

$$\int d\psi = \int (3x^2 - 3y^2)\, dy$$

or

$$\psi = 3\left(x^2y - \frac{y^3}{3}\right) + f_1(x) \qquad (1)$$

Similarly,

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = -6xy$$

and integrating with respect to x yields

$$\int d\psi = \int 6xy\, dx$$

or

$$\psi = 3x^2y + f_2(y) \qquad (2)$$

To satisfy both Eqs. (1) and (2)

$$\psi = 3x^2y - y^3 + C$$

where C is an arbitrary constant. Since the streamline $\psi = 0$ passes through the origin $(x=0, y=0)$ it follows that $C = 0$ and

$$\underline{\underline{\psi = 3x^2y - y^3}} \qquad (3)$$

The equation of the streamline passing through the origin is found by setting $\psi = 0$ in Eq. (3) to yield

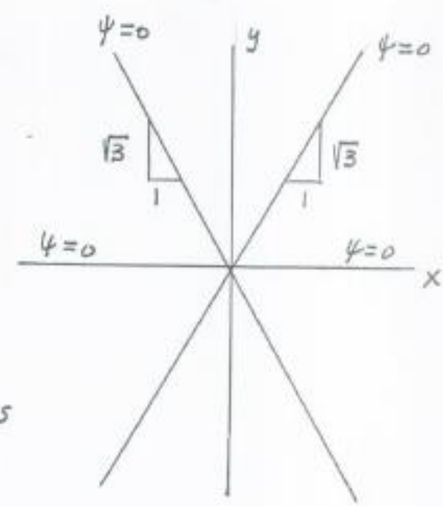
$$y(3x^2 - y^2) = 0$$

which is satisfied for $y = 0$

and

$$y = \pm\sqrt{3}\, x$$

A sketch of the $\psi = 0$ streamlines are shown in the figure.



6. The velocity potential for a certain inviscid flow field is

$$\phi = -(3x^2y - y^3)$$

where ϕ has the units of ft^2/s when x and y are in feet. Determine the pressure difference (in psi) between the points (1,2) and (4,4), where the coordinates are in feet, if the fluid is water and elevation changes are negligible.

Since the flow field is described by a velocity potential the flow is irrotational and the Bernoulli equation can be applied between any two points. Thus,

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} \qquad (1)$$

Also,

$$u = \frac{\partial \phi}{\partial x} = -6xy \qquad v = \frac{\partial \phi}{\partial y} = -3x^2 + 3y^2$$

At $x = 1\,ft$, $y = 2\,ft$

$$u_1 = -6(1)(2) = -12\,\frac{ft}{s}$$

$$v_1 = -3(1)^2 + 3(2)^2 = 9\,\frac{ft}{s}$$

So that $V_1^2 = u_1^2 + v_1^2 = \left(-12\,\frac{ft}{s}\right)^2 + \left(9\,\frac{ft}{s}\right)^2 = 225\left(\frac{ft}{s}\right)^2$

At $x = 4\,ft$, $y = 4\,ft$

$$u_2 = -6(4)(4) = -96\,\frac{ft}{s}$$

$$v_2 = -3(4)^2 + 3(4)^2 = 0$$

So that $V_2^2 = \left(-96\,\frac{ft}{s}\right)^2$

Thus, from Eq. (1)

$$p_1 - p_2 = \frac{1}{2}\frac{\gamma}{g}\left[V_2^2 - V_1^2\right]$$

$$= \frac{1}{2}\,\frac{\left(62.4\,\frac{lb}{ft^3}\right)}{\left(32.2\,\frac{ft}{s^2}\right)}\left[\left(-96\,\frac{ft}{s}\right)^2 - 225\left(\frac{ft}{s}\right)^2\right]$$

$$= 8710\,\frac{lb}{ft^2} = \left(8710\,\frac{lb}{ft^2}\right)\left(\frac{ft^2}{144\,in.^2}\right) = \underline{\underline{60.5\,psi}}$$

7. A The velocity potential for a certain inviscid, incompressible flow field is given by the equation

$$\phi = 2x^2y - \left(\frac{2}{3}\right)y^3$$

where ϕ has the units of m²/s when x and y are in meters. Determine the pressure at the point x = 2 m, y = 2 m if the pressure at x = 1 m, y = 1 m is 200 kPa. Elevation changes can be neglected, and the fluid is water.

Since the flow is irrotational,

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} \qquad (1)$$

with $V^2 = u^2 + v^2$. For the velocity potential given,

$$u = \frac{\partial \phi}{\partial x} = 4xy \qquad v = \frac{\partial \phi}{\partial y} = 2x^2 - 2y^2$$

At point 1 let $x = 1\,m$ and $y = 1\,m$ so that

$$u_1 = 4(1)(1) = 4\,\frac{m}{s} \qquad v_1 = 2(1)^2 - 2(1)^2 = 0$$

and $V_1^2 = \left(4\,\frac{m}{s}\right)^2 = 16\,\frac{m^2}{s^2}$

At point 2 $x = 2\,m$ and $y = 2\,m$ so that

$$u_2 = 4(2)(2) = 16\,\frac{m}{s} \qquad v_2 = 2(2)^2 - 2(2)^2 = 0$$

and $V_2^2 = \left(16\,\frac{m}{s}\right)^2 = 256\,\frac{m^2}{s^2}$

Thus, from Eq. (1)

$$p_2 = p_1 + \frac{\gamma}{2g}\left(V_1^2 - V_2^2\right)$$

$$= 200 \times 10^3\,\frac{N}{m^2} + \frac{\left(9.80 \times 10^3\,\frac{N}{m^3}\right)}{2\left(9.81\,\frac{m}{s^2}\right)}\left(16\,\frac{m^2}{s^2} - 256\,\frac{m^2}{s^2}\right)$$

$$= 80.1\ kPa$$

8. Consider the two-dimensional flow air flow around the corner shown in in Fig. 2. The x- and y-direction velocities are

$$u = \frac{v_0}{l} \sin h\left(\frac{x}{l}\right) \cos h\left(\frac{y}{l}\right)$$

and

$$u = -\frac{v_0}{l} \cos h\left(\frac{x}{l}\right) \sin h\left(\frac{y}{l}\right)$$

respectively. Assume constant density, steady flow, negligible gravity and inviscid flow. Find $p(x,y)$.

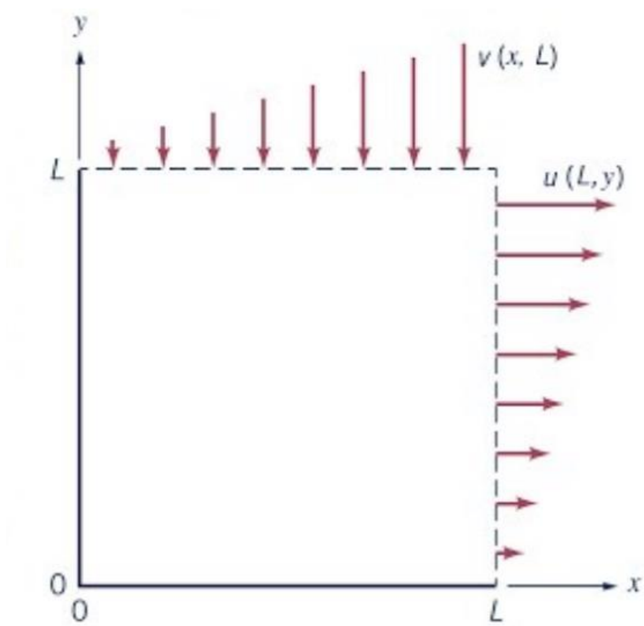


Fig. 2.

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9. The velocity potential

$$\phi = -k(x^2 - y^2) \quad (k = \text{constant})$$

may be used to represent the flow against an infinite plane boundary, as illustrated in Fig. 3. For flow in the vicinity of a stagnation point, it is frequently assumed that the pressure gradient along the surface is of the form

$$\frac{\delta p}{\delta x} = Ax$$

where A is a constant. Use the given velocity potential to show that this is true.

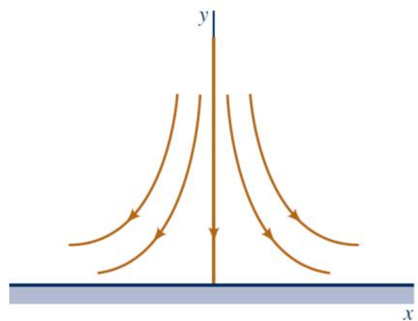


Fig. 3.

For the velocity potential given

$$u = \frac{\partial \phi}{\partial x} = -2kx \qquad (1)$$

$$v = \frac{\partial \phi}{\partial y} = -2ky \qquad (2)$$

and the stagnation point occurs at the origin.

For this steady, two-dimensional flow

$$-\frac{\partial p}{\partial x} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \qquad (\text{Eq. 6.51a})$$

and along the surface $(y=0)$ $v=0$ so that

$$\frac{\partial p}{\partial x} = \rho u \frac{\partial u}{\partial x} \qquad (3)$$

From Eq. (1) $u = -2kx$ and therefore

$$\frac{\partial u}{\partial x} = -2k$$

and Eq. (3) becomes

$$\frac{\partial p}{\partial x} = \rho(-2kx)(-2k) = 4k^2 x$$

or

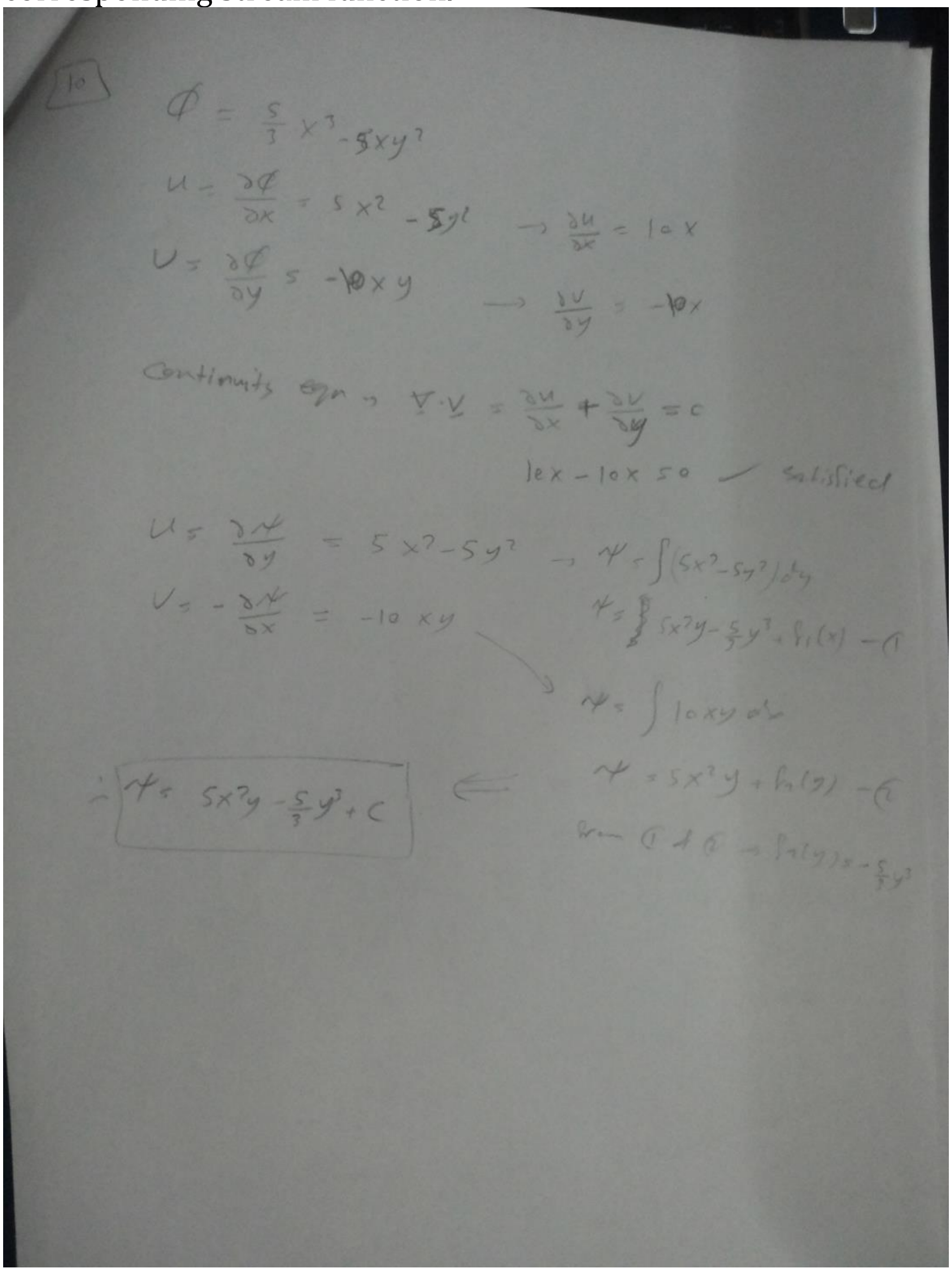
$$\frac{\partial p}{\partial x} = A x$$

where $A = 4k^2$.

10. The velocity potential for a given two-dimensional flow field is

$$\phi = \left(\frac{5}{3}\right)x^3 - 2xy^2$$

Show that the continuity equation is satisfied and determine the corresponding stream function.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$\phi = \frac{5}{3}x^3 - 5xy^2$

$u = \frac{\partial \phi}{\partial x} = 5x^2 - 5y^2 \rightarrow \frac{\partial u}{\partial x} = 10x$

$v = \frac{\partial \phi}{\partial y} = -10xy \rightarrow \frac{\partial v}{\partial y} = -10x$

Continuity eqn $\rightarrow \underline{\nabla} \cdot \underline{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$10x - 10x = 0$ ✓ satisfied

$u = \frac{\partial \psi}{\partial y} = 5x^2 - 5y^2 \rightarrow \psi = \int (5x^2 - 5y^2)\, dy$

$\psi = 5x^2 y - \frac{5}{3}y^3 + f_1(x)$ —(1)

$v = -\frac{\partial \psi}{\partial x} = -10xy$

$\psi = \int 10xy\, dx$

$\psi = 5x^2 y + f_2(y)$ —(2)

from (1) & (2) $\rightarrow f_2(y) = -\frac{5}{3}y^3$

$\therefore \boxed{\psi = 5x^2 y - \frac{5}{3}y^3 + C}$