SPC 407

Supersonic & Hypersonic Fluid Dynamics

Lecture 4

October 23, 2016

Isentropic flow inside a nozzle/ diffuser Equation Summary and tables

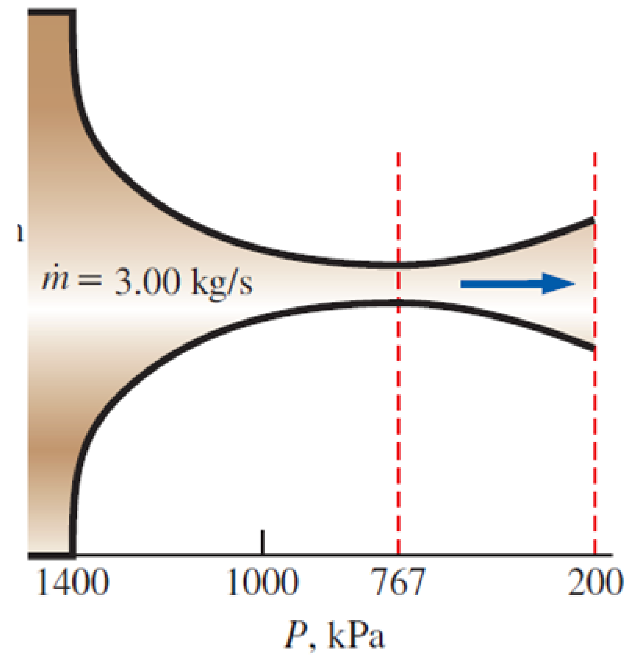
$$\text{Ma}^* = \text{Ma}\sqrt{\frac{k+1}{2+(k-1)\text{Ma}^2}}$$

$$\frac{A}{A^*} = \frac{1}{\text{Ma}}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}\text{Ma}^2\right)\right]^{0.5(k+1)/(k-1)}$$

$$\frac{P}{P_0} = \left(1+\frac{k-1}{2}\text{Ma}^2\right)^{-k/(k-1)}$$

$$\frac{\rho}{\rho_0} = \left(1+\frac{k-1}{2}\text{Ma}^2\right)^{-1/(k-1)}$$

$$\frac{T}{T_0} = \left(1+\frac{k-1}{2}\text{Ma}^2\right)^{-1}$$



$$\mathrm{Ma}^* = \mathrm{Ma}\sqrt{\frac{k+1}{2+(k-1)\mathrm{Ma}^2}}$$

$$\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)\right]^{0.5(k+1)/(k-1)}$$

$$\frac{P}{P_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{-k/(k-1)}$$

$$\frac{\rho}{\rho_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{-1/(k-1)}$$

$$\frac{T}{T_0} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{-1}$$

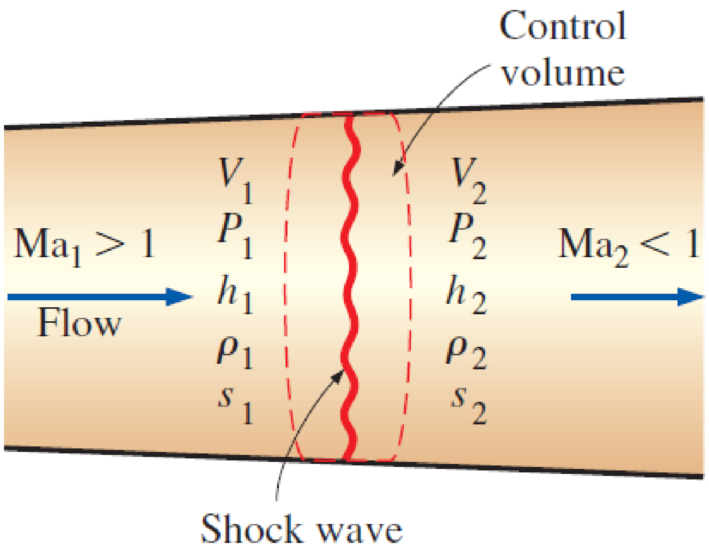
TABLE A–13

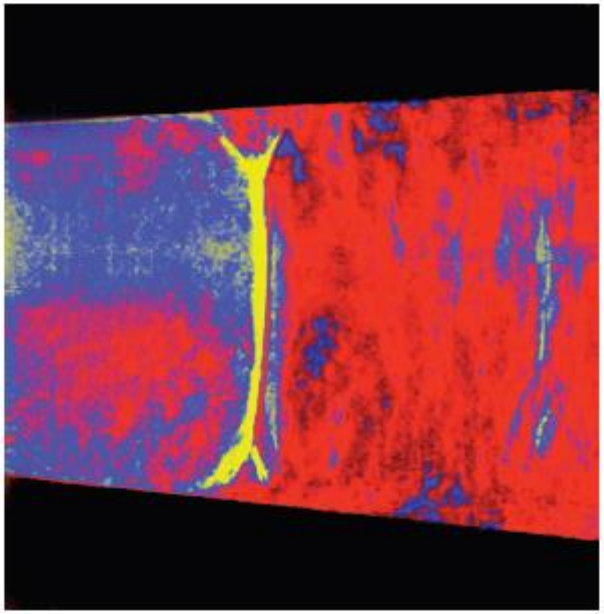
One-dimensional isentropic compressible flow functions for an ideal gas with $k = 1.4$

Ma	Ma*	A/A^*	P/P_0	ρ/ρ_0	T/T_0
0	0	∞	1.0000	1.0000	1.0000
0.1	0.1094	5.8218	0.9930	0.9950	0.9980
0.2	0.2182	2.9635	0.9725	0.9803	0.9921
0.3	0.3257	2.0351	0.9395	0.9564	0.9823
0.4	0.4313	1.5901	0.8956	0.9243	0.9690
0.5	0.5345	1.3398	0.8430	0.8852	0.9524
0.6	0.6348	1.1882	0.7840	0.8405	0.9328
0.7	0.7318	1.0944	0.7209	0.7916	0.9107
0.8	0.8251	1.0382	0.6560	0.7400	0.8865
0.9	0.9146	1.0089	0.5913	0.6870	0.8606
1.0	1.0000	1.0000	0.5283	0.6339	0.8333
1.2	1.1583	1.0304	0.4124	0.5311	0.7764
1.4	1.2999	1.1149	0.3142	0.4374	0.7184
1.6	1.4254	1.2502	0.2353	0.3557	0.6614
1.8	1.5360	1.4390	0.1740	0.2868	0.6068
2.0	1.6330	1.6875	0.1278	0.2300	0.5556
2.2	1.7179	2.0050	0.0935	0.1841	0.5081
2.4	1.7922	2.4031	0.0684	0.1472	0.4647
2.6	1.8571	2.8960	0.0501	0.1179	0.4252
2.8	1.9140	3.5001	0.0368	0.0946	0.3894
3.0	1.9640	4.2346	0.0272	0.0760	0.3571
5.0	2.2361	25.000	0.0019	0.0113	0.1667
∞	2.2495	∞	0	0	0

Normal Shockwave

Consider the steady flow of a fluid through a duct such as a nozzle, diffuser, or some other flow passage where the flow takes place adiabatically and with no shaft or electrical work, as shown in Fig





Assuming the fluid experiences little or no change in its elevation

Governing Equations

Conservation of mass::

$$\rho_1 A V_1 = \rho_2 A V_2$$

Or

$$\rho_1 V_1 = \rho_2 V_2$$

Conservation of energy:

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

$$h_{01} = h_{02}$$

$$T_{01} = T_{02}$$

Governing Equations

Linear momentum equation:

CONSERVATION OF ENERGY
(steady flow, $w = 0$, $q = 0$, $\Delta pe = 0$)

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$$

or

$$h + \frac{V^2}{2} = \text{constant}$$

Differentiate,

$$dh + V\,dV = 0$$

Also,

$$T\,\overset{0\ \text{(isentropic)}}{ds} = dh - v\,dP$$

$$dh = v\,dP = \frac{1}{\rho}\,dP$$

Substitute,

$$\frac{dP}{\rho} + V\,dV = 0$$

Governing Equations

Linear momentum equation:

$$A(P_1 - P_2) = \dot{m}(V_2 - V_1)$$

Second Law of Thermodynamic

$$s_2 - s_1 \geq 0$$

$$s_2 - s_1 = c_P \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$$

Governing Equations

From previous lecture:

$$T_0 = T + \frac{V^2}{2c_p}$$

$$\frac{T_0}{T} = 1 + \left(\frac{k-1}{2}\right)\mathrm{Ma}^2$$

So

$$\frac{T_{01}}{T_1} = 1 + \left(\frac{k-1}{2}\right)\mathrm{Ma}_1^2 \quad \text{and} \quad \frac{T_{02}}{T_2} = 1 + \left(\frac{k-1}{2}\right)\mathrm{Ma}_2^2$$

Dividing the first equation by the second one and noting that $T_{01} = T_{02}$ **we have**

$$\frac{T_2}{T_1} = \frac{1 + \mathrm{Ma}_1^2(k-1)/2}{1 + \mathrm{Ma}_2^2(k-1)/2}$$

Governing Equations

From the ideal-gas equation of state,

$$\rho_1 = \frac{P_1}{RT_1} \quad \text{and} \quad \rho_2 = \frac{P_2}{RT_2}$$

Substituting these into the conservation of mass relation $\rho_1 V_1 = \rho_2 V_2$ and noting that $\mathrm{Ma} = V/c$ and $c = \sqrt{kRT}$, we have

$$\frac{T_2}{T_1} = \frac{P_2 V_2}{P_1 V_1} = \frac{P_2 \mathrm{Ma}_2 c_2}{P_1 \mathrm{Ma}_1 c_1} = \frac{P_2 \mathrm{Ma}_2 \sqrt{T_2}}{P_1 \mathrm{Ma}_1 \sqrt{T_1}} = \left(\frac{P_2}{P_1}\right)^2 \left(\frac{\mathrm{Ma}_2}{\mathrm{Ma}_1}\right)^2$$

The pressure ratio across the shock

$$\frac{P_2}{P_1} = \frac{\mathrm{Ma}_1 \sqrt{1 + \mathrm{Ma}_1^2 (k-1)/2}}{\mathrm{Ma}_2 \sqrt{1 + \mathrm{Ma}_2^2 (k-1)/2}}$$

Governing Equations

by combining the conservation of mass and momentum equations

$$P_1 - P_2 = \frac{\dot{m}}{A}(V_2 - V_1) = \rho_2 V_2^2 - \rho_1 V_1^2$$

However,

$$\rho V^2 = \left(\frac{P}{RT}\right)(\mathrm{Ma}\, c)^2 = \left(\frac{P}{RT}\right)(\mathrm{Ma}\sqrt{kRT})^2 = Pk\,\mathrm{Ma}^2$$

Thus,

$$P_1(1 + k\mathrm{Ma}_1^2) = P_2(1 + k\mathrm{Ma}_2^2)$$

$$\frac{P_2}{P_1} = \frac{1 + k\mathrm{Ma}_1^2}{1 + k\mathrm{Ma}_2^2}$$

$$\mathrm{Ma}_2^2 = \frac{\mathrm{Ma}_1^2 + 2/(k - 1)}{2\mathrm{Ma}_1^2 k/(k - 1) - 1}$$

Normal Shockwave Equation Summary and tables

$$T_{01} = T_{02}$$

$$\mathrm{Ma}_2 = \sqrt{\frac{(k-1)\mathrm{Ma}_1^2 + 2}{2k\mathrm{Ma}_1^2 - k + 1}}$$

$$\frac{P_2}{P_1} = \frac{1 + k\mathrm{Ma}_1^2}{1 + k\mathrm{Ma}_2^2} = \frac{2k\mathrm{Ma}_1^2 - k + 1}{k + 1}$$

$$\frac{\rho_2}{\rho_1} = \frac{P_2/P_1}{T_2/T_1} = \frac{(k+1)\mathrm{Ma}_1^2}{2 + (k-1)\mathrm{Ma}_1^2} = \frac{V_1}{V_2}$$

$$\frac{T_2}{T_1} = \frac{2 + \mathrm{Ma}_1^2(k-1)}{2 + \mathrm{Ma}_2^2(k-1)}$$

$$\frac{P_{02}}{P_{01}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left[\frac{1 + \mathrm{Ma}_2^2(k-1)/2}{1 + \mathrm{Ma}_1^2(k-1)/2}\right]^{(k+1)/[2(k-1)]}$$

$$\frac{P_{02}}{P_1} = \frac{(1 + k\mathrm{Ma}_1^2)[1 + \mathrm{Ma}_2^2(k-1)/2]^{k/(k-1)}}{1 + k\mathrm{Ma}_2^2}$$

TABLE A–14

One-dimensional normal shock functions for an ideal gas with $k = 1.4$

Ma_1	Ma_2	P_2/P_1	ρ_2/ρ_1	T_2/T_1	P_{02}/P_{01}	P_{02}/P_1
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.8929
1.1	0.9118	1.2450	1.1691	1.0649	0.9989	2.1328
1.2	0.8422	1.5133	1.3416	1.1280	0.9928	2.4075
1.3	0.7860	1.8050	1.5157	1.1909	0.9794	2.7136
1.4	0.7397	2.1200	1.6897	1.2547	0.9582	3.0492
1.5	0.7011	2.4583	1.8621	1.3202	0.9298	3.4133
1.6	0.6684	2.8200	2.0317	1.3880	0.8952	3.8050
1.7	0.6405	3.2050	2.1977	1.4583	0.8557	4.2238
1.8	0.6165	3.6133	2.3592	1.5316	0.8127	4.6695
1.9	0.5956	4.0450	2.5157	1.6079	0.7674	5.1418
2.0	0.5774	4.5000	2.6667	1.6875	0.7209	5.6404
2.1	0.5613	4.9783	2.8119	1.7705	0.6742	6.1654
2.2	0.5471	5.4800	2.9512	1.8569	0.6281	6.7165
2.3	0.5344	6.0050	3.0845	1.9468	0.5833	7.2937
2.4	0.5231	6.5533	3.2119	2.0403	0.5401	7.8969
2.5	0.5130	7.1250	3.3333	2.1375	0.4990	8.5261
2.6	0.5039	7.7200	3.4490	2.2383	0.4601	9.1813
2.7	0.4956	8.3383	3.5590	2.3429	0.4236	9.8624
2.8	0.4882	8.9800	3.6636	2.4512	0.3895	10.5694
2.9	0.4814	9.6450	3.7629	2.5632	0.3577	11.3022
3.0	0.4752	10.3333	3.8571	2.6790	0.3283	12.0610
4.0	0.4350	18.5000	4.5714	4.0469	0.1388	21.0681
5.0	0.4152	29.000	5.0000	5.8000	0.0617	32.6335
∞	0.3780	∞	6.0000	∞	0	∞

$$T_{01} = T_{02}$$

$$\mathrm{Ma}_2 = \sqrt{\frac{(k-1)\mathrm{Ma}_1^2 + 2}{2k\mathrm{Ma}_1^2 - k + 1}}$$

$$\frac{P_2}{P_1} = \frac{1 + k\mathrm{Ma}_1^2}{1 + k\mathrm{Ma}_2^2} = \frac{2k\mathrm{Ma}_1^2 - k + 1}{k + 1}$$

$$\frac{\rho_2}{\rho_1} = \frac{P_2/P_1}{T_2/T_1} = \frac{(k+1)\mathrm{Ma}_1^2}{2 + (k-1)\mathrm{Ma}_1^2} = \frac{V_1}{V_2}$$

$$\frac{T_2}{T_1} = \frac{2 + \mathrm{Ma}_1^2(k-1)}{2 + \mathrm{Ma}_2^2(k-1)}$$

$$\frac{P_{02}}{P_{01}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left[\frac{1 + \mathrm{Ma}_2^2(k-1)/2}{1 + \mathrm{Ma}_1^2(k-1)/2}\right]^{(k+1)/[2(k-1)]}$$

$$\frac{P_{02}}{P_1} = \frac{(1 + k\mathrm{Ma}_1^2)[1 + \mathrm{Ma}_2^2(k-1)/2]^{k/(k-1)}}{1 + k\mathrm{Ma}_2^2}$$

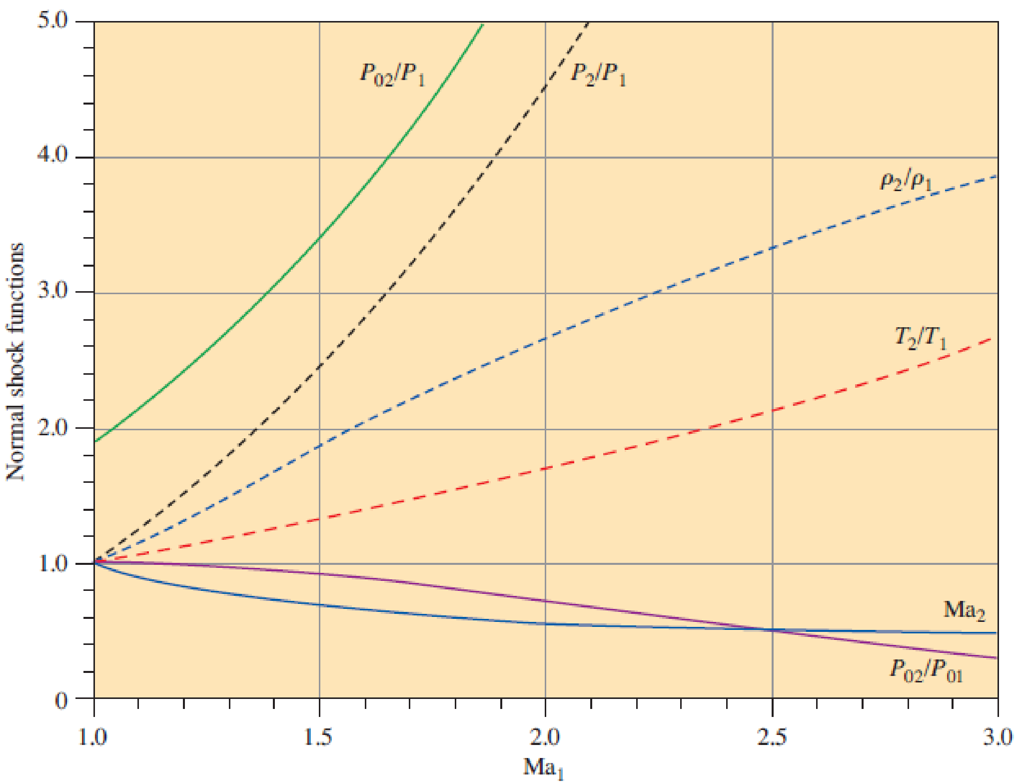


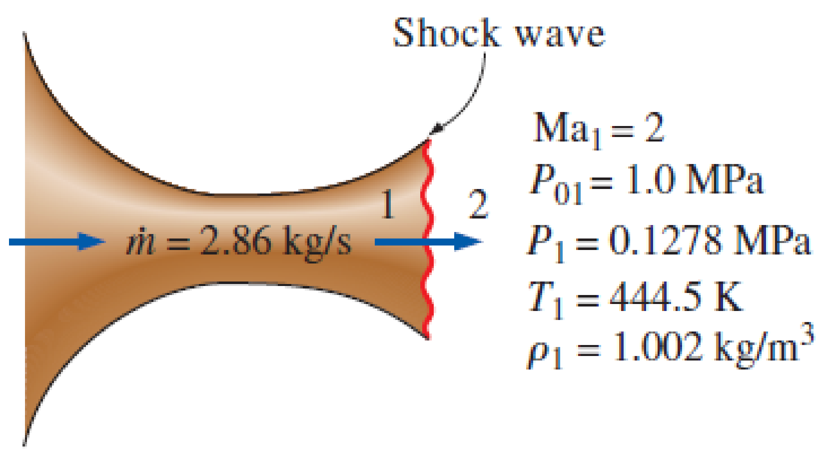
TABLE A–14

One-dimensional normal shock functions for an ideal gas with $k = 1.4$

Ma_1	Ma_2	P_2/P_1	ρ_2/ρ_1	T_2/T_1	P_{02}/P_{01}	P_{02}/P_1
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.8929
1.1	0.9118	1.2450	1.1691	1.0649	0.9989	2.1328
1.2	0.8422	1.5133	1.3416	1.1280	0.9928	2.4075
1.3	0.7860	1.8050	1.5157	1.1909	0.9794	2.7136
1.4	0.7397	2.1200	1.6897	1.2547	0.9582	3.0492
1.5	0.7011	2.4583	1.8621	1.3202	0.9298	3.4133
1.6	0.6684	2.8200	2.0317	1.3880	0.8952	3.8050
1.7	0.6405	3.2050	2.1977	1.4583	0.8557	4.2238
1.8	0.6165	3.6133	2.3592	1.5316	0.8127	4.6695
1.9	0.5956	4.0450	2.5157	1.6079	0.7674	5.1418
2.0	0.5774	4.5000	2.6667	1.6875	0.7209	5.6404
2.1	0.5613	4.9783	2.8119	1.7705	0.6742	6.1654
2.2	0.5471	5.4800	2.9512	1.8569	0.6281	6.7165
2.3	0.5344	6.0050	3.0845	1.9468	0.5833	7.2937
2.4	0.5231	6.5533	3.2119	2.0403	0.5401	7.8969
2.5	0.5130	7.1250	3.3333	2.1375	0.4990	8.5261
2.6	0.5039	7.7200	3.4490	2.2383	0.4601	9.1813
2.7	0.4956	8.3383	3.5590	2.3429	0.4236	9.8624
2.8	0.4882	8.9800	3.6636	2.4512	0.3895	10.5694
2.9	0.4814	9.6450	3.7629	2.5632	0.3577	11.3022
3.0	0.4752	10.3333	3.8571	2.6790	0.3283	12.0610
4.0	0.4350	18.5000	4.5714	4.0469	0.1388	21.0681
5.0	0.4152	29.000	5.0000	5.8000	0.0617	32.6335
∞	0.3780	∞	6.0000	∞	0	∞

Example: Shock Wave in a Converging–Diverging Nozzle

If the air flowing through the converging–diverging nozzle of Example 12–6 experiences a normal shock wave at the nozzle exit plane (Fig. 12–30), determine the following after the shock: (*a*) the stagnation pressure, static pressure, static temperature, and static density; (*b*) the entropy change across the shock; (*c*) the exit velocity; and (*d*) the mass flow rate through the nozzle. Approximate the flow as steady, one-dimensional, and isentropic with $k = 1.4$ from the nozzle inlet to the shock location.



Example: Shockwave in a Converging–Diverging Nozzle

SOLUTION Air flowing through a converging–diverging nozzle experiences a normal shock at the exit. The effect of the shock wave on various properties is to be determined.

Assumptions **1** Air is an ideal gas with constant specific heats at room temperature. **2** Flow through the nozzle is steady, one-dimensional, and isentropic before the shock occurs. **3** The shock wave occurs at the exit plane.

Properties The constant-pressure specific heat and the specific heat ratio of air are c_p = 1.005 kJ/kg·K and k = 1.4. The gas constant of air is 0.287 kJ/kg·K.

Analysis (a) The fluid properties at the exit of the nozzle just before the shock (denoted by subscript 1) are those evaluated in Example 12–6 at the nozzle exit to be

$$P_{01} = 1.0 \text{ MPa} \quad P_1 = 0.1278 \text{ MPa} \quad T_1 = 444.5 \text{ K} \quad \rho_1 = 1.002 \text{ kg/m}^3$$

The fluid properties after the shock (denoted by subscript 2) are related to those before the shock through the functions listed in Table A–14. For Ma_1 = 2.0, we read

$$\text{Ma}_2 = 0.5774 \quad \frac{P_{02}}{P_{01}} = 0.7209 \quad \frac{P_2}{P_1} = 4.5000 \quad \frac{T_2}{T_1} = 1.6875 \quad \frac{\rho_2}{\rho_1} = 2.6667$$

Example: Shock Wave in a Converging–Diverging Nozzle

Then the stagnation pressure P_{02}, static pressure P_2, static temperature T_2, and static density ρ_2 after the shock are

$$P_{02} = 0.7209 P_{01} = (0.7209)(1.0 \text{ MPa}) = \mathbf{0.721\ MPa}$$

$$P_2 = 4.5000 P_1 = (4.5000)(0.1278 \text{ MPa}) = \mathbf{0.575\ MPa}$$

$$T_2 = 1.6875 T_1 = (1.6875)(444.5 \text{ K}) = \mathbf{750\ K}$$

$$\rho_2 = 2.6667 \rho_1 = (2.6667)(1.002 \text{ kg/m}^3) = \mathbf{2.67\ kg/m^3}$$

(b) The entropy change across the shock is

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$$

$$= (1.005 \text{ kJ/kg·K}) \ln (1.6875) - (0.287 \text{ kJ/kg·K}) \ln (4.5000)$$

$$= \mathbf{0.0942\ kJ/kg·K}$$

Example: Shock Wave in a Converging–Diverging Nozzle

(*c*) The air velocity after the shock is determined from $V_2 = \text{Ma}_2 c_2$, where c_2 is the speed of sound at the exit conditions after the shock:

$$V_2 = \text{Ma}_2 c_2 = \text{Ma}_2 \sqrt{kRT_2}$$

$$= (0.5774)\sqrt{(1.4)(0.287 \text{ kJ/kg·K})(750.1 \text{ K})\left(\frac{1000 \text{ m}^2/\text{s}^2}{1 \text{ kJ/kg}}\right)}$$

$$= \mathbf{317 \text{ m/s}}$$

(*d*) The mass flow rate through a converging–diverging nozzle with sonic conditions at the throat is not affected by the presence of shock waves in the nozzle. Therefore, the mass flow rate in this case is the same as that determined in Example 12–6:

$$\dot{m} = \mathbf{2.86 \text{ kg/s}}$$

Oblique Shockwave