SPC 407
Sheet 6 - Solution
Compressible Flow – Fanno Flow

1. What is the effect of friction on flow velocity in subsonic and supersonic Fanno flow?

   Solution:

   Friction increases the flow velocity in subsonic Fanno flow, but decreases the flow velocity in supersonic flow.

   These results may not be intuitive, but they come from following the Fanno line, which satisfies the conservation equations.

2. Consider supersonic Fanno flow that is decelerated to sonic velocity (Ma=1) at the duct exit as a result of frictional effects. If the duct length is increased further, will the flow at the duct exit be supersonic, subsonic, or remain sonic? Will the mass flow rate of the fluid increase, decrease, or remain constant as a result of increasing the duct length?

   Solution:

   The flow at the duct exit remains sonic. The mass flow rate must remain constant since upstream conditions are not affected by the added duct length.

   The mass flow rate is fixed by the upstream stagnation conditions and the size of the throat – therefore, the mass flow rate does not change by extending the duct. However, a shock wave appears in the duct when it is extended.

3. Consider supersonic Fanno flow of air with an inlet Mach number of 1.8. If the Mach number decreases to 1.2 at the duct exit as a result of friction, does the (a) stagnation temperature T_0, (b) stagnation pressure P_0, and (c) entropy as of the fluid increase, decrease, or remain constant during this process?

   Solution:

   During supersonic Fanno flow, the stagnation temperature T0 remains constant, stagnation pressure P_0 decreases, and entropy s increases.

   Friction leads to irreversible losses, which are felt as a loss of stagnation pressure and an increase of entropy. However, since the flow is adiabatic, the stagnation temperature does not change downstream.

4. What is the characteristic aspect of Fanno flow? What are the main approximations associated with Fanno flow?

Solution:

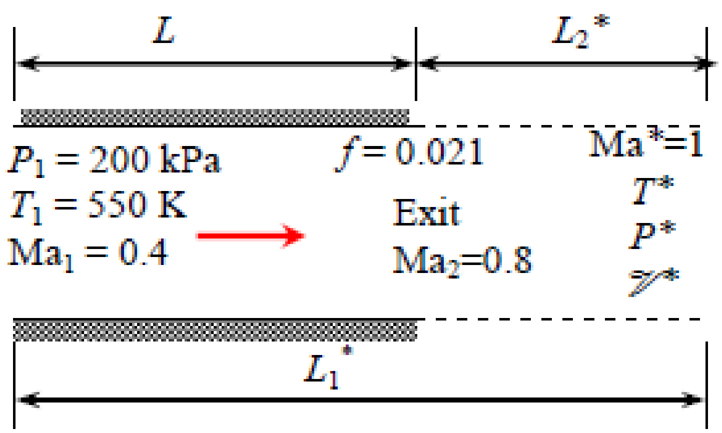
The characteristic aspect of Fanno flow is its consideration of friction. The main assumptions associated with Fanno flow are: the flow is steady, one-dimensional, and adiabatic through a constant-area duct, and the fluid is an ideal gas with constant specific heats.

Compared to Rayleigh flow, Fanno flow accounts for friction but neglects heat transfer effects, whereas Rayleigh flow accounts for heat transfer but neglects frictional effects.

5. Air enters a 12-cm-diameter adiabatic duct at Ma_1=0.4, T_1=550 K, and P_1= 200 kPa. The average friction factor for the duct is estimated to be 0.021. If the Mach number at the duct exit is 0.8, determine the duct length, temperature, pressure, and velocity at the duct exit.

Solution:

Take the properties of air to be k = 1.4, c_p = 1.005 kJ/kg-K, and R = 0.287 kJ/kg-K. The average friction factor is given to be f = 0.021.



The inlet velocity is

$$c_1 = \sqrt{kRT_1} = \sqrt{(1.4)(0.287\ \text{kJ/kg}\cdot\text{K})(550\ \text{K})\left(\frac{1000\ \text{m}^2/\text{s}^2}{1\ \text{kJ/kg}}\right)} = 470.1\ \text{m/s}$$

$$V_1 = \text{Ma}_1 c_1 = 0.4(470.1\ \text{m/s}) = 188.0\ \text{m/s}$$

The Fanno flow functions corresponding to the inlet and exit Mach numbers are, from Table A-16,

$$\text{Ma}_1 = 0.4: \quad (fL^*/D_h)_1 = 2.3085 \quad T_1/T^* = 1.1628,\ P_1/P^* = 2.6958,\ V_1/V^* = 0.4313$$

$$\text{Ma}_2 = 0.8: \quad (fL^*/D_h)_2 = 0.0723 \quad T_2/T^* = 1.0638,\ P_2/P^* = 1.2893,\ V_2/V^* = 0.8251$$

Then the temperature, pressure, and velocity at the duct exit are determined to be

$$\frac{T_2}{T_1} = \frac{T_2/T^*}{T_1/T^*} = \frac{1.0638}{1.1628} = 0.9149 \quad \rightarrow \quad T_2 = 0.9149T_1 = 0.9149(550\text{ K}) = \mathbf{503.2K}$$

$$\frac{P_2}{P_1} = \frac{P_2/P^*}{P_1/P^*} = \frac{1.2893}{2.6958} = 0.4783 \quad \rightarrow \quad P_2 = 0.4783P_1 = 0.4783(200\text{ kPa}) = \mathbf{95.65kPa}$$

$$\frac{V_2}{V_1} = \frac{V_2/V^*}{V_1/V^*} = \frac{0.8251}{0.4313} = 1.9131 \quad \rightarrow \quad V_2 = 1.9131V_1 = 1.9131(188.0\text{ m/s}) = \mathbf{359.7m/s}$$

Finally, the actual duct length is determined to be

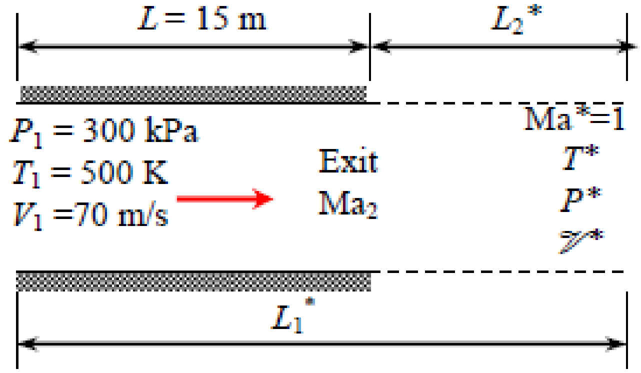
$$L = L_1^* - L_2^* = \left(\frac{fL_1^*}{D_h} - \frac{fL_2^*}{D_h}\right)\frac{D_h}{f} = (2.3085 - 0.0723)\frac{0.12\text{ m}}{0.021} = \mathbf{12.8m}$$

Note that it takes a duct length of 12.8 m for the Mach number to increase from 0.4 to 0.8. The Mach number rises at a much higher rate as sonic conditions are approached. The maximum (or sonic) duct lengths at the inlet and exit states in this case are L_1^* = 13.2 m and L_2^* = 0.413 m. Therefore, the flow would reach sonic conditions if a 0.413 m long section were added to the existing duct.

6. Air enters a 15-m-long, 4-cm-diameter adiabatic duct at V_1=70 m/s, T_1=500 K, and P_1=300 kPa. The average friction factor for the duct is estimated to be 0.023. Determine the Mach number at the duct exit, the exit velocity, and the mass flow rate of air.

Solution:

Take the properties of air to be k = 1.4, c_p = 1.005 kJ/kg-K, and R = 0.287 kJ/kg-K. The average friction factor is given to be f = 0.023.



The first thing we need to know is whether the flow is choked at the exit or not. Therefore, we first determine the inlet Mach number and the corresponding value of the function fL^*/D_h ,

$$c_1 = \sqrt{kRT_1} = \sqrt{(1.4)(0.287\ \text{kJ/kg}\cdot\text{K})(500\ \text{K})\left(\frac{1000\ \text{m}^2/\text{s}^2}{1\ \text{kJ/kg}}\right)} = 448.2\ \text{m/s}$$

$$\text{Ma}_1 = \frac{V_1}{c_1} = \frac{70\ \text{m/s}}{448.2\ \text{m/s}} = 0.1562$$

Corresponding to this Mach number we calculate (or read) from Table A-16), $(fL^*/D_h)1$ = 25.540. Also, using the actual duct length L, we have

$$\frac{fL}{D_h} = \frac{(0.023)(15\ \text{m})}{0.04\ \text{m}} = 8.625 \ < \ 25.540$$

Therefore, flow is not choked and exit Mach number is less than 1. Noting that $L = L_1^* - L_2^*$, the function fL^*/D_h at the exit state is calculated from

$$\left(\frac{fL^*}{D_h}\right)_2 = \left(\frac{fL^*}{D_h}\right)_1 - \frac{fL}{D_h} = 25.540 - 8.625 = 16.915$$

The Mach number corresponding to this value of fL^*/D is obtained from Table A-16 to be

Ma_2 = 0.187

which is the Mach number at the duct exit.

The mass flow rate of air is determined from the inlet conditions to be

$$\rho_1 = \frac{P_1}{RT_1} = \frac{300 \text{ kPa}}{(0.287 \text{ kJ/kgK})(500 \text{ K})}\left(\frac{1 \text{ kJ}}{1 \text{ kPa}\cdot\text{m}^3}\right) = 2.091 \text{ kg/m}^3$$

$$\dot{m}_{air} = \rho_1 A_{c1} V_1 = (2.091 \text{ kg/m}^3)[\pi(0.04 \text{ m})^2/4](70 \text{ m/s}) = \mathbf{0.184\ kg/s}$$

It can be shown that $L_2^* = 29.4$ m, indicating that it takes a duct length of 15 m for the Mach number to increase from 0.156 to 0.187, but only 29.4 m to increase from 0.187 to 1. Therefore, the Mach number rises at a much higher rate as sonic conditions are approached.

7. Consider the adiabatic flow of air through a pipe of 0.2-ft inside diameter and 3-ft length. The inlet flow conditions are $M_1 = 2.5$, $P_1 = 0.5$ atm. And $T_1 = 520R$. Assuming the local friction coefficient equals a constant of 0.005, calculate the following flow conditions at the exit: M_2, P_2, T_2, and P_{02}?.

Solution:

From Table A.4: For $M_1 = 2.5$, $\frac{4fL_1^*}{D} = 0.432$, $\frac{p_1}{p^*} = 0.2921$, $\frac{T_1}{T^*} = 0.5333$, $p_{o_1}/p_o^* = 2.637$.

From Table A.1: For $M_1 = 2.5$, $\frac{p_{o_1}}{p_1} = 17.09$

$$\frac{4fL_2^*}{D} = \frac{4fL_1^*}{D} + \frac{4fL}{D} = 0.432 + \frac{4(0.005)(3)}{0.2} = 0.132$$

From Table A.4: For $\frac{4fL_2^*}{D} = 0.132$, $\boxed{M_2 = 1.489}$, $p_2/p^* = 0.6125$, $T_2/T^* = 0.8314$, $p_{o_1}/p_o^* = 1.169$

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = (0.6125)\left(\frac{1}{0.2921}\right)(0.5) = \boxed{1.048 \text{ atm}}$$

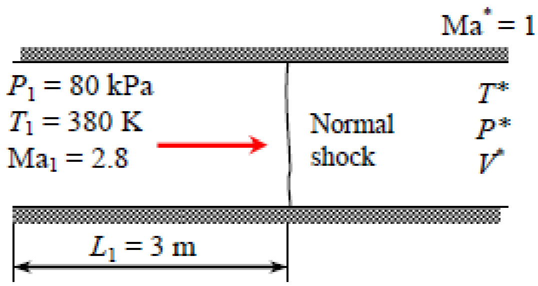
$$T_2 = \frac{T_2}{T^*}\frac{T^*}{T_1}T_1 = (0.8314)\left(\frac{1}{0.5333}\right)(520) = \boxed{810.7°R}$$

$$p_{o_2} = \frac{p_{o_2}}{p_o^*}\frac{p_o^*}{p_{o_1}}\frac{p_{o_1}}{p_1}p_1 = (1.169)\left(\frac{1}{2.637}\right)(17.09)(0.5) = \boxed{3.788 \text{ atm}}$$

8. Air enters a 5-cm-diameter, 4-m-long adiabatic duct with inlet conditions of $Ma_1 = 2.8$, $T_1 = 380$ K, and $P_1 = 80$ kPa. It is observed that a normal shock occurs at a location 3 m from the inlet. Taking the average friction factor to be 0.007, determine the velocity, temperature, and pressure at the duct exit.

Solution:

Take the properties of air to be k = 1.4, c_p = 1.005 kJ/kg-K, and R = 0.287 kJ/kg-K. The average friction factor is given to be f = 0.007.



The Fanno flow functions corresponding to the inlet Mach number of 2.8 are, from Table A-16,

$$Ma_1 = 2.8: \quad (fL^*/D_h)_1 = 0.4898 \qquad T_1/T^* = 0.4673, \quad P_1/P^* = 0.2441$$

First we check to make sure that the flow everywhere upstream the shock is supersonic. The required duct length from the inlet L_1* for the flow to reach sonic conditions is

$$L_1^* = 0.4898\frac{D}{f} = 0.4898\frac{0.05\text{ m}}{0.007} = 3.50\text{ m}$$

which is greater than the actual length 3 m. Therefore, the flow is indeed supersonic when the normal shock occurs at the indicated location. Also, using the actual duct length L_1, we have 0.4200

$$\frac{fL_1}{D_h} = \frac{(0.007)(3\text{ m})}{0.05\text{ m}} = 0.4200$$

Noting that $L = L_1^* - L_2^*$, the function fL^*/D_h at the exit state and the corresponding Mach number are

$$\left(\frac{fL^*}{D_h}\right)_2 = \left(\frac{fL^*}{D_h}\right)_1 - \frac{fL_1}{D_h} = 0.4898 - 0.4200 = 0.0698 \quad \rightarrow \quad Ma_2 = 1.315$$

From Table A-16, at Ma_2 =1.315: T_2/T^* = 0.8918 and P_2/P^* = 0.7183. Then the temperature, pressure, and velocity before the shock are determined to be

$$\frac{T_2}{T_1} = \frac{T_2/T^*}{T_1/T^*} = \frac{0.8918}{0.4673} = 1.9084 \qquad \rightarrow \quad T_2 = 1.9084T_1 = 1.9084(380\text{ K}) = 725.2\text{ K}$$

$$\frac{P_2}{P_1} = \frac{P_2/P^*}{P_1/P^*} = \frac{0.7183}{0.2441} = 2.9426 \qquad \rightarrow \quad P_2 = 2.9426P_1 = 2.9426(80\text{ kPa}) = 235.4\text{ kPa}$$

The normal shock functions corresponding to a Mach number of 1.315 are, from Table A-14,

$$\mathrm{Ma}_2 = 1.315:\ \mathrm{Ma}_3 = 0.7786, \quad T_3/T_2 = 1.2001, \quad P_3/P_2 = 1.8495$$

Then the temperature and pressure after the shock become

$$T_3 = 1.2001 T_2 = 1.2001(725.2\ \mathrm{K}) = 870.3\ \mathrm{K} \quad \text{and} \quad P_3 = 1.8495 P_2 = 1.8495(235.4\ \mathrm{kPa}) = 435.4\ \mathrm{kPa}$$

Sonic conditions exist at the duct exit, and the flow downstream the shock is still Fanno flow. From Table A-16,

$$\mathrm{Ma}_3 = 0.7786: \quad T_3/T^* = 1.0702, \quad P_3/P^* = 1.3286$$

$$\mathrm{Ma}_4 = 1: \quad T_4/T^* = 1, \quad P_4/P^* = 1$$

Then the temperature, pressure, and velocity at the duct exit are determined to be

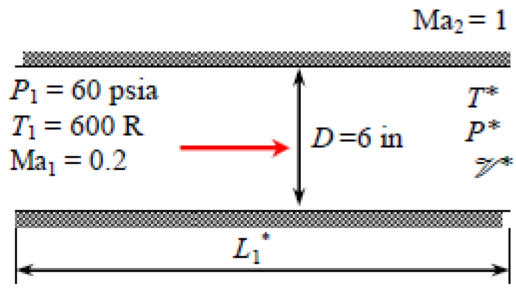
$$\frac{T_4}{T_3} = \frac{T_4/T^*}{T_3/T^*} = \frac{1}{1.0702} \quad \to \quad T_4 = T_3/1.0702 = (870.3\ \mathrm{K})/1.0702 = \mathbf{813\ K}$$

$$\frac{P_4}{P_3} = \frac{P_4/P^*}{P_3/P^*} = \frac{1}{1.3286} \quad \to \quad P_4 = P_3/1.3286 = (435.4\ \mathrm{kPa})/1.3286 = \mathbf{328\ kPa}$$

$$V_4 = \mathrm{Ma}_4 c_4 = (1)\sqrt{kRT_4} = \sqrt{(1.4)(0.287\ \mathrm{kJ/kg\cdot K})(813\ \mathrm{K})\left(\frac{1000\ \mathrm{m^2/s^2}}{1\ \mathrm{kJ/kg}}\right)} = \mathbf{572\ m/s}$$

It can be shown that $L_3^* = 0.67$ m, and thus the total length of this duct is 3.67 m. If the duct is extended, the normal shock will move further upstream, and eventually to the inlet of the duct.

9. Helium gas with k=1.667 enters a 6-in-diameter duct at Ma_1=0.2, P_1=60 psia, and T_1=600 R. For an average friction factor of 0.025, determine the maximum duct length that will not cause the mass flow rate of helium to be reduced.
Solution:
Take the properties of helium to be k = 1.667, cp = 1.2403 Btu/lbm-R, and R = 0.4961 Btu/lbm-R. The friction factor is given to be f = 0.025.



The Fanno flow function fL*/D corresponding to the inlet Mach number of 0.2 is (Table A-16)

$$\frac{fL_1^*}{D} = 14.5333$$

Noting that * denotes sonic conditions, which exist at the exit state, the duct length is determined to be

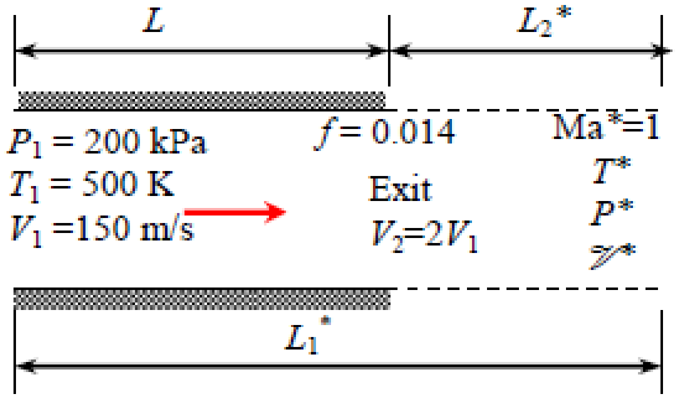
$$L_1^* = 14.5333 D / f = 14.5333(6/12 \text{ ft})/0.025 = \mathbf{291 ft}$$

Thus, for the given friction factor, the duct length must be 291 ft for the Mach number to reach Ma = 1 at the duct exit.
This problem can also be solved using equations instead of tabulated values for the Fanno functions.

10. Air enters a 15-cm-diameter adiabatic duct with inlet conditions of V_1 = 150 m/s, T_1 = 500 K, and P_1 = 200 kPa. For an average friction factor of 0.014, determine the duct length from the inlet where the inlet velocity doubles. Also determine the pressure drop along that section of the duct.

Solution:

Take the properties of air to be k = 1.4, c_p = 1.005 kJ/kg-K, and R = 0.287 kJ/kg-K. The average friction factor is given to be f = 0.014.



The inlet Mach number is

$$c_1 = \sqrt{kRT_1} = \sqrt{(1.4)(0.287 \text{ kJ/kg·K})(500 \text{ K})\left(\frac{1000 \text{ m}^2/\text{s}^2}{1 \text{ kJ/kg}}\right)} = 448.2 \text{ m/s} \quad \rightarrow \quad \text{Ma}_1 = \frac{V_1}{c_1} = \frac{150 \text{ m/s}}{448.2 \text{ m/s}} = 0.3347$$

The Fanno flow functions corresponding to the inlet and exit Mach numbers are, from Table A-16,

$$\text{Ma}_1 = 0.3347: \qquad (fL^*/D_h)_1 = 3.924 \qquad P_1/P^* = 3.2373, \quad V_1/V^* = 0.3626$$

Therefore, $V_1 = 0.3626V^*$. Then the Fanno function V_2/V^* becomes

$$\frac{V_2}{V^*} = \frac{2V_1}{V^*} = \frac{2 \times 0.3626V^*}{V^*} = 0.7252$$

The corresponding Mach number and Fanno flow functions are, from Table A-16,

$$\text{Ma}_2 = 0.693, \ (fL^*/D_h)_1 = 0.2220, \ \text{and} \quad P_2/P^* = 1.5099.$$

Then the duct length where the velocity doubles, the exit pressure, and the pressure drop become

$$L = L_1^* - L_2^* = \left(\frac{fL_1^*}{D_h} - \frac{fL_2^*}{D_h}\right)\frac{D_h}{f} = (3.924 - 0.2220)\frac{0.15 \text{ m}}{0.014} = \mathbf{39.7 \text{ m}}$$

$$\frac{P_2}{P_1} = \frac{P_2/P^*}{P_1/P^*} = \frac{1.5099}{3.2373} = 0.4664 \qquad \rightarrow P_2 = 0.4664P_1 = 0.4664(200 \text{ kPa}) = 93.3 \text{ kPa}$$

$$\Delta P = P_1 - P_2 = 200 - 93.3 = 106.7 \text{ kPa} \cong \mathbf{107 \ kPa}$$

Note that it takes a duct length of 39.7 m for the velocity to double, and the Mach number to increase from 0.3347 to 0.693. The maximum (or sonic) duct lengths at the inlet and exit states in this case are $L_1^* = 42.1$ m and $L_2^* = 2.38$ m. Therefore, the flow would reach sonic conditions if there is an additional 2.38 m of duct length.

11. The stagnation chamber of a wind tunnel is connected to a high-pressure air bottle farm which is outside the laboratory building. The two are connected by a long pipe of 4-in inside diameter. If the static pressure ratio between the bottle farm and the stagnation chamber is 10, and the bottle-farm static pressure is 100 atm, how long can the pipe be without choking? Assume adiabatic, subsonic, one-dimensional flow with a friction coefficient of 0.005.

Solution:

$\frac{p_1}{p_2} = 10.$ Also, p_1 = 100 atm. Hence, p_2 = 10 atm. If the exit is choked, then $p^* = p_2$ = 10 atm. Thus, p_1/p^* = 100/10 = 10.

From Table A.4, for p_1/p^* = 10,

$$\frac{4fL_1{}^*}{D} = 55.83$$

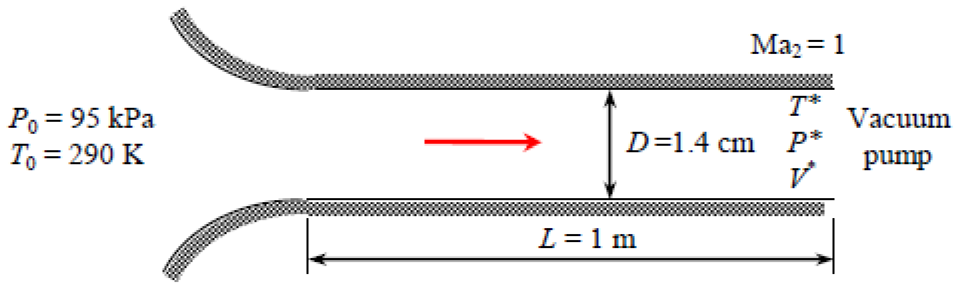
D = 4 inches = 0.333 ft.

$$L_1{}^* = \frac{55.83D}{4f} = \frac{55.83(0.333)}{4(0.005)} = \boxed{930 \text{ ft}}$$

12. Air in a room at T_0 = 300 K and P_0 = 100 kPa is drawn steadily by a vacuum pump through a 1.4-cm-diameter, 35-cm-long adiabatic tube equipped with a converging nozzle at the inlet. The flow in the nozzle section can be approximated as isentropic, and the average friction factor for the duct can be taken to be 0.018. Determine the maximum mass flow rate of air that can be sucked through this tube and the Mach number at the tube inlet.

Solution:

Take the properties of air to be k = 1.4, c_p = 1.005 kJ/kg-K, and R = 0.287 kJ/kg-K. The average friction factor is given to be f = 0.018.



The mass flow rate will be maximum when the flow is choked, and thus the exit Mach number is Ma2 = 1.

In that case we have

$$\frac{fL_1^*}{D}=\frac{fL_1}{D}=\frac{(0.025)(1\,\text{m})}{0.014\,\text{m}}=1.786$$

The Mach number corresponding to this value of fL*/D at the tube inlet is obtained from Table A-16 to be Ma1 = 0.4422.

Noting that the flow in the nozzle section is isentropic, the thermodynamic temperature, pressure, and density at the tube inlet become

$$T_1=T_{01}\left(1+\frac{k-1}{2}\text{Ma}_1^2\right)^{-1}=(290\,\text{K})\left(1+\frac{1.4-1}{2}(0.4422)^2\right)^{-1}=279.1\,\text{K}$$

$$P_1=P_{01}\left(1+\frac{k-1}{2}\text{Ma}_1^2\right)^{-k/(k-1)}=(95\,\text{kPa})\left(1+\frac{1.4-1}{2}(0.4422)^2\right)^{-1.4/0.4}=83.06\,\text{kPa}$$

$$\rho_1=\frac{P_1}{RT_1}=\frac{83.06\,\text{kPa}}{(0.287\,\text{kJ/kgK})(279.1\,\text{K})}=1.037\,\text{kg/m}^3$$

Then the inlet velocity and the mass flow rate become

$$c_1 = \sqrt{kRT_1} = \sqrt{(1.4)(0.287\,\text{kJ/kg}\cdot\text{K})(279.1\,\text{K})\left(\frac{1000\,\text{m}^2/\text{s}^2}{1\,\text{kJ/kg}}\right)} = 334.9\,\text{m/s}$$

$$V_1 = \text{Ma}_1 c_1 = 0.4422(334.9\,\text{m/s}) = 148.1\,\text{m/s}$$

$$\dot{m}_{air} = \rho_1 A_{c1} V_1 = (1.037\,\text{kg/m}^3)[\pi(0.014\,\text{m})^2/4](148.1\,\text{m/s}) = \mathbf{0.0236 kg/s}$$

This is the maximum mass flow rate through the tube for the specified stagnation conditions at the inlet.
The flow rate will remain at this level even if the vacuum pump drops the pressure even further.