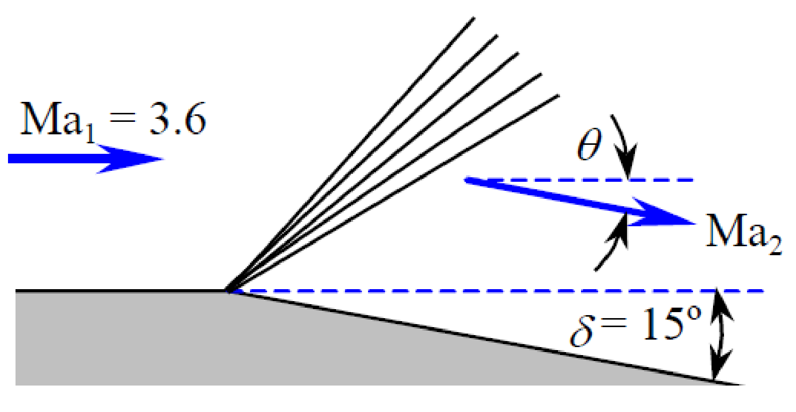
SPC 407
Sheet 7 - Solution
Compressible Flow - Normal Shock wave

Compressible Flow – Expansion Waves

1. Are the isentropic relations of ideal gases applicable for flows across (a) normal shock waves, (b) oblique shock waves, and (c) Prandtl–Meyer expansion waves?
   Solution:
   The isentropic relations of ideal gases are not applicable for flows across (a) normal shock waves and (b) oblique shock waves, but they are applicable for flows across (c) Prandtl-Meyer expansion waves.
   Flow across any kind of shock wave involves irreversible losses – hence, it cannot be isentropic.

2. Air flowing at 32 kPa, 240 K, and $Ma_1 = 3.6$ is forced to undergo an expansion turn of 15°. Determine the Mach number, pressure, and temperature of air after the expansion.
   Solution:
   Air flowing at a specified supersonic Mach number undergoes an expansion turn. The Mach number, pressure, and temperature downstream of the sudden expansion along a wall are to be determined.
   Assumptions 1. The flow is steady.
   2. The boundary layer on the wedge is very thin.
   3. Air is an ideal gas with constant specific heats.
   Properties The specific heat ratio of air is k = 1.4.



On the basis of Assumption #2, we take the deflection angle as equal to the wedge half-angle, i.e.,$\theta \approx \delta = 15°$. Then the upstream and downstream Prandtl-Meyer functions are determined to be

$$\nu(\mathrm{Ma}) = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(\mathrm{Ma}^2-1)}\right) - \tan^{-1}\left(\sqrt{\mathrm{Ma}^2-1}\right)$$

Upstream:

$$\nu(\mathrm{Ma}_1) = \sqrt{\frac{1.4+1}{1.4-1}}\tan^{-1}\left(\sqrt{\frac{1.4-1}{1.4+1}(3.6^2-1)}\right) - \tan^{-1}\left(\sqrt{3.6^2-1}\right) = 60.09°$$

Then the downstream Prandtl-Meyer function becomes

$$\nu(\mathrm{Ma}_2) = \theta + \nu(\mathrm{Ma}_1) = 15° + 60.09° = 75.09°$$

Ma_2 is found from the Prandtl-Meyer relation, which is now implicit:
Downstream:

$$\nu(\mathrm{Ma}_2) = \sqrt{\frac{1.4+1}{1.4-1}}\tan^{-1}\left(\sqrt{\frac{1.4-1}{1.4+1}\mathrm{Ma}_2^2-1)}\right) - \tan^{-1}\left(\sqrt{\mathrm{Ma}_2^2-1}\right) = 75.09°$$

Solution of this implicit equation gives Ma2 = 4.81. Then the downstream pressure and temperature are determined from the isentropic flow relations:

$$P_2 = \frac{P_2/P_0}{P_1/P_0}P_1 = \frac{[1+\mathrm{Ma}_2^2(k-1)/2]^{-k/(k-1)}}{[1+\mathrm{Ma}_1^2(k-1)/2]^{-k/(k-1)}}P_1 = \frac{[1+4.81^2(1.4-1)/2]^{-1.4/0.4}}{[1+3.6^2(1.4-1)/2]^{-1.4/0.4}}(32\,\mathrm{kPa}) = \mathbf{6.65kPa}$$

$$T_2 = \frac{T_2/T_0}{T_1/T_0}T_1 = \frac{[1+\mathrm{Ma}_2^2(k-1)/2]^{-1}}{[1+\mathrm{Ma}_1^2(k-1)/2]^{-1}}T_1 = \frac{[1+4.81^2(1.4-1)/2]^{-1}}{[1+3.6^2(1.4-1)/2]^{-1}}(240\,\mathrm{K}) = \mathbf{153K}$$

Note that this is an expansion, and Mach number increases while pressure and temperature decrease, as expected.

3. For a given Prandtl-Meyer expansion, the upstream Mach number is 3 and the pressure ratio across the wave is p2/p1 = 0.4. Calculate the angles of the forward and rearward Mach lines of the expansion fan relative to the free-stream direction.

Solution:

From Table A.5, for $M_1 = 1.5$: $\nu_1 = 11.91^\circ$ and $\mu_1 = 41.81^\circ$. So

$$\nu_2 = \nu_1 + \theta_1 = 11.91 + 20 = 31.91^\circ$$

From Table A.5, for $\nu_2 = 31.91^\circ$:

$$\boxed{M_2 = 2.207} \quad \text{and} \quad \mu_2 = 26.95^\circ$$

From Table A.1, for $M_1 = 1.5$:

$$\frac{p_{o_1}}{p_1} = 3.671 \quad \text{and} \quad \frac{T_{o_1}}{T_1} = 1.45$$

From Table A.1, for $M_2 = 2.207$:

$$\frac{p_{o_2}}{p_2} = 10.81 \quad \text{and} \quad \frac{T_{o_2}}{T_2} = 1.974$$

The flow through an expansion wave is isentropic; hence $p_{o_2} = p_{o_1}$ and $T_{o_2} = T_{o_1}$. Thus,

$$p_2 = \frac{p_2}{p_{o_2}} \frac{p_{o_2}}{p_{o_1}} \frac{p_{o_1}}{p_1} p_1 = (10.81)^{-1}(1)(3.671)(1700) = \boxed{577.3 \text{ lb/ft}^2}$$

$$T_2 = \frac{T_2}{T_{o_2}} \frac{T_{o_2}}{T_{o_1}} \frac{T_{o_1}}{T_1} T_1 = (1.975)^{-1}(1)(1.45)(460) = \boxed{337.9^\circ\text{R}}$$

$$p_{o_2} = p_{o_1} = \frac{p_{o_1}}{p_1} p_1 = (3.671)(1700) = \boxed{6241 \text{ lb/ft}^2}$$

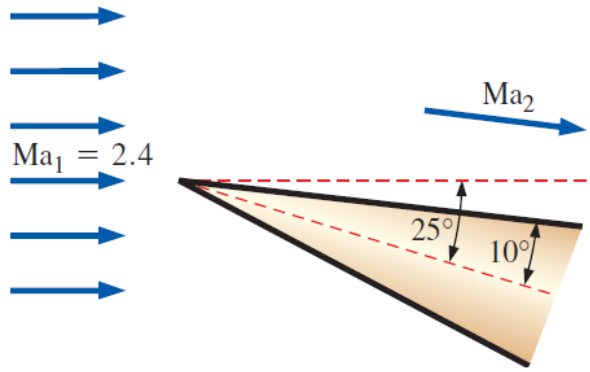
$$T_{o_2} = T_{o_1} = \frac{T_{o_1}}{T_1} T_1 = (1.45)(460) = \boxed{667^\circ\text{R}}$$

Returning to Fig. 4.32:

$$\text{Angle of forward Mach line} = \mu_1 = \boxed{41.81^\circ}$$

$$\text{Angle of rearward Mach line} = \mu_2 - \theta_2 = 26.95 - 20 = \boxed{6.95^\circ}$$

4. Consider the supersonic flow of air at upstream conditions of 70 kPa and 260 K and a Mach number of 2.4 over a two-dimensional wedge of half-angle 108. If the axis of the wedge is tilted 258 with respect to the upstream air flow, determine the downstream Mach number, pressure, and temperature above the wedge.



Solution:

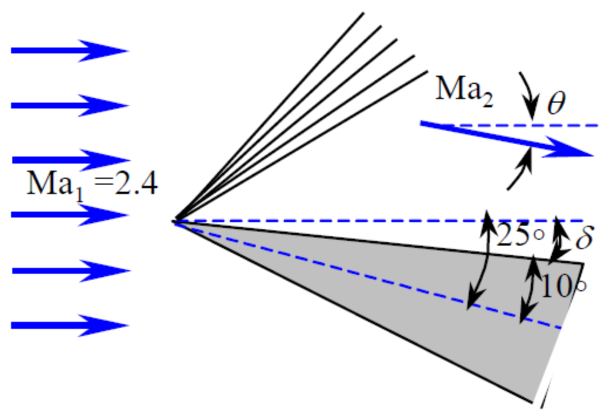
Air flowing at a specified supersonic Mach number undergoes an expansion turn over a tilted wedge. The Mach number, pressure, and temperature downstream of the sudden expansion above the wedge are to be determined.

Assumptions 1. The flow is steady.

2. The boundary layer on the wedge is very thin.

3. Air is an ideal gas with constant specific heats.

Properties The specific heat ratio of air is k = 1.4.



On the basis of Assumption #2, the deflection angle is determined to be $\theta \approx \delta = 25 - 10 = 15°$. Then the upstream and downstream Prandtl-Meyer functions are determined to be

$$\nu(\mathrm{Ma}) = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}(\mathrm{Ma}^2-1)}\right) - \tan^{-1}\left(\sqrt{\mathrm{Ma}^2-1}\right)$$

Upstream:

$$\nu(\mathrm{Ma}_1) = \sqrt{\frac{1.4+1}{1.4-1}}\tan^{-1}\left(\sqrt{\frac{1.4-1}{1.4+1}(2.4^2-1)}\right) - \tan^{-1}\left(\sqrt{2.4^2-1}\right) = 36.75°$$

Then the downstream Prandtl-Meyer function becomes

$$\nu(\mathrm{Ma}_2) = \theta + \nu(\mathrm{Ma}_1) = 15° + 36.75° = 51.75°$$

Now Ma_2 is found from the Prandtl-Meyer relation, which is now implicit:
Downstream:

$$\nu(\mathrm{Ma}_2) = \sqrt{\frac{1.4+1}{1.4-1}}\tan^{-1}\left(\sqrt{\frac{1.4-1}{1.4+1}(\mathrm{Ma}_2^2-1)}\right) - \tan^{-1}\left(\sqrt{\mathrm{Ma}_2^2-1}\right) = 51.75°$$

It gives $\mathrm{Ma}_2 = 3.105$. Then the downstream pressure and temperature are determined from the isentropic flow relations

$$P_2 = \frac{P_2/P_0}{P_1/P_0}P_1 = \frac{[1+\mathrm{Ma}_2^2(k-1)/2]^{-k/(k-1)}}{[1+\mathrm{Ma}_1^2(k-1)/2]^{-k/(k-1)}}P_1 = \frac{[1+3.105^2(1.4-1)/2]^{-1.4/0.4}}{[1+2.4^2(1.4-1)/2]^{-1.4/0.4}}(70\text{ kPa}) = \mathbf{23.8kPa}$$

$$T_2 = \frac{T_2/T_0}{T_1/T_0}T_1 = \frac{[1+\mathrm{Ma}_2^2(k-1)/2]^{-1}}{[1+\mathrm{Ma}_1^2(k-1)/2]^{-1}}T_1 = \frac{[1+3.105^2(1.4-1)/2]^{-1}}{[1+2.4^2(1.4-1)/2]^{-1}}(260\text{ K}) = \mathbf{191\ K}$$

Note that this is an expansion, and Mach number increases while pressure and temperature decrease, as expected.

5. Reconsider Prob. 4. Determine the downstream Mach number, pressure, and temperature below the wedge for a strong oblique shock for an upstream Mach number of 5.

Solution:

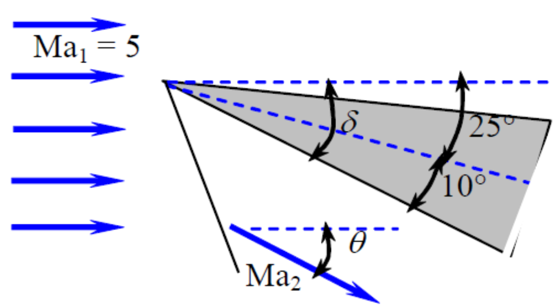
Air flowing at a specified supersonic Mach number undergoes a compression turn (an oblique shock) over a tilted wedge. The Mach number, pressure, and temperature downstream of the shock below the wedge are to be determined.

Assumptions 1. The flow is steady.

2. The boundary layer on the wedge is very thin.

3. Air is an ideal gas with constant specific heats.

Properties The specific heat ratio of air is k = 1.4.



On the basis of Assumption #2, the deflection angle is determined to $\theta \approx \delta = 25 + 10 = 35°$. Then the two values of oblique shock angle β are determined from

$$\tan\theta = \frac{2(\mathrm{Ma}_1^2 \sin^2\beta - 1)/\tan\beta}{\mathrm{Ma}_1^2(k + \cos 2\beta) + 2} \qquad \rightarrow \qquad \tan 12° = \frac{2(3.4^2 \sin^2\beta - 1)/\tan\beta}{3.4^2(1.4 + \cos 2\beta) + 2}$$

which is implicit in β . From β curve, β_{weak} = 49.86° and β_{strong} = 77.66°. Then for the case of strong oblique shock, the upstream “normal” Mach number $\mathrm{Ma}_{1,n}$ becomes

$$\mathrm{Ma}_{1,n} = \mathrm{Ma}_1 \sin\beta = 5\sin 77.66° = 4.884$$

Also, the downstream normal Mach numbers Ma2,n become

$$\mathrm{Ma}_{2,n} = \sqrt{\frac{(k-1)\mathrm{Ma}_{1,n}^2 + 2}{2k\mathrm{Ma}_{1,n}^2 - k + 1}} = \sqrt{\frac{(1.4-1)(4.884)^2 + 2}{2(1.4)(4.884)^2 - 1.4 + 1}} = 0.4169$$

The downstream pressure and temperature are determined to be

$$P_2 = P_1 \frac{2k\mathrm{Ma}_{1,n}^2 - k + 1}{k+1} = (70\,\mathrm{kPa})\frac{2(1.4)(4.884)^2 - 1.4 + 1}{1.4 + 1} = \mathbf{1940kPa}$$

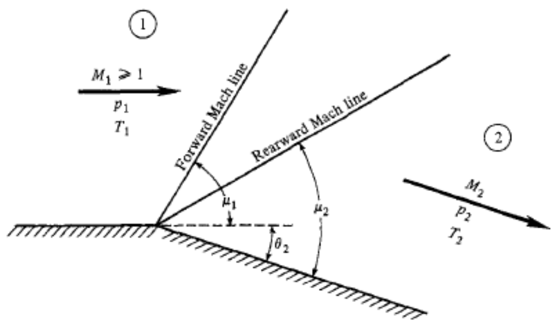
$$T_2 = T_1 \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = T_1 \frac{P_2}{P_1}\frac{2 + (k-1)\mathrm{Ma}_{1,n}^2}{(k+1)\mathrm{Ma}_{1,n}^2} = (260\,\mathrm{K})\frac{1940\,\mathrm{kPia}}{70\,\mathrm{kPa}}\frac{2 + (1.4-1)(4.884)^2}{(1.4+1)(4.884)^2} = \mathbf{1450K}$$

The downstream Mach number is determined to be

$$\mathrm{Ma}_2 = \frac{\mathrm{Ma}_{2,n}}{\sin(\beta - \theta)} = \frac{0.4169}{\sin(77.66° - 35°)} = \mathbf{0.615}$$

Note that $\mathrm{Ma}_{1,n}$ is supersonic and $\mathrm{M}_{a2,n}$ and Ma_2 are subsonic. Also note the huge rise in temperature and pressure across the strong oblique shock, and the challenges they present for spacecraft during reentering the earth’s atmosphere.

6. A uniform supersonic stream with M_1 = 1.5, p_1 = 17001b/ft2, and T_1 = 460° R encounters an expansion comer (see Fig.) which deflects the stream by an angle θ_2 = 20°. Calculate M_2, p_2, T_2, p_{O2}, T_{o2}, and the angles the forward and rearward Mach lines make with respect to the upstream flow direction.



Solution:

7. Consider a supersonic flow with an upstream Mach number of 4 and pressure of 1 atm. This flow is first expanded around an expansion comer with θ = 15°, and then compressed through a compression comer with equal angle θ = 15° so that it is returned to its original upstream direction. Calculate the Mach number and pressure downstream of the compression corner.

Solution:

From Table A.5, for M_1 = 4, ν_1 = 65.78

$$\nu_2 = \nu_1 + \theta = 65.78 + 15 = 80.78°$$

From Table A.5, M_2 = 5.44.

From Table A.1, for M_1 = 4, $\frac{p_{o_1}}{p_1}$ = 151.8

for M_2 = 5.44 $\frac{p_{o_2}}{p_2}$ = 871.3

$$p_2/p_1 = \frac{p_2}{p_{o_2}}\frac{p_{o_2}}{p_{o_1}}\frac{p_{o_1}}{p_1} = \left(\frac{1}{871.3}\right)(1)(151.8) = 0.1742$$

From the θ-β-M diagram, at the compression corner for M_2 = 5.44 and θ = 15°, β = 23.6°

$$M_{n_2} = 5.44 \sin 23.6° = 2.18$$

From Table A.2: M_{n_3} = 0.5498 and p_3/p_2 = 5.378

$$M_3 = \frac{M_{n_3}}{\sin(\beta - \theta)} = \frac{0.5498}{\sin(23.6 - 15)} = \boxed{3.67}$$

$$P_3 = \frac{p_3}{p_2}\frac{p_2}{p_1} p_1 = (5.378)(0.1742)(1) = \boxed{0.937 \text{ atm}}$$

Note that, although the flow directions in regions 1 and 3 are the same, the properties in region 3 are different than in region 1 due to the losses (entropy increase) across the shock wave.

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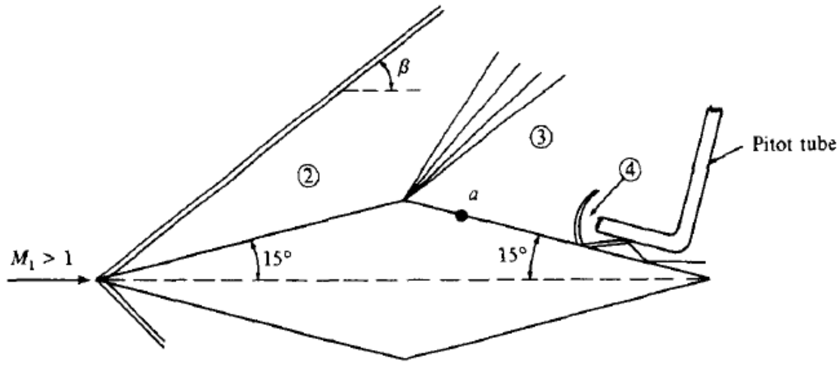
8. Consider the arrangement shown in Fig. A 15° half-angle diamond wedge airfoil is in a supersonic flow at zero angle of attack. A Pitot tube is inserted into the flow at the location shown in Fig. The pressure measured by the Pitot tube is 2.596 atm. At point a on the backface, the pressure is 0.1 atm. Calculate the free-stream Mach number M_1.



Solution:

There will be a normal shock wave in front of the face of the Pitot tube immersed in region 3 in Fig. 4.34. Let the region immediately behind this normal shock be denoted as region 4. The Pitot tube senses the total pressure in region 4, i.e., p_{o_4}. The pressure at point a is the static pressure in region 3. Thus

$$\frac{p_{o4}}{p_3} = \frac{2.596}{0.1} = 25.96$$

From Table A.2, for $p_{o4}/p_3 = 25.96$: $M_3 = 4.45$. From Table A.5, for $M_3 = 4.45$, we have $\nu_3 = 71.27°$. From Eq. (4.45)

$$\nu_2 = \nu_3 - \theta = 71.27 - 30 = 41.27°$$

From Table A.5, for $\nu_2 = 41.27°$: $M_2 = 2.6$. In region 2, we have

$$M_{n_2} = M_2 \sin(\beta - \theta) = 2.6 \sin(\beta - 15°) \qquad \text{(E.1)}$$

In this equation, both M_{n_2} and β are unknown. We must solve by trial and error, as follows.

Assume $M_1 = 4$. Then $\beta = 27°$, $M_{n_1} = M_1 \sin\beta = 4 \sin 27° = 1.816$. Hence, from Table A.2, $M_{n_2} = 0.612$. Putting these results into Eq. (E.1) above,

$$0.612 \stackrel{?}{=} 2.6 \sin 12° = 0.54$$

This does *not* check.

Assume $M_1 = 4.5$. Then $\beta = 25.5°$, $M_{n_1} = 4.5 \sin 25.5° = 1.937$. Hence, from Table A.2, $M_{n_2} = 0.588$. Putting these results into Eq. (E.1),

$$0.588 \stackrel{?}{=} 2.6 \sin 10.5° = 0.47$$

This does *not* check. We are going in the wrong direction.

Assume $M_1 = 3.5$. Then $\beta = 29.2^\circ$, $M_{n_1} = 3.5 \sin 29.2^\circ = 1.71$. Hence, from Table A.2, $M_{n_2} = 0.638$. Putting these results into Eq. (E.1),

$$0.638 \stackrel{?}{=} 2.6 \sin 14.2^\circ = 0.638$$

This *checks*. Thus

$$\boxed{M_1 = 3.5}$$