Sheet 6 Solution
Two Degree of Freedom Modal Analysis

Problem 1

1. Consider the system of Figure 1 consisting of two pendulums coupled by a spring. Determine the natural frequency and mode shapes. Plot the mode shapes as well as the solution to an initial condition consisting of the first mode shape for k = 20 N/m, l = 0.5 m and m1 = m2 = 10 kg, a = 0.1 m along the pendulum.

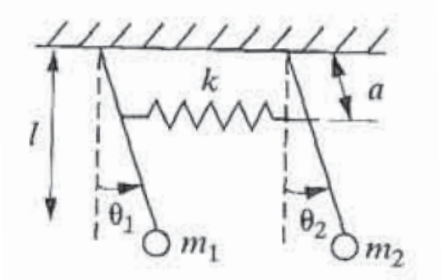


Figure 1. Two Pendulums of the same length connected by a spring used to model a machine part.

Solution: Given:

$$k = 20 \text{ N/m} \quad m_1 = m_2 = 10 \text{ kg}$$
$$a = 0.1 \text{ m} \qquad l = 0.5 \text{ m}$$

For gravity use $g = 9.81$ m/s^2. For a mass on a pendulum, the inertia is: $I = ml^2$

Calculate mass and stiffness matrices (for small θ). The equations of motion are:

$$\begin{aligned} I_1\ddot{\theta}_1 &= ka^2(\theta_2 - \theta_1) - m_1 gl\theta_1 \\ I_2\ddot{\theta}_2 &= -ka^2(\theta_2 - \theta_1) - m_2 gl\theta_2 \end{aligned} \Rightarrow ml^2\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} mgl + ka^2 & -ka^2 \\ -ka^2 & mgl + ka^2 \end{bmatrix}\begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

Substitution of the given values yields:

$$\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} 49.05 & -0.2 \\ -0.2 & 49.05 \end{bmatrix}\theta = 0$$

Natural frequencies:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 19.7 & -0.08 \\ -0.08 & 19.7 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 19.54 \text{ and } \lambda_2 = 19.7 \Rightarrow \omega_1 = 4.42 \text{ rad/s and } \omega_2 = 4.438 \text{ rad/s}$$

Eigenvectors:

$$\lambda_1 = 19.54$$

$$\begin{bmatrix} 0.08 & -0.08 \\ -0.08 & 0.08 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 19.7$$

$$\begin{bmatrix} -0.08 & -0.08 \\ -0.08 & -0.08 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

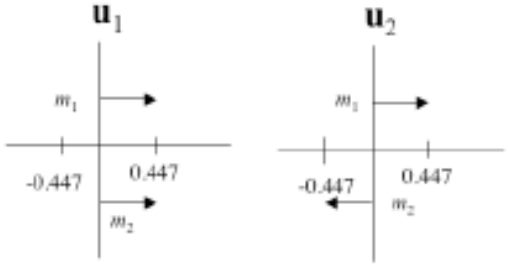
Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Mode shapes:

$$\mathbf{u}_1 = M^{-1/2}\mathbf{v}_1 = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix}$$

$$\mathbf{u}_2 = M^{-1/2}\mathbf{v}_2 = \begin{bmatrix} 0.4472 \\ -0.4472 \end{bmatrix}$$

A plot of the mode shapes is simply



This shows the first mode vibrates in phase and in the second mode the masses vibrate out of phase.

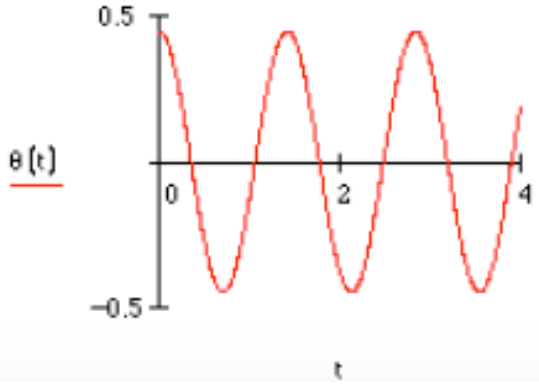
$$\theta(0) = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix} \qquad \dot{\theta}(0) = 0, \;\; S = M^{-1/2}P = \begin{bmatrix} 0.4472 & 0.4472 \\ 0.4472 & -0.4472 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 1.118 & 1.118 \\ 1.118 & -1.118 \end{bmatrix}, \;\; \mathbf{r}(0) = S^{-1}\theta(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \dot{\mathbf{r}}(0) = 0$$

From Eq. (4.67) and (4.68): $r_1(t) = \sin\left(4.42t + \frac{\pi}{2}\right) = \cos 4.45t, \;\; r_2(t) = 0$

Convert to physical coordinates: $\theta(t) = Sr(t) = \begin{bmatrix} 0.4472\cos 4.42t \\ 0.4472\cos 4.42t \end{bmatrix}$ rad

$\theta(t) := 0.4472 \cdot \cos(4.429 \cdot t)$



Problem 2

Find the response of the system shown in Figure 2 by modal analysis for the case where the rods have equal stiffness (i.e., $k_1 = k_2$), $J_1 = 3J_2$, and the initial conditions are $x(0) = [0\ 1]^T$ and $\dot{x}(0) = 0$.

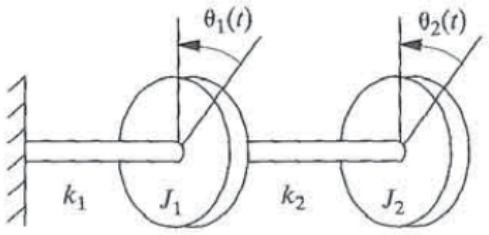


Figure 2.

The equations of motion are

$$J_2\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\ddot{\theta} + k\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\theta = 0$$

Calculate eigenvalues and eigenvectors:

$$J^{-1/2} = J_2^{-1/2}\begin{bmatrix}\frac{1}{\sqrt{3}} & 0\\ 0 & 1\end{bmatrix}$$

$$\tilde{K} = J^{-1/2}KJ^{-1/2} = \frac{k}{J_2}\begin{bmatrix}\frac{2}{3} & \frac{-1}{\sqrt{3}}\\ \frac{-1}{\sqrt{3}} & 1\end{bmatrix} \Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - \frac{5k}{3J_2}\lambda + \frac{k^2}{3J_2^2} = 0$$

$$\lambda_1 = \frac{\left(5-\sqrt{13}\right)k}{6J_2} \Rightarrow \omega_1 = \sqrt{\lambda_1}, \text{ and } \frac{\left(5+\sqrt{13}\right)k}{6J_2} \Rightarrow \omega_2 = \sqrt{\lambda_2}$$

$$\lambda_1 = \frac{\left(5-\sqrt{13}\right)k}{6J_2} \Rightarrow \begin{bmatrix}\frac{\left(5+\sqrt{13}\right)k}{6J_2} & \frac{-k}{\sqrt{3J_2}}\\ \frac{-k}{\sqrt{3J_2}} & \frac{\left(5+\sqrt{13}\right)k}{6J_2}\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix} = 0$$

$$\Rightarrow v_{11} = 1.3205v_{12} \Rightarrow \mathbf{v}_1 = \begin{bmatrix}0.7992\\ 0.6011\end{bmatrix}$$

$$\lambda_2 = \frac{\left(5+\sqrt{13}\right)k}{6J_2} \Rightarrow \begin{bmatrix} \frac{\left(-1-\sqrt{13}\right)k}{6J_2} & \frac{-k}{\sqrt{3J_2}} \\ \frac{-k}{\sqrt{3J_2}} & \frac{\left(1-\sqrt{13}\right)k}{6J_2} \end{bmatrix} \begin{bmatrix} v2_{11} \\ v_{22} \end{bmatrix} = 0$$

$$\Rightarrow v_{21} = -0.7522 v_{22} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.6011 \\ 0.7992 \end{bmatrix}$$

Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.7992 & -0.6011 \\ 0.6011 & 0.7992 \end{bmatrix}$

Calculate S and S^{-1}:

$$S = J^{-1/2} P = \frac{1}{\sqrt{J_2}} \begin{bmatrix} 0.4614 & -0.3470 \\ 0.6011 & 0.7992 \end{bmatrix}$$

$$S^{-1} = P^T J^{1/2} = J_2^{1/2} \begin{bmatrix} 1.3842 & 0.6011 \\ -1.0411 & 0.7992 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\theta(0) = S^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = J_2^{1/2}\begin{bmatrix} 0.6011 \\ 0.7992 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\theta}(0) = 0$$

Modal solution:

$$r_1(t) = \frac{\sqrt{\omega_1^2 r_{10}^2 + \dot{r}_{10}^2}}{\omega_1} \sin\left[\omega_1 t + \tan^{-1}\frac{\omega_1 r_{10}}{\dot{r}_{10}}\right]$$

$$r_2(t) = \frac{\sqrt{\omega_2^2 r_{20}^2 + \dot{r}_{20}^2}}{\omega_2} \sin\left[\omega_2 t + \tan^{-1}\frac{\omega_2 r_{10}}{\dot{r}_{20}}\right]$$

$$r_1(t) = 0.6011 J_2^{1/2} \sin\left[\omega_1 t + \frac{\pi}{2}\right] = 0.6011 J_2^{1/2} \cos\omega_1 t$$

$$r_2(t) = 0.7992 J_2^{1/2} \sin\left[\omega_2 t + \frac{\pi}{2}\right] = 0.6011 J_2^{1/2} \cos\omega_2 t$$

$$\mathbf{r}(t) = \begin{bmatrix} 0.6011 J_2^{1/2} \cos\omega_1 t \\ 0.7992 J_2^{1/2} \cos\omega_2 t \end{bmatrix}$$

Convert to physical coordinates:

$$\theta(t) = S\mathrm{r}(t) = J_2^{1/2}\begin{bmatrix} 0.4614 & -0.3470 \\ 0.6011 & 0.7992 \end{bmatrix}\begin{bmatrix} 0.6011 J_2^{1/2}\cos\omega_1 t \\ 0.7992 J_2^{1/2}\cos\omega_2 t \end{bmatrix}$$

$$\theta(t) = \begin{bmatrix} 0.2774\cos\omega_1 t - 0.2774\cos\omega_2 t \\ 0.3613\cos\omega_1 t + 0.6387\cos\omega_2 t \end{bmatrix}$$

where $\omega_1 = 0.4821\sqrt{\dfrac{k}{J_2}}$ and $\omega_2 = 1.1976\sqrt{\dfrac{k}{J_2}}$,

Problem 3

A slightly more sophisticated model of a vehicle suspension system is given in Figure 3. Calculate the natural frequencies for k1 =103 N/m, k2 = 104 N/m, m2 = 50 kg, and m1 = 2000 kg. Suppose that the tire hits a bump which corresponds to an initial condition of $x(0) = [0\ 0.01]^T$ and $\dot{x}(0) = 0$. Use modal analysis to calculate the response of the car $x_1(t)$. Plot the response for three cycles.

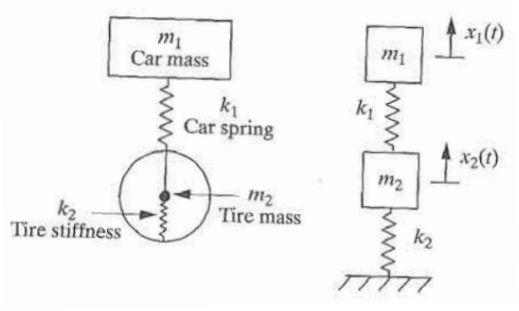


Figure 3.

The equations of motion are

$$2000\ddot{x}_1 + 1000x_1 - 1000x_2 = 0$$
$$50\ddot{x}_2 - 1000x_1 + 11{,}000x_2 = 0$$

In matrix form this becomes:

$$\begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 1000 & -1000 \\ -1000 & 11{,}000 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Natural frequencies:

$$\det\left(-\omega^2 M + K\right) = 0$$

$$\begin{vmatrix} -2000\omega^2 + 1000 & -1000 \\ -1000 & -50\omega^2 + 11{,}000 \end{vmatrix} = 100{,}000\omega^4 - 2.205\times10^7\omega^2 + 10^7 = 0$$

$$\omega_{1,2}^2 = 0.454,\ \ 220.046 \ \Rightarrow \omega_1 = 0.674 \text{ rad/s} \ \text{ and } \ \omega_2 = 14.8 \text{ rad/s}$$

Calculate the eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.0224 & 0 \\ 0 & 0.1414 \end{bmatrix}, \quad \tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 0.5 & -3.1623 \\ -3.1623 & 0.1414 \end{bmatrix}$$

$$\Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 220.05\lambda + 100 = 0 \Rightarrow \begin{matrix} \lambda_1 = 0.4545 & \omega_1 = 0.6741 \text{ rad/s} \\ \lambda_2 = 220.05 & \omega_2 = 14.834 \text{ rad/s} \end{matrix}$$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.9999 & -0.0144 \\ 0.0144 & 0.9999 \end{bmatrix}$$

Calculate S and S^{-1}:

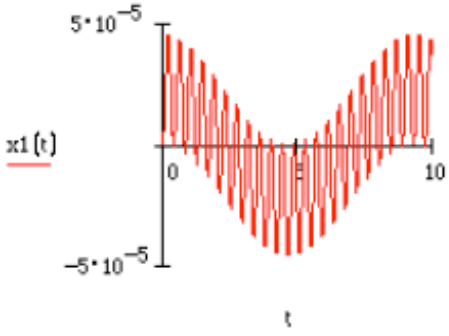
$$S = M^{-1/2} P = \begin{bmatrix} 0.0224 & -0.003 \\ 0.0020 & 0.1414 \end{bmatrix}, \quad S^{-1} = P^T M^{1/2} = \begin{bmatrix} 44.7167 & 0.1018 \\ -0.6441 & 7.0703 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = S^{-1}\begin{bmatrix} 0 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.001018 \\ 0.07070 \end{bmatrix}, \quad \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solution (from equations (4.67) and (4.68)): $\mathbf{r}(t) = \begin{bmatrix} 0.001018\cos 0.6741t \\ 0.07070\cos 14.834t \end{bmatrix}$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.0224 & -0.0003 \\ 0.0020 & 0.1414 \end{bmatrix}\begin{bmatrix} 0.001018\cos 0.6741t \\ 0.07070\cos 14.834t \end{bmatrix} = \begin{bmatrix} 2.277\times10^{-5}\cos 0.6741t - 2.277\times10^{-5}\cos 14.834t \\ 2.074\times10^{-6}\cos 0.6741t + 9.998\times10^{-3}\cos 14.834t \end{bmatrix}$$

$$x1(t) := 2.277\cdot10^{-5}\cdot\cos(0.674\cdot t) - 2.277\cdot10^{-5}\cdot\cos(14.834\cdot t)$$