Sheet 7 Solution
Multiple Degree of Freedom

Problem 1

The vibration is the vertical direction of an airplane and its wings can be modeled as a three-degree-of-freedom system with one mass corresponding to the right wing, one mass for the left wing, and one mass for the fuselage. The stiffness connecting the three masses corresponds to that of the wing and is a function of the modulus E of the wing. Find the equations of motion of the system. Will this system have a rigid body mode? If so, why?

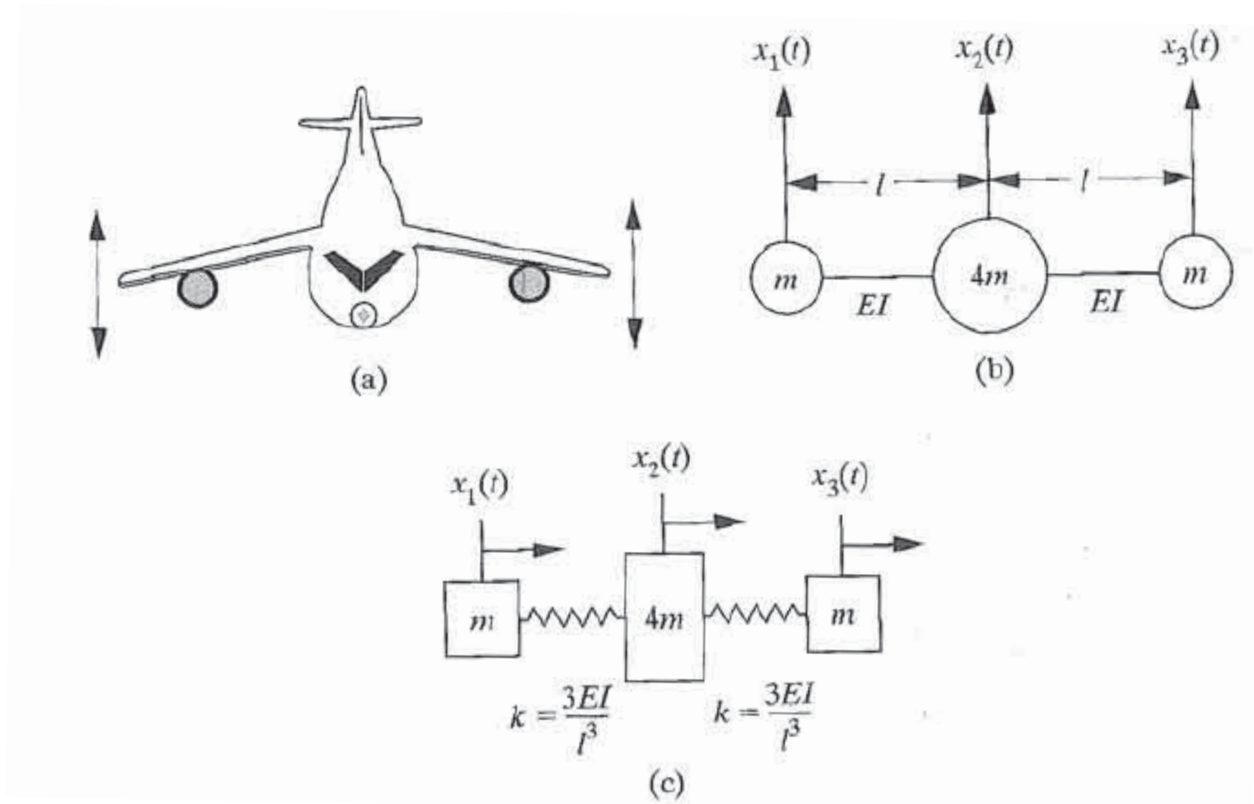


Figure 1. Model of the wing vibration of an airplane: (a) vertical wing vibration; (b) lumped mass/beam deflection model and (c) spring-mass model.

Solution

The equation of motion is

$$m\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{bmatrix}+\frac{EI}{l^3}\begin{bmatrix}3 & -3 & 0\\ -3 & 6 & -3\\ 0 & -3 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

The system will have a rigid body mode because the system is not connected to the ground.

Problem 2

Consider the two-mass system of Figure 2. This system is free to move in the x_1-x_2 plane. Hence each mass has two degrees of freedom. Derive the linear equations of motion and write them in matrix form for m = 10 kg and k = 100 N/m.

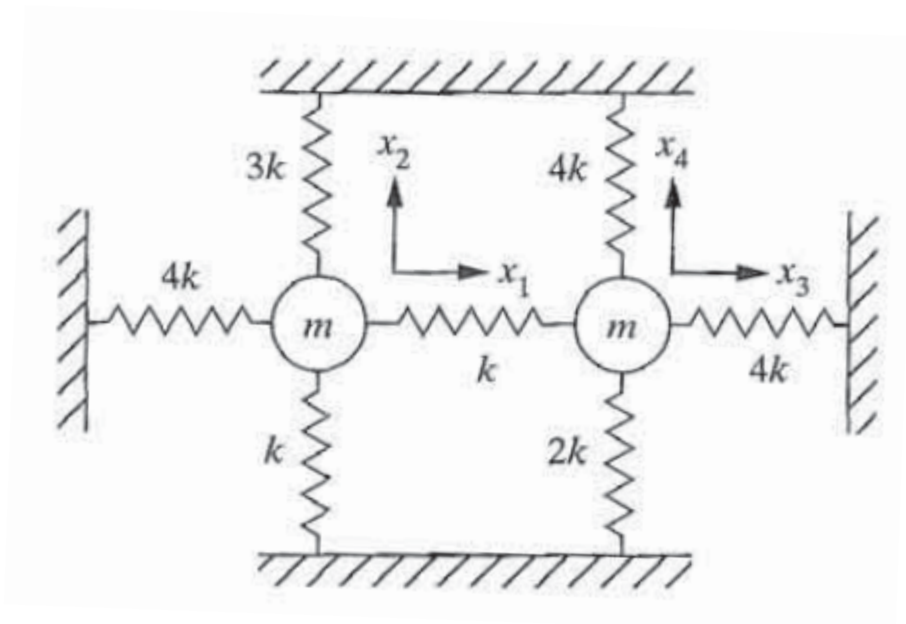


Figure 2. Two mass system free to move in two directions

Solution

Mass 1

$$x_1 - \text{direction:}\quad m\ddot{x}_1 = -4kx_1 + k\left(x_3 - x_1\right) = -5kx_1 + kx_3$$

$$x_2 - \text{direction:}\quad m\ddot{x}_2 = -3kx_2 - kx_2 = -4kx_2$$

Mass 2

$$x_3 - \text{direction:}\quad m\ddot{x}_3 = -4kx_3 - k\left(x_3 - x_1\right) = -kx_1 - 5kx_3$$

$$x_4 - \text{direction:}\quad m\ddot{x}_4 = -4kx_4 - 2kx_4 = -6kx_4$$

In matrix form with the values given:

$$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 500 & 0 & -100 & 0 \\ 0 & 400 & 0 & 0 \\ -100 & 0 & 500 & 0 \\ 0 & 0 & 0 & 600 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Problem 3

Use Lagrange's equation to derive the equations of motion of the lathe of Fig. 3.

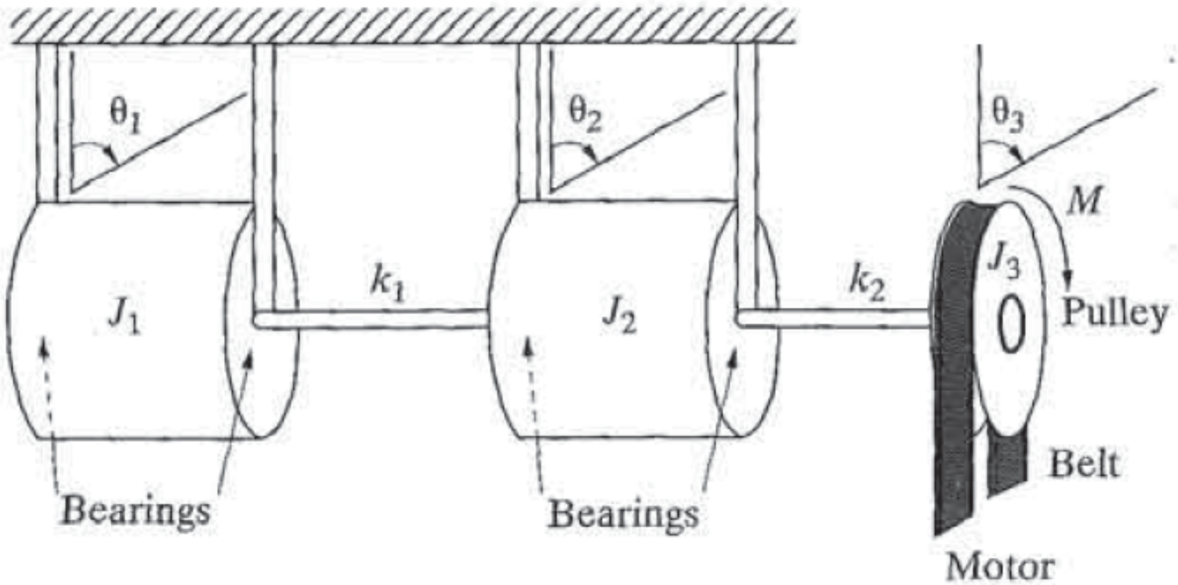


Figure 3.

Solution

Let the generalized coordinates be θ_1, $\theta_2$and θ_3

The kinetic energy is

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}J_3\dot{\theta}_3^2$$

The potential energy is

$$U = \frac{1}{2}k_1\left(\theta_2 - \theta_2\right)^2 + \frac{1}{2}k_2\left(\theta_3 - \theta_2\right)^2$$

There is a non-conservative moment M(t) on inertia 3.

$$J_1\ddot{\theta}_1 + k_1\theta_1 - k_2\theta_2 = 0$$

$$J_2\ddot{\theta}_2 - k_1\theta_1 + \left(k_1 + k_2\right)\theta_2 - k_2\theta_3 = 0$$

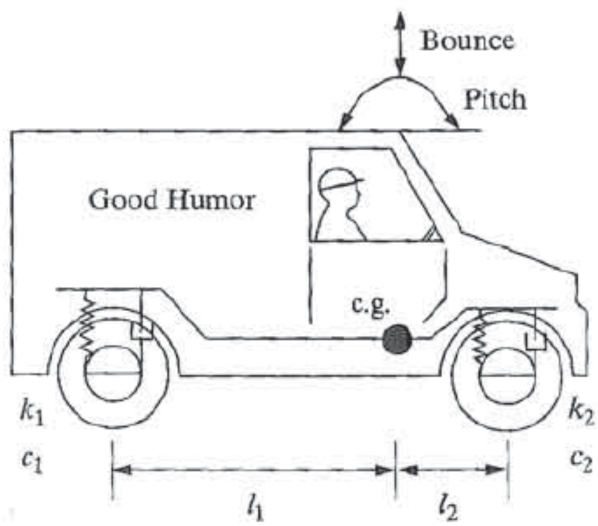
$$J_3\ddot{\theta}_3 - k_2\theta_2 + k_2\theta_3 = M\left(t\right)$$

In matrix form this yields

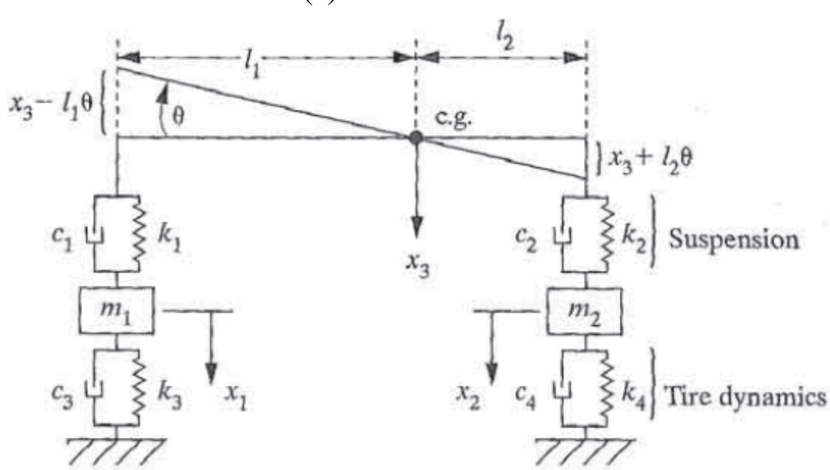
$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \ddot{\theta} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \theta = \begin{bmatrix} 0 \\ 0 \\ M(t) \end{bmatrix}$$

Problem 4

Consider the model of the vibration of an automobile of Fig. 4 (a) that allows for bounce and pitch motion. Include the tire dynamics as indicated in Fig 4 (b). Derive the equations of motion using Lagrange formulation for the undamped case. Let m3 denote the mass of the car acting at c.g.



(a)



(b)

Figure 4. Model of vibration for an automobile including tire dynamics

Solution

Let the generalized coordinates be x_1, x_2, x_3 and θ. The kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}J\dot{\theta}^2$$

The potential energy is (ignoring gravity)

$$U = \frac{1}{2}k_1\left(x_3 - \ell_1\theta - x_1\right)^2 + \frac{1}{2}k_2(x_3 - \ell_2\theta - x_2)^2 + \frac{1}{2}k_3x_1^2 + \frac{1}{2}k_4x_2^2$$

$$m_1\ddot{x}_1 + \left(k_3 + k_1\right)x_1 - k_1x_3 + k_1l_1\theta = 0$$
$$m_2\ddot{x}_2 + \left(k_4 + k_2\right)x_2 - k_2x_3 - k_2l_2\theta = 0$$
$$m_3\ddot{x}_3 - k_1x_1 - k_2x_2 + \left(k_1 + k_2\right)x_3 - \left(k_1l_1 - k_2l_2\right)\theta = 0$$
$$J\ddot{\theta} + k_1l_1x_1 - k_2l_2x_2 - \left(k_1l_1 - k_2l_2\right)x_3 + \left(k_1l_1^2 + k_2l_2^2\right) = 0$$

in matrix form

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} \left(k_3 + k_1\right) & 0 & -k_1 & k_1l_1 \\ 0 & \left(k_4 + k_2\right) & -k_2 & k_2l_2 \\ -k_1 & -k_2 & \left(k_1 + k_2\right) & -\left(k_2l_2 + k_1l_1\right) \\ k_1l_1 & k_2l_2 & -\left(k_2l_2 + k_1l_1\right) & \left(k_1l_1^2 + k_2l_2^2\right) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \theta \end{Bmatrix} = 0$$