Sheet 8 Solution
Continuous Systems

Problem 1

Find the coefficient matrix of the transverse vibration of a beam with fixed-fixed ends

Solution:

The general form of the solution of the differential equation is

$$y(x,t) = (C_1\cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x)(A\cos \omega t + B \sin \omega t)$$

For a fixed-fixed beam we apply the following boundary conditions at both ends:

- Deflection =0 $y(x,t) = 0$
- Slope =0 $\frac{\partial y}{\partial x} = y_x(x,t) = 0$
- $y(x,t)|_{x=0}$ gives $C_1 \cos 0 + C_2 \sin 0 + C_3 \cosh 0 + C_4 \sinh 0 = 0$
- $y_x(x,t)|_{x=0}$ gives $-C_1 \sin 0 + C_2 \cos 0 - C_3 \sinh 0 + C_4 \cosh 0 = 0$
- $y(x,t)|_{x=L}$ gives $C_1 \cos \beta L + C_2 \sin \beta L + C_3 \cosh \beta L + C_4 \sinh \beta L = 0$
- $y_x(x,t)|_{x=L}$ gives $-C_1 \sin \beta L + C_2 \cos \beta L - C_3 \sinh \beta L + C_4 \cosh \beta L = 0$

$y(x,t)|_{x=0}$ gives

$$C_1 + C_3 = 0$$

$y_x(x,t)|_{x=0}$ gives

$$C_2 + C_4 = 0$$

In matrix form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cos \beta L & \sin \beta L & \cosh \beta L & \sinh \beta L \\ -\sin \beta L & \cos \beta L & -\sinh \beta L & \cosh \beta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 2

Find the coefficient matrix of the transverse vibration of a beam with free ends.

Solution:

The general form of the solution of the differential equation is

$$y(x,t) = (C_1\cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x)(A\cos \omega t + B \sin \omega t)$$

For a free-free beam we apply the following boundary conditions at both ends:

Moment $=0 \Rightarrow \frac{\partial^2 y}{\partial x^2} = y_{xx}(x,t) = 0$

Shear $=0 \Rightarrow \frac{\partial^3 y}{\partial x^3} = y_{xxx}(x,t) = 0$

- $y_{xx}(x,t)|_{x=0}$ gives $-C_1 \cos 0 - C_2 \sin 0 + C_3 \cosh 0 + C_4 \sinh 0 = 0$
- $y_{xxx}(x,t)|_{x=0}$ gives $C_1 \sin 0 - C_2 \cos 0 + C_3 \sinh 0 + C_4 \cosh 0 = 0$
- $y_{xx}(x,t)|_{x=L}$ gives $-C_1 \cos \beta L - C_2 \sin \beta L + C_3 \cosh \beta L + C_4 \sinh \beta L = 0$
- $y_{xxx}(x,t)|_{x=L}$ gives $C_1 \sin \beta L - C_2 \cos \beta L + C_3 \sinh \beta L + C_4 \cosh \beta L = 0$

$y_{xx}(x,t)|_{x=0}$ gives

$$0 = -C_1 + C_3$$

$y_{xxx}(x,t)|_{x=0}$ gives

$$0 = C_1 + C_3$$

In matrix form

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -\cos \beta L & -\sin \beta L & \cosh \beta L & \sinh \beta L \\ \sin \beta L & -\cos \beta L & \sinh \beta L & \cosh \beta L \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$